Effect of fluctuations on the transition temperature in quasi-one-dimensional superconductors

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(Submitted January 29, 1974)
Zh. Eksp. Teor. Fiz. **66**, 2290–2302 (June 1974)

The temperature of the transition to the superconducting state of quasi-one-dimensional metals has a power-law dependence on the hopping probability and can be considerably lower than that predicted by the BCS theory. The power exponent in this dependence is calculated in a model with a large number of conducting bands. Its connection with the form of the correlation function of a one-dimensional conductor at zero temperature is elucidated.

1. INTRODUCTION

The interest in one-dimensional conducting structures is connected to a considerable extent with the fact that, in some of them, we may expect strong attraction between the electrons and, consequently, a high superconducting-transition temperature. However, there cannot be phase transitions in one-dimensional systems. This statement follows from the general theorem on the impossibility of coexistence of phases in one-dimensional systems[1].

Real solids are quasi-one-dimensional. There is a finite probability that an electron will hop from one filament to another. This hopping can proceed by tunneling under the barrier, or via impurities. If its probability is small the superconducting-transition temperature can be considerably lower than that which follows from the BCS formula. In certain cases, apparently, this temperature can become equal to zero.

The appearance of the superconducting transition at low temperatures can be understood from the following considerations. In the one-dimensional case, fluctuations of the phase of the order parameter destroy the long-range order, but short-range order exists over a certain distance r_c. As the temperature is lowered, r_c increases and the probability of an electron hop from filament to filament over the length r_c increases. The system becomes three-dimensional and a superconducting transition occurs in it at those temperatures for which this probability becomes sufficiently large.

In the one-dimensional case, interaction of the electrons with phonons facilitates not only the superconducting transition but also the transition to the dielectric state (the Peierls transition). Treating these transitions simultaneously makes the problem considerably more complicated and leads to the necessity, even in the self-consistent field approximation, of solving complicated "parquet" equations[2,3]. Since we wish to study the influence of the phase fluctuations on the superconductivity, we have considered a model in which the Peierls instability does not arise. In this model, there are a large number of conducting bands in each filament. Such a model gives a quantitative description of systems in which superconducting metal, in the form of thin filaments with a cross section of a few atomic units, is situated in a dielectric asbestos matrix[4]. It is possible that it is applicable to those superconductors which have a β-tungsten structure, in which all five sub-bands of the d-band are conducting.

In this model, the correlation function at zero temperature and with no hopping has a power-law dependence on the coordinates. The dependence of the exponent on the number of bands is found. An analogous situation exists in scaling theory, in which the assumption of a large number of fields permits one to find the form of the singularity at the phase-transition point[5]. This same exponent appears in the dependence of the transition temperature on the hopping probability.

2. CHOICE OF MODEL

We shall consider a system of parallel filaments. Let there be n bands in each filament. The one-electron states in each filament are characterized by the longitudinal quasi-momentum and the band index. In the case of metallic filaments with a cross-section radius of a few atomic units the different bands correspond to different states of the transverse motion. We assume the hopping probability to be small. We shall not elucidate the nature of the actual mechanisms of attraction between the electrons. We assume that the interaction is described by an effective potential $V(\mathbf{r}-\mathbf{r}')$, which includes the Coulomb repulsion. The Hamiltonian of such a system is of the form

$$\hat{H}=\sum_{i,\alpha,p}\varepsilon_\alpha(p)a_{i\alpha}^+(p)a_{i\alpha}(p)+\sum_{i,j,\alpha,\beta,p}T_{ij\alpha\beta}a_{i\alpha}^+(p)a_{j\beta}(p)+\int V(\mathbf{r}-\mathbf{r}')\psi^+(\mathbf{r})\psi^+(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r})\,d^3\mathbf{r}\,d^3\mathbf{r}', \tag{1}$$

where

$$T_{ij\alpha\beta}=T_{ij\alpha\beta}^+,\qquad \psi(\mathbf{r})=\sum_{i,\alpha,p}u_\alpha(\mathbf{r}-\mathbf{r}_i)e^{ipx}a_{i\alpha}(p).$$

The indices i, j label the filaments; α, β define the band indices and spin-state indices; $u_\alpha(\mathbf{r}-\mathbf{r}_i)\exp(ipx)$ are the eigenfunctions of an electron in a filament.

The second term in the Hamiltonian describes the hopping of electrons from filament to filament. Near the Fermi surfaces, the electron energies $\epsilon_\alpha(p)$ are equal to

$$\varepsilon_\alpha(p)=v_\alpha(|p|-p_\alpha),$$

where v_α and p_α are the longitudinal velocity and momentum at the Fermi surface in the α-th band, and p is the longitudinal momentum of the electron.

We shall consider the case when the hopping amplitude $T_{ij\alpha\beta}$ equals zero. The BCS equation determining the superconducting order parameter Δ_α is written in the form

$$\Delta_\alpha=\frac{T}{\pi}\sum_\beta V_{\alpha\beta}\sum_\omega\int\frac{\Delta_\beta\,dp}{\omega^2+\varepsilon_\beta^2(p)+\Delta_\beta^2}, \tag{2}$$

where

$$V_{\alpha\beta}=\int V(\mathbf{r}-\mathbf{r}')u_\alpha(\mathbf{r})u_{-\alpha}(\mathbf{r}')u_\beta^*(\mathbf{r})u_{-\beta}^*(\mathbf{r}')\,d^3\mathbf{r}\,d^3\mathbf{r}'.$$

The states α and $-\alpha$ are conjugate in time. The $V_{\alpha\beta}$ all have the same order of magnitude, and, therefore, we shall assume for the estimates that they are all equal to the same value, λ. Thus, in the self-consistent field approximation, for the superconducting-transition temperature T_{c0} we obtain the equation

$$1=\frac{\lambda}{\pi}\sum_{\beta}\frac{1}{v_\beta}\ln\frac{\omega_D}{T_{c0}}\equiv\frac{\lambda n}{\pi v_0}\ln\frac{\omega_D}{T_{c0}}. \tag{3}$$

We shall see that the Peierls transition does not arise in the case when the number of bands is large, and the corrections to the interaction, which have a Kohn singularity when the momentum equals $2p_F$, are small. The singularities in the "zero-sound" channel lead to a renormalization of the effective interaction, which, at frequencies of the order of ω_D, takes the form $-\lambda + g\delta_{\alpha,-\alpha}$, where

$$g=\frac{V_{\alpha,-\alpha}}{1+V_{\alpha,-\alpha}\ln(\varepsilon_F/\omega_D)}+\lambda.$$

For frequencies less than ω_D and for an arbitrary number of bands, the self-consistent field approximation corresponds to the "parquet" equations

$$\Gamma_{\alpha\beta}^{ij}=V_{\alpha\beta}^{ij}-\int_0^\xi[\Gamma_{\alpha\mu}^{ij}\Gamma_{\mu\beta}^{ij}+2\Gamma_{\alpha,-\alpha}^{ik}\Gamma_{\alpha,-\alpha}^{kj}\delta_{\beta,-\alpha} \tag{4}$$

$$-2\Gamma_{\alpha\alpha}^{ii}\Gamma_{\alpha,-\alpha}^{ij}\delta_{\beta,-\alpha}-\Gamma_{\alpha\alpha}^{ij}\Gamma_{\alpha\alpha}^{ij}\delta_{\beta,\alpha}]d\xi,$$

where

$$\xi=\ln\frac{\omega_D}{\max(\varepsilon,p^2/2m)}.$$

For large n, the most important term in the integrand in (4) is the first term, in which the summation is performed over all the bands. Allowance for only this term is equivalent to the BCS approximation and yields Eq. (3) for the transition temperature. In the next order in $1/n$ we obtain for the transition temperature, in place of formula (3),

$$1=\frac{\lambda n+|g|}{\pi v_0}\ln\frac{\omega_D}{T}.$$

Thus, for $\lambda n \gg |g|$ the "particle-hole" channel can be regarded as a perturbation.

3. FLUCTUATIONS IN ONE-DIMENSIONAL SYSTEMS FOR $T_{c0}-T \ll T_{c0}$

Fluctuations of the superconducting order parameter Δ in the one-dimensional case lead to the disappearance of the phase transition. For temperatures close to T_{c0}, these fluctuations can be described by means of the Ginzburg-Landau equation. The free energy F for fluctuations that vary slowly with distance has the form

$$F[\Delta]=\sum_i\int dx\left[A|\Delta_i(x)|^2+C\left|\frac{\partial\Delta_i(x)}{\partial x}\right|^2+\frac{B}{2}|\Delta_i(x)|^4\right]. \tag{5}$$

In formula (5) the integration is taken along a filament and the sum is taken over all the filaments. The coefficients A, B and C in our model are equal to

$$A=\frac{n}{\pi v_0}\frac{T-T_{c0}}{T_{c0}},\quad B=\frac{7\zeta(3)}{8\pi^3}\frac{n}{v_0T_{c0}^2},$$
$$C=\frac{7\zeta(3)}{16\pi^3T_{c0}^2}\sum_\alpha v_\alpha\equiv\frac{7\zeta(3)nv_{av}}{16\pi^3T_{c0}^2}. \tag{6}$$

The velocities v_{av} and v_0 are of the same order of magnitude.

Using (5), we can write an expression for the correlator $\langle\Delta(x)\Delta^*(0)\rangle$ by means of continuous integrals:

$$\langle\Delta_i(x)\Delta_i^*(0)\rangle=\int\Delta_i(x)\Delta_i^*(0)\exp\left(-\frac{F[\Delta(x)]}{T}\right)D\Delta(x)$$
$$\times\left[\int\exp\left(-\frac{F[\Delta(x)]}{T}\right)D\Delta(x)\right]^{-1}. \tag{7}$$

Calculating the continuous integral by Feynman's method[6], we obtain for large distances x

$$\langle\Delta_i(x)\Delta_i^*(0)\rangle\sim e^{-|x|/r_c}, \tag{8}$$

where r_c^{-1} is proportional to the spacing between the levels of the anharmonic oscillator[7].

Thus, in the one-dimensional case a phase transition does not occur and the correlator falls off exponentially with distance. Below we shall be interested in the region of temperatures not too close to T_{c0}, where the deviations of $|\Delta|^2$ from the value $\Delta_0^2(T)=-A/B$ that minimizes the free energy are small. Calculating these deviations by means of the formula (7), we obtain

$$\overline{(|\Delta|-\Delta_0)^2}\sim\Delta_0^2T_{c0}^{1/2}/n(T_{c0}-T)^{1/2}. \tag{9}$$

It follows from the formula (9) that fluctuations of the modulus can be neglected when

$$(T_{c0}-T)/T_{c0}\gg\frac{1}{n^{1/3}}. \tag{10}$$

Nevertheless, even when the condition (10) is fulfilled, the expression (8) indicates the absence of long-range order, this absence being due to the strong phase fluctuations[8].

The possibility of electron hops from filament to filament has little effect on the fluctuations of the modulus, but has a strong influence on the existence of the long-range order. We shall assume below that the condition (10) is fulfilled and shall study fluctuations of the phase of the order parameter for constant modulus in a model with hopping.

4. REGION OF CLASSICAL FLUCTUATIONS

Let each filament be in the superconducting state. The modulus of the order parameter experiences small deviations from its equilibrium value, and we do not take these into account.

Starting from the expression (1) for the Hamiltonian, we can find the free energy of phase fluctuations that vary slowly along the filaments, for a nonzero hopping amplitude $T_{ij\alpha\beta}$. This energy is the sum of the energy of the motion of the electrons along the filament and the tunneling energy:

$$F[\varphi]=\int\left[\frac{N_s(T)}{8m}\sum_i\left(\frac{\partial}{\partial x}\varphi_i(x)\right)^2+\sum_{i,j}W_{ij}(1-\cos(\varphi_i(x)-\varphi_j(x)))\right]dx, \tag{11}$$

where $N_S(T)$ is the linear density of "superconducting electrons," determined by the formula

$$N_s(T)=\frac{2mv_{av}n}{\pi}\sum_{k=-\infty}^{\infty}\frac{\pi T\Delta_0^2(T)}{[(2k+1)^2\pi^2T^2+\Delta_0^2(T)]^{1/2}}. \tag{12}$$

The mean effective mass m equals

$$m=\left(\sum_\alpha p_\alpha\right)\left(\sum_\alpha v_\alpha\right)^{-1}.$$

The tunneling energy W_{ij}, which is analogous to the energy of Josephson junctions, can be expressed in terms of the hopping amplitude $T_{ij\alpha\beta}$ as follows:

$$W_{ij}=\sum_\alpha|T_{ij\alpha\alpha}|^2\frac{N_s(T)}{mv_{av}v_\alpha}. \tag{13}$$

The temperature dependence of W is not the same as in the Josephson effect. This is connected with the fact that the spacing between the bands is greater than the order parameter Δ.

The formulas (12), (13) are valid for clean superconductors. Impurities change the coefficients N_S in the usual way. More important is the influence of impurities on W_{ij}. A term proportional to the impurity concentration is added to the hopping amplitude T_{ij}.

The total free energy $\mathcal{F}$ of the system is connected with the free-energy functional $F[\varphi]$ by the following relation:

$$\mathcal{F}=-T\ln\int\exp\left(-\frac{F[\varphi]}{T}\right)D\varphi. \tag{14}$$

The expression (14) for the free energy of the system is valid for sufficiently high temperatures, when the fluctuations become classical. We shall indicate the limits of the classical region later.

The exact calculation of the continuous integral in (14) with a functional $F[\varphi]$ having the form (11) is impossible. However, we can show that a system having the free energy (14) possesses a phase transition. In order to estimate the temperature T_c of this transition in order of magnitude, we shall use the self-consistent field method. In this method, the value of the order parameter $\langle\cos\varphi\rangle$ is determined from the equation

$$\langle\cos\varphi\rangle=\left[\int\exp\left(-\frac{F[\varphi]}{T}\right)D\varphi\right]^{-1}\int\cos\varphi\exp\left(-\frac{F[\varphi]}{T}\right)D\varphi, \tag{15}$$

where

$$F[\varphi]=\int\left[\frac{N_s}{8m}\left(\frac{\partial\varphi}{\partial x}\right)^2-W\langle\cos\varphi\rangle\cos\varphi\right]dx,\qquad W=\sum_j W_{ij}. \tag{16}$$

The transition temperature is found from the condition for which a nonzero solution of Eq. (15) appears. Near the transition point the order parameter is small, and we can expand in it in the right-hand side of (15). After this, we obtain

$$1=\frac{W}{T}\int\langle\cos\varphi(0)\cos\varphi(x)\rangle_0\,dx. \tag{17}$$

The averaging in (17) is performed for W = 0. The correlator appearing in formula (17) is a Gaussian integral and equals

$$\langle e^{i\varphi(0)-i\varphi(x)}\rangle_0=e^{-|x|/r_c}, \tag{18}$$

where

$$r_c=N_s(T)/2mT. \tag{19}$$

Substituting (18) in (17), we find

$$T_c^2=N_s(T_c)W/2m. \tag{20}$$

For a low hopping probability, T_c is small compared with T_{c0}. In this case, $N_S(T_c)$ coincides with the linear electron density N, and W coincides with its value W_0 at zero temperature. Then the relation (20) takes the form

$$T_c^2=NW_0/2m. \tag{21}$$

If this probability is not too small, so that

$$W_0\gg mT_{c0}^2/N,\quad nT_{ij}\gg T_{c0}, \tag{22}$$

the transition temperature is close to T_{c0} and Eq. (20) is written as follows:

$$\frac{T_{c0}-T_c}{T_{c0}}=\sqrt{\frac{T_{c0}^2m}{2W_0N}}\approx\frac{T_{c0}}{nT_{ij}}. \tag{23}$$

In the derivation of (23) it was assumed that the condition (10) is fulfilled.

An analysis in the framework of self-consistent field theory does not prove the existence of a phase transition. In our case, however, we can convince ourselves that the transition temperature is correctly determined, in order of magnitude, by the formulas (20)—(23). In fact, for $T^2\gg N_S(T)W/m$, the free energy can be calculated by expanding (14) in a series in W. The correction to the linear energy-density $\mathcal{F}'$ of an isolated filament has, in this case, the form of a series in the small parameter:

$$\mathcal{F}'=-\frac{N_s}{16m}\left(\frac{W}{T}\right)^2\left(1-\frac{3}{128}\left(\frac{WN_s}{2mT^2}\right)^2+\ldots\right). \tag{24}$$

The numerical coefficients have been obtained for a square lattice and nearest-neighbor interaction. It can be seen from (24) that for $WN_S/mT^2\ll 1$ the free energy has no singularities.

In the region of low temperatures $T\ll T_c$ the phase fluctuations are small and in formula (11) we can expand $\cos(\varphi_i-\varphi_j)$ as a series in its argument. The fluctuations then become three-dimensional and the mean order parameter $\langle\Delta\rangle$ equals

$$\langle\Delta\rangle=\Delta_0(T)\left(1-I\frac{T}{T_c}+\ldots\right). \tag{25}$$

For a square lattice and nearest-neighbor interaction,

$$I=\frac{1}{(2\pi)^2}\int_0^\pi\int_0^\pi\frac{dq_y\,dq_z}{(\sin^2q_y+\sin^2q_z)^{1/2}}.$$

Thus, for temperatures below T_c the order parameter does not vanish. For very small W the transition temperature is low and the estimates obtained are incorrect because of the neglected contribution of the quantum fluctuations, which lead to a further decrease of the transition temperature.

5. PARTITION FUNCTION OF THE SYSTEM WHEN QUANTUM FLUCTUATIONS ARE TAKEN INTO ACCOUNT

To calculate the contribution of the quantum fluctuations we shall start from the general expression for the partition function Z in terms of the Hamiltonian:

$$Z=\mathrm{Sp}\exp(-\hat{H}/T). \tag{26}$$

Using the form of the Hamiltonian (1) we rewrite the expression (26) as follows[9]:

$$Z=\mathrm{Sp}\int\left[\exp\left(-\frac{\hat{H}_0}{T}\right)T_\tau\exp\left(-\int_0^{1/T}\tilde{H}(\tau,\Delta)\,d\tau\right)\right]D\Delta(\mathbf{r},\mathbf{r}',\tau), \tag{27}$$

where T_τ is the chronological operator. In formula (27),

$$\hat{H}_0=\sum_{i,\alpha,p}\varepsilon_\alpha(p)a_{i\alpha}^+(p)a_{i\alpha}(p)+\sum_{i,j,\alpha,\beta}T_{ij\alpha\beta}a_{i\alpha}^+(p)a_{j\beta}(p),$$
$$\tilde{H}(\tau,\Delta)=\iint\left[\frac{\Delta(\mathbf{r},\mathbf{r}',\tau)\Delta^*(\mathbf{r}',\mathbf{r},\tau)}{V(\mathbf{r}-\mathbf{r}')}+\Delta(\mathbf{r},\mathbf{r}',\tau)\bar{\psi}(\mathbf{r},\tau)\bar{\psi}(\mathbf{r}',\tau)\right.$$
$$\left.+\Delta^*(\mathbf{r}',\mathbf{r},\tau)\psi(\mathbf{r}',\tau)\psi(\mathbf{r},\tau)\right]d^3\mathbf{r}\,d^3\mathbf{r}', \tag{28}$$
$$\psi(\mathbf{r},\tau)=e^{\tau\hat{H}_0}\psi(\mathbf{r})e^{-\tau\hat{H}_0},\qquad\bar{\psi}(\mathbf{r},\tau)=e^{\tau\hat{H}_0}\psi^+(\mathbf{r})e^{-\tau\hat{H}_0}.$$

We can convince ourselves of the equivalence of (26) and (27) by expanding the exponential in (27) in a series in the term that is linearly dependent on Δ and calculating the Gaussian integral. We represent the expression (27) in the form

$$Z=\int\exp(-F[\Delta(\mathbf{r},\mathbf{r}',\tau)])D\Delta(\mathbf{r},\mathbf{r}',\tau). \tag{29}$$

The functional $F[\Delta]$ is found from (27) by calculating the trace over the fermion operators. Although $\hat{H}$ in (28) depends quadratically on the fermion operators, because of the arbitrary dependence of Δ on τ it is impossible to calculate this trace in general form. We shall be interested in small deviations of the functional $F[\Delta]$ from its minimum value. The point of the minimum is determined from the equation

$$\frac{\delta F}{\delta \Delta}=\int_0^{1/T}\left\langle \frac{\delta \hat{H}}{\delta \Delta}\right\rangle d\tau=0. \tag{30}$$

Hence it follows that the value of Δ that minimizes the functional $F[\Delta]$ is equal to

$$\Delta(\mathbf{r},\mathbf{r}')=V(\mathbf{r}-\mathbf{r}')\langle\psi(\mathbf{r})\psi(\mathbf{r}')\rangle. \tag{31}$$

In the absence of hopping, Eq. (2) follows from formula (31).

It is important that Eq. (31) determines only $|\Delta|$. In formula (29) the integration is performed over the real and imaginary parts of Δ. It is convenient to change to integration over the modulus $|\Delta|$ and over the phase φ. If the condition (10) is fulfilled, the dependence of the functional F on the modulus is steep and the integration over $|\Delta|$ in (29) can be performed by the method of steepest descents. Therefore, the modulus $|\Delta|$ is assumed below to be independent of the coordinates and equal to its value determined from Eq. (2).

The integration over the phase φ in (29) cannot be performed by the method of steepest descents, since the dependence of F on φ is flat. Only those fluctuations for which the phase φ is a slow function of the coordinates and of the "time" τ are important. For these flat fluctuations, the functional $F[\varphi]$ can be written in explicit form:

$$\begin{aligned}F[\varphi]=&\iint\sum_{i,j}\frac{1}{8}K_{ij}\dot{\varphi}_i(x)\dot{\varphi}_j(x')\,dx\,dx'\,d\tau\\&+\int\left[\frac{N_s}{8m}\left(\frac{\partial\varphi_i}{\partial x}\right)^2+\sum_{i,j}W_{ij}(1-\cos(\varphi_i-\varphi_j))\right]dx\,d\tau.\end{aligned} \tag{32}$$

The last two terms in (32) have the same meaning as in (11). The coefficient K_{ij} in the first term of the formula (32) represents the susceptibility, since the time derivative of the phase is equivalent to a scalar potential. If the interaction between the electrons is weak, only the Coulomb interaction, because of its long-range character, influences the susceptibility. The expression for the Fourier component of the static susceptibility has the form

$$K(\mathbf{q})=\sum_j dx\,K_j(x)\,e^{i(q_\parallel x+\mathbf{q}_\perp\mathbf{r}_j)}=\frac{\Pi}{1+V_q\Pi}, \tag{33}$$

where Π is the compressibility of the ideal electron gas, V_q is the Fourier component of the Coulomb interaction,

$$\Pi=\frac{n}{v_0\pi},\quad V_q=\sum_{\varkappa}\frac{4\pi e^2}{(\mathbf{q}_\perp+\varkappa)^2+q_\parallel^2}, \tag{34}$$

and $\varkappa$ is a reciprocal-lattice vector. Below, values $q_\perp \sim d^{-1}$ and $q_\parallel \ll q_\perp$ will be important, where d is the distance between filaments. Therefore, with logarithmic accuracy, we have

$$K=\frac{n}{\pi v_0+ne^2\ln(d/a)}, \tag{35}$$

where a is the thickness of a filament.

This form of the coefficients in the first two terms of the functional (32) leads to the correct expression for the pair correlation function $\langle\varphi(0,0)\varphi(x,\tau)\rangle$ and the collective-excitation spectrum, coinciding with the expression that is obtained by summation of the diagrams for the two-electron Green functions [10].

Appearing in the exponent in formula (29) is the classical action as a function of the imaginary "time" τ. The functional integral is reduced by Feynman's method to the trace of an operator:

$$Z=\mathrm{Sp}\exp(-\hat{H}_{\mathrm{eff}}/T), \tag{36}$$

where

$$\begin{aligned}\hat{H}_{\mathrm{eff}}=&\iint\sum_{i,j}2K_{ij}^{-1}\hat{\rho}_i(x)\hat{\rho}_j(x')\,dx\,dx'\\&+\int\left[\sum_i\frac{N_s}{8m}\left(\frac{\partial\hat{\varphi}_i}{\partial x}\right)^2+\sum_{i,j}W_{ij}(1-\cos(\hat{\varphi}_i-\hat{\varphi}_j))\right]dx.\end{aligned} \tag{37}$$

The density operator $\hat{\rho}$ and phase operator $\hat{\varphi}$ satisfy the following commutation relation:

$$[\hat{\rho}_i(x)\hat{\varphi}_j(x')]=\delta_{ij}\delta(x-x'). \tag{38}$$

For the subsequent calculations it is more convenient to use the continuous integral (29) with the functional $F[\varphi]$ defined by formula (32).

6. EFFECT OF QUANTUM FLUCTUATIONS ON THE TRANSITION TEMPERATURE

We now examine how the quantum fluctuations influence the superconducting-transition temperature. As in the classical case, the interaction between the filaments will be taken into account in the self-consistent field approximation. For the superconducting transition temperature we obtain, in analogy with formula (17), the equation

$$1=W\int_0^{1/T}d\tau\int dx\langle\cos\varphi(0,0)\cos\varphi(x,\tau)\rangle_0. \tag{39}$$

The correlation function appearing in this expression is of the form

$$\begin{aligned}&\langle\exp\{i\varphi(0,0)-i\varphi(x,\tau)\}\rangle_0\\&=\exp\left\{-\frac{2T}{\pi}\sum_{\omega_n}\int\frac{dq[1-\cos(\omega_n\tau+qx)]}{K\omega_n^2+Nq^2/m}\right\}.\end{aligned} \tag{40}$$

In the limiting cases, the expression (40) equals

$$\begin{aligned}&\langle\exp\{i\varphi(0,0)-i\varphi(x,\tau)\}\rangle_0\\&=\begin{cases}\left(\frac{T}{T_{c0}}\right)^{2\alpha}\exp\left\{-\frac{|x|}{r_c}\right\}, & |x|T\sqrt{\frac{N}{mK}}\gg 1,\quad \tau T\sim 1\\ \left[T_{c0}^2\left(\tau^2+\frac{Nx^2}{K}\right)\right]^{-\alpha}, & |x|T\sqrt{\frac{N}{mK}}\ll 1,\quad \tau T\ll 1\end{cases},\end{aligned} \tag{41}$$

where

$$\alpha=\frac{1}{\pi}\left(\frac{KN}{m}\right)^{-1/2}=\left(\frac{v_0}{v_{\mathrm{av}}n^2}+\frac{e^2}{\pi n v_{\mathrm{av}}}\ln\frac{d}{a}\right)^{1/2}, \tag{42}$$

and r_c is given by formula (19).

At short distances the correlation function does not depend on the temperature and falls off with distance according to a power law. At large distances the correlation function, as in the classical case, depends exponentially on the coordinates. The quantum effects corresponding to the terms with $\omega_n \neq 0$ in formula (40) lead only to a pre-exponential factor.

Substituting the expression (40) into (39), we find the transition temperature

$$T_c=\left(\frac{WN}{2m}\right)^{1/2}\left(\frac{WN}{T_{c0}^2m}\right)^{\alpha/2(1-\alpha)}. \tag{43}$$

Thus, allowance for the quantum fluctuations leads to an extra factor, less than unity, in the formula for T_c.

The expression (43) gives a value for the transition temperature that is lower than the lower limit for the transition temperature that was obtained by Dzyaloshinskiĭ and Kats [11]. This discrepancy is connected with the fact that quantum fluctuations were not taken into account in the latter work.

If W and, consequently, T_c are not too small, so that

$$\alpha \ln (T_{c0}/T_c) \ll 1, \tag{44}$$

the second factor in (43) is close to unity, and the formula (43) goes over into the classical formula (20) for T_c.

In a model with a large number of bands, the parameter α is small. The Coulomb interaction increases α. For $\alpha \sim 1$ the expression (43) is incorrect. However, in the spirit of scaling theory, we assume that the correlator (40) also has a power form for $\alpha \sim 1$. An analogous situation occurs in classical two-dimensional systems [12]. The expression (43) for T_c preserves its form if by the value of α we mean the power in the correlator (41). If $\alpha < 1$, there is a superconducting transition for arbitrarily small W. If $\alpha > 1$, the integral in (39) is determined by the short distances and depends weakly on the temperature. In this case, there exists a critical value $W_c \sim mT_{c0}^2/N$ at which the transition temperature T_c vanishes. For W less than this value, there is no superconducting transition at any temperature. For W much greater than W_c, the condition (22) is fulfilled and the transition temperature is close to T_{c0}.

It may be thought that the estimate given above for the transition temperature is also valid when the number of bands is small. It is clear that in this case too, for a small hopping probability, a transition exists if, and only if, the integral of the pair correlation function $\langle \psi^+(0, 0)\psi^+(0, 0)\psi(x, \tau)\psi(x, \tau)\rangle_0$ diverges at zero temperature in the absence of hopping. The "parquet" approximation is inapplicable for the calculation of this correlator, and it is necessary to sum all the diagrams. The following possibilities can present themselves:

1. The effective interaction tends to a certain constant. In this case the correlator has a power form, with an index, of order unity, determined by this constant. The convergence of the integral of the correlator is determined by the magnitude of the index.

2. The effective interaction tends to infinity. An analogous situation occurs in the theory of the Kondo effect [13]. In this case, the correlator falls off exponentially and the integral of the correlator converges.

When the number of bands is large, the first possibility is realized. The correlator has a power form, with exponent given by formula (42). The case with a small number of bands requires further study.

7. EXPANSION IN THE LIMITS OF LOW AND HIGH TEMPERATURES

In the preceding section, all the estimates for the temperature were obtained in the self-consistent field approximation. The proof of the correctness of these estimates can be carried through, as in Sec. 4, by expanding the physical quantities in a series in a small parameter for high and low temperatures.

We introduce the Green function $G_{ij}(x, x')$ by the formula

$$G_{ij}(x, x') = \langle \varphi_i(x)\varphi_j(x')\rangle, \tag{45}$$

where

$$\langle \varphi_i(x)\varphi_j(x')\rangle = \int \varphi_i(x)\varphi_j(x') \times \exp(-F[\varphi])D\varphi \Big[\int \exp(-F[\varphi])D\varphi\Big]^{-1}. \tag{46}$$

The functional $F[\varphi]$ in (46) is defined by the expression (32).

To calculate the integral in (46), we expand the exponential of the cosine in a series in $(\varphi_i - \varphi_j)$. Each term of this series is a Gaussian integral over φ. For such averages Wick's theorem is fulfilled and each average of a product is broken down into a product of pair averages of the form

$$G_{0ij}(x, x') = \langle \varphi_i(x)\varphi_j(x')\rangle_0 = G_0(x, x')\delta_{ij}. \tag{47}$$

In formula (47) G_{0ij} is the bare Green function. In the momentum representation it is equal to

$$G_0(q, i\omega_n) = \frac{1}{{}^1/_4 K\omega_n^2 + Nq^2/4m}. \tag{48}$$

Any term of the series can be represented graphically in the usual way [14]. The simplest diagrams for the self-energy part Σ are shown in the figure.

In the limit of low temperatures compared with T_c, it is sufficient to confine ourselves to the simplest graph (see Fig. a) in the expression for Σ. For the function $G(q, i\omega_n)$ we then obtain the equation

$$G_{ij}^{-1}(q, i\omega_n) = \delta_{ij}\left(\frac{1}{4}K\omega_n^2 + \frac{1}{4}\frac{N}{m}q^2\right) + (W\delta_{ij} - W_{ij})\exp\left\{-\frac{T}{2}\sum_{\omega_n}\int G_{ii}(q, i\omega_n)\frac{dq}{2\pi}\right\}. \tag{49}$$

With the same accuracy, we find the form of the mean order parameter

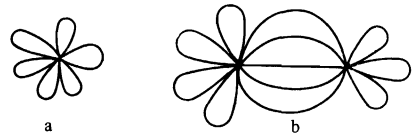
$$\langle\Delta\rangle = \Delta_0\langle e^{i\varphi}\rangle = \Delta_0 \exp\left\{-\frac{T}{2}\sum_{\omega_n}\int G_{ii}(q, i\omega_n)\frac{dq}{2\pi}\right\}. \tag{50}$$

Solving Eq. (49) and substituting the solution into (50), we have

$$\langle\Delta\rangle = \Delta_0\left(\frac{WN}{T_{c0}^2 m}\right)^{\alpha/2(1-\alpha)}\left(1 - I\frac{T}{T_c} + \ldots\right). \tag{51}$$

Allowance for the next diagrams in Σ and in the expression (50) for $\langle\Delta\rangle$ leads to corrections of the next order in α and in T/T_c. Consequently, for low temperatures the mean value of the order parameter Δ can be much smaller than the value that follows from the BCS theory, but for $\alpha \ll 1$ and $T \ll T_c$ it does not vanish.

At high temperatures, the free energy can be found by expanding the expression (29), with the functional $F[\varphi]$ defined by the formula (32), in a series in $W(1 - \cos(\varphi_i - \varphi_j))$. The correlators which then appear are calculated by means of the relations (40) and (41). At the important large distances, they differ from their classical values only by an extra factor. It is this factor which will appear in the coefficients in the expansion of the free energy. As a result, for $T_c \ll T \ll T_{c0}$ we obtain, for a square lattice and nearest-neighbor interaction,



a b

$$\mathcal{F}=-\frac{1}{16}\left(\frac{T}{T_{c0}}\right)^{4\alpha}\frac{N}{m}\left(\frac{W}{T}\right)^{2}\left(1-\frac{3}{128}\left(\frac{T_c}{T}\right)^{4-4\alpha}+\ldots\right) \quad (52)$$

In this temperature region, the free energy is expanded in a series and has no singularities in the temperature.

The expansions made for $T \ll T_c$ and $T \gg T_c$ show that the expression (43) gives a correct estimate for the superconducting-transition temperature.

8. CONCLUSION

Thus, for a high transition temperature in quasi-one-dimensional systems, a comparatively high probability of electron hops from filament to filament is necessary. This may be achieved by the inclusion of impurities. When (22) is fulfilled, the transition temperature is close to T_{c0}. In the opposite limiting case, it is considerably lower and can even vanish. At the transition point all the physical quantities have power singularities. The resistivity at this point vanishes. In the model considered, with a large number of bands, the resistivity is also low at temperatures appreciably above the transition point. In fact, near T_{c0} the conductivity σ of the fluctuational pairs has the form

$$\sigma=0.02\frac{e^2v_{av}}{T_c}f\left[n^{1/3}\left(\frac{T-T_{c0}}{T_{c0}}\right)\right], \quad (53)$$

where the function f(x) has the following asymptotic forms[15,16]:

$$f(x)=\begin{cases}x^{-1/2}, & x\to+\infty,\\ 0.9|x|^{-3/4}\exp(|x|^{3/2}), & x\to-\infty.\end{cases} \quad (54)$$

When the condition (10) is fulfilled, the conductivity σ is considerably greater than the conductivity of the normal metal.

We have considered above the usual pairing, in which the attraction between electrons in one filament is the strongest. In quasi-one-dimensional systems the opposite case is possible: the attraction between electrons in neighboring filaments is the strongest. Electrons in the same filament can even repel each other. In isotropic systems, e.g., in ^{3}He, such interaction leads to pairing with nonzero angular momentum. In quasi-one-dimensional systems with such interaction, it may be thought that a superconducting phase transition exists even in the absence of hopping. The nature of this state requires further study.

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Translated by P. J. Shepherd
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