Interaction of Langmuir solitons with plasma particles

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Evolution of the electron distribution function of a plasma is investigated on the basis of the theory developed by Rudakov.[1] A power-law distribution function is established at velocities considerably exceeding the thermal electron velocities. The dynamics of the interaction between solitons and thermal ions is considered. It is shown that nonlinear damping of solitons by ions results in slowing down of the solitons without appreciable variation of their amplitude. The soliton mean free path is calculated.

The purpose of the present study was to investigate relaxation processes with participation of charge particles in a plasma with a high level of Langmuir turbulence. We specify a homogeneous model of a strongly-turbulent state as aggregates of Langmuir solitons, and use this model to investigate effects typical of a system of strongly interacting collective degrees of freedom of a plasma. This approach was first developed by Rudakov[1] who obtained, in particular a number of important results on the interaction of electron beams with a strongly turbulent plasma. Strictly speaking, a soliton is only a particular solution of the equations of Langmuir turbulence, and the dynamics of this turbulence can be very complicated even in the hydrodynamic description (see[2, 3]). It is possible, however, to use these particular solutions to construct a strongly turbulent state[4], which explains the spectral distribution of the noise in high-turbulence-level systems investigated by the method of numerical simulation.

The electric field of a Langmuir soliton is given by

$$E(x,t)=E_0\,\mathrm{ch}^{-1}k_0\xi\cos(kx-\omega t),$$
$$k_0=\frac{eE_0}{\sqrt{6}\,T},\quad \omega=\omega_{pe}\left(1+\frac{3}{2}k^2r_D^2-\frac{3}{2}k_0^2r_D^2\right),\tag{1}$$
$$\xi=x-v_gt,\quad v_g=\partial\omega/\partial k,$$

and its spectral resolution is

$$E(x,t)=\cos(kx-\omega t)\int_0^\infty dk'E_0\left(k_0\,\mathrm{ch}\frac{\pi k'}{2k_0}\right)^{-1}\cos k'\xi$$
$$=\mathrm{Re}\int_{-\infty}^{+\infty}dk'E_0\left(2k_0\,\mathrm{ch}\frac{\pi k'}{2k_0}\right)^{-1}\exp[i(k'+k)x-i(\omega+k'v_g)t].\tag{2}$$

We note that the solution (1) is valid both at $k_0 \gg k$ (or, equivalently, $W/nT \gg k^2r_D^2$) and in the weak-turbulence approximation $k \gg k_0$; in the latter case, the soliton is simply a packet of Langmuir waves with self-consistent shape and amplitude. In the present paper, in which we consider the interaction of solitons with charged particles, we neglect the collisions of the solitons with one another; this is permissible if the characteristic number N of solitons per unit length is small enough, namely $1/Nv_g \gg \tau_E$, where τ_E is the time of the relaxation processes with particle participation. Under this assumption, we shall consider the interaction of solitons with resonant electrons, and also their nonlinear damping by thermal ions of a plasma.

1. HEATING OF RESONANT ELECTRONS BY INTERACTION WITH LANGMUIR SOLITONS

This interaction can be qualitatively understood in the following manner. Assume that an external source excites in a plasma Langmuir localized noises at sufficiently high energy density, $W/nT > k^2r_D^2$, in the form of solitons. The characteristic phase velocity in the spectral expansion of the soliton (2) is of the order of $\omega_{pe}/k_0 \sim v_{Te}\sqrt{24\pi T}/E_0$. At a sufficiently high soliton amplitude it is possible to satisfy the phase-resonance condition $\omega/k \sim v$ for a fraction of the epithermal electrons and harmonics of the spectrum, meaning that quasilinear diffusion of the resonant electrons in velocity space is possible. Effects of this type were actually observed in experiments on numerical simulation of a strongly turbulent plasma with a computer[5-7], namely, high-energy "tails" appeared in the electron distribution function at $W/nT \sim 1$.

We present in this paper a solution of a simple model problem. We exclude from consideration the Langmuir-wave source and assume that all the solitons have at a given instant of time equal amplitudes $E_0(t)$; at a specified average energy density, this means also that $N(x)$ is constant. In the absence of a source, the solitons will attenuate when they interact with the electrons, and the quantity ω/k_0 will increase with time. In order for any noticeable number of electrons to become involved in the acceleration process it is necessary to have at the initial instant of time $\omega/k_0 \sim v_{Te}$, i.e., $k_0r_D \sim 1$. At the same time, the group velocity of the soliton cannot exceed the velocity of sound, hence the limitation $kr_D < \sqrt{m/M} \ll 1$. To simplify the calculations, we put $k = 0$, and we shall show subsequently the results that follow from allowance for the finite character of k.

Owing to the strong correlation between the harmonics in the spectral expansion of the soliton, the change of the amplitude of one harmonic leads to a restructuring of the entire spectrum in similar fashion. To describe the interaction of the electrons with the solitons we shall use the system of equations of quasilinear theory, averaged over an interval whose length exceeds the average distance between solitons[1]. In our case it can be represented in the form

$$\frac{\partial f}{\partial t}=\frac{12}{\pi^2}Nv_{Te}\frac{E_0^3}{(24\pi nT)^{3/2}}\frac{\partial}{\partial\rho}\frac{1}{\rho\,\mathrm{ch}^2(1/\rho)}\frac{\partial f}{\partial\rho},\tag{3}$$

$$\frac{dE_0}{dt}=\sqrt{\frac{3}{2}}\pi^3ev_{Te}^2\int_0^\infty\frac{d\rho}{\mathrm{ch}^2(1/\rho)}\frac{\partial f}{\partial\rho},\tag{4}$$

where

$$\rho=\frac{2k_0}{\pi k}=\left(\frac{2}{\sqrt{6}\,\pi}\frac{e}{\omega_{pe}T}\right)vE_0\equiv\chi vE_0.$$

As seen from (3), the diffusion coefficient $D(\rho, t)$ has a maximum at a certain point $\rho = \rho_0 \sim 0.8$, i.e.,

$$v_0=\frac{2\sqrt{6}\,\pi}{5}\frac{\omega_{pe}T}{eE_0}$$

and is exponentially small at $\rho < \rho_0$. At $\rho > \rho_0$ we have

$D(\rho) \sim 1/\rho$. We shall henceforth assume, when solving the system (3) and (4), that

$$D(\rho) \sim \frac{1}{\rho}\theta(\rho-\rho_0),$$

where

$$\theta(\rho-\rho_0) = \begin{cases} 0, & \rho<\rho_0, \\ 1, & \rho>\rho_0. \end{cases}$$

Thus, we obtain for the boundary of the resonance region in space

$$v_0(t)E_0(t) = \text{const} \tag{5}$$

and as the soliton is damped the boundary of the resonance region moves towards the higher velocities. We can simplify also Eq. (4) in similar fashion. As a result, the system (3) and (4) takes the form

$$\frac{\partial f}{\partial t} = \psi_1 \frac{\partial}{\partial \rho}\frac{\theta(\rho-\rho_0)}{\rho}\frac{\partial f}{\partial \rho}, \quad \psi_1 = \frac{12}{\pi^3} N v_{Te} \frac{E_0^3(t)}{(24\pi nT)^{3/2}}, \tag{6}$$

$$dE_0/dt = \psi_2 f(\rho_0), \quad \psi_2 = -\sqrt{6}\,\pi^3 e v_{Te}^2. \tag{7}$$

We seek a self-similar solution of the system (6) and (7). (The form of the distribution function during the initial stage of the process, at arbitrarily initial conditions, can be established only by numerical calculation). We put

$$f(\rho,\ t) = \varphi(t)\Phi(\rho), \quad E_0 = E_0(t). \tag{8}$$

Substitution of (8) in (6) and (7) yields

$$\dot{\varphi}\Phi + \rho\Phi_\rho'\varphi\frac{\dot{E}_0}{E_0} = \psi_1\varphi(t)\frac{\partial}{\partial\rho}\frac{\theta(\rho-\rho_0)}{\rho}\frac{\partial\Phi}{\partial\rho}, \tag{9}$$

$$dE_0/dt = \psi_2\varphi(t)\Phi(\rho_0). \tag{10}$$

The condition for the separation of the variables is

$$\dot{\varphi} \sim \varphi\dot{E}_0/E_0 \sim E_0^3\varphi,$$

from which we have

$$E_0 \sim t^{-1/3}, \quad \varphi(t) \sim t^{-4/3}.$$

We denote by $f_\infty(v)$ the distribution function that is formed as the soliton becomes damped in the nonresonant region $v < v_0(t)$. As $t \to \infty$, the distribution $f_\infty(v)$ becomes established in all of velocity space. Since the diffusion coefficient is exponentially small at $v < v_0(t)$, we get from the particle conservation law

$$\int_{v_0}^{\infty} f_\infty(v)\,dv = \int_{v_0}^{\infty}\varphi(t)\Phi(\rho)\,dv = \frac{\varphi(t)}{\chi E_0(t)}\int_1^{\infty}\Phi(\rho)\,d\rho, \tag{11}$$

from which we obtain $f_\infty = \alpha/v^4$, where $\alpha = 3n_r v_0^3$ and $n_r(t) \sim 1/t$ is the density of the resonant electrons. Taking (1) into account, Eq. (10) becomes

$$\dot{E}_0 = -\sqrt{6}\,\pi^3 e v_{Te}^2 \alpha/v_0^4. \tag{12}$$

Equation (12) allows us to determine the constant ψ_1 in Eqs. (6) and (9). It is difficult to obtain an exact solution of (9), but in the asymptotic region $\rho \gg 1$ we can use the approximate solution

$$\Phi(\rho) \sim \text{const}\cdot\exp(-G\rho^3/3), \tag{13}$$

where

$$G = \frac{2e\alpha}{3\sqrt{nT}\,N v_{Te}^3} = \frac{1}{\pi}\frac{n_r}{n}\left(\frac{v_0}{v_{Te}}\right)^3 (N r_D)^{-1}.$$

Thus, heating of the electrons leads to establishment of the following distribution function:

$$f = \begin{cases} f_1 = \alpha/v^4, & v<v_0(t), \\ f_2 = t^{-4/3}\exp\left(-\dfrac{G\rho^3}{3}\right)F(\rho), & v \gg v_0(t), \end{cases} \tag{14}$$

where $F(\rho)$ is a function that varies slowly in comparison with the exponential function.

$$\alpha/v_0^4 = \varphi(t)\Phi(\rho_0).$$

At the same time, the functions f_1 and f_2 should also satisfy the particle-number conservation law (11). The asymptotic form of the solution (13) enables us to estimate the integrals in (11). It turns out that the conditions (11) and (15) can be satisfied simultaneously only at $G \sim 1$, so that it is precisely at this value of the parameter G that a self-similar solution of the problem actually exists. The point is that the quantity G has the physical meaning of the ratio of the characteristic diffusion time τ_D to the characteristic soliton-damping time τ_E:

$$G \approx \tau_D/\tau_E.$$

In the self-similar solution, the characteristic diffusion and soliton-damping time should coincide, and this corresponds to the condition $G \approx 1$.

Although the quantity G can be arbitrary at the initial instant of time, the system evolves in such a way that $G \to 1$ as $t \to \infty$. If $G \gg 1$ at the initial instant of time, then the characteristic soliton-damping time, and consequently also the time of variation of $v_0(t)$, is much shorter than the characteristic quasilinear-diffusion time: $\tau_D \gg \tau_E$. Then

$$G = \frac{1}{\pi}\frac{n_r}{n}\left(\frac{v_0}{v_{Te}}\right)^3 (N r_D)^{-1} \to 1 \tag{15}$$

as a result of the rapid decrease of n_r at a slight increase of v_0. On the other hand, if $G \ll 1$ at the initial instant of time, then the particles have time to diffuse during the characteristic damping time τ_E to velocities much larger than v_0, so that $f(v_0, \tau_E) \ll n_r/v_0$ and subsequently as $v_0 \to \infty$ the number of resonant particles changes weakly, so that $G \to 1$ in this case, too.

As a result, a self-similar solution is established and can be expressed in terms of the physical variables in the form

$$f \approx t^{-4/3}\exp\left[-\frac{1}{3}\left(\frac{2}{\sqrt{6}\pi}\frac{eE_0}{\omega_{pe}T}\right)^3 v^3\right], \quad v \gg v_0(t), \tag{16}$$

$$f \approx \frac{\alpha}{v^4} = \frac{3\sqrt{nT}\,N v_{Te}^3}{2e}v^{-4}, \quad v \leqslant v_0(t), \tag{17}$$

$$E_0(t) = (\sqrt{6}\,\pi^3 e v_{Te}^2 e^{-1/3})\,t^{-1/3}. \tag{18}$$

As $t \to \infty$, the solitons attenuate completely, and the electron distribution function tends to the form (17) in the velocity interval $v < v_{max}$, where the quantity v_{max} is determined by the limits of applicability of our analysis. Namely, the solitons do not overlap in space under the condition $N \ll k_0(t) \sim E_0(t)$. Thus, the maximum velocity of the accelerated electrons should be estimated at $v_{max} \approx \omega_{pe}/N$.

2. NONLINEAR DAMPING OF LANGMUIR SOLITON BY THERMAL IONS OF A PLASMA

In the weakly turbulent state, the principal nonlinear effect for Langmuir waves is induced scattering by ions. The maximum characteristic value of the increment is in this case $\gamma \sim \omega_{pe}W/nT$. Scattering by ions makes an appreciable contribution to the evolution of the spectrum if a field exists with values of ω, **k**, ω', and **k**' such that

$$(\omega-\omega')/|\mathbf{k}-\mathbf{k}'|v_{Ti} \leqslant 1.$$

For a Langmuir soliton this condition can be written in the form $v_g \lesssim v_{Ti}$, which follows directly from the spectral expansion (2). Thus, the soliton should be attenuated by particles that are at resonance with the group velocity of the packet. We shall show that it is possible to obtain in a rather simple manner equations that describe this damping.

For processes with characteristic times larger than $1/\omega_{pe}$, the interaction of the electron with the soliton field can be described with the aid of the averaged force $-\nabla(e^2E^2(\xi)/2m\omega^2)$. When the electrons move under the influence of this force, a charge-separation field is produced, which we shall describe by the potential ϕ. Assuming the process to be quasistatic also for the ions, we can use the following simple equations for the particle densities:

$$n_e=n_0\exp\left[\frac{e\phi}{T_e}-\frac{e^2E^2(\xi)}{2m\omega^2T_e}\right],\qquad n_i=n_0\exp\left[-\frac{e\phi}{T_i}\right]. \tag{19}$$

Using the quasineutrality condition $n_i - n_0 = n_e - n_0$ and substituting in (19) the soliton field from (1), we obtain

$$\phi(\xi)\approx\frac{E_0^2}{8\pi ne}\frac{T_i}{T_i+T_e}\operatorname{ch}^{-2}k_0\xi. \tag{20}$$

Thus, in the time scale of the nonlinear interaction, the soliton represents for the ions a rarefaction wave whose field is given by the potential (20). The damping of such a wave can be obtained by considering the reflection of particles from a potential barrier[8]. In the coordinate system connected with the wave, the ions that are trapped are those with velocities $v < v_m(\xi)$, where

$$v_m(\xi)=\left[\frac{2e(\phi_{max}-\phi(\xi))}{M}\right]^{1/2}=v_{Ti}\frac{E_0}{\sqrt{4\pi nT}}\operatorname{th}k_0\xi. \tag{21}$$

If the characteristic time of reflection of the particle from the barrier is shorter than the characteristic damping time

$$k_0v_{Ti}E_0/\sqrt{4\pi nT}>\gamma, \tag{22}$$

then we can consider particles that arrive from infinity and go off to infinity, for which we should put $\tanh(k_0\xi) = 1$ in (21).

In the laboratory frame, the trapped particles are those in the velocity interval

$$v_g-v_m<v<v_g+v_m,\qquad v_m=v_{Ti}E_0/\sqrt{4\pi nT}.$$

The absolute value of the ion-energy increment as a result of reflection is $2Mv_g|v_g - v|$. The total rate of change of the soliton energy (per unit area) can be obtained from the relation

$$\frac{d\mathscr{E}}{dt}=Mn_0\int_0^{v_m}[f^i(v_g+\delta v)-f^i(v_g-\delta v)]2v_g\delta v^2\,d\delta v.$$

Assuming f^i to be a Maxwellian function, we arrive at the following expression:

$$\frac{d\mathscr{E}}{dt}\approx-\frac{9}{4(2\pi)^{1/2}}\omega_{pe}\frac{T_e}{e}\left[\frac{MT_i}{m(T_i+T_e)}\right]^{1/2}(kr_D)^2\frac{E_0^4}{(4\pi nT)^{3/2}}. \tag{23}$$

In the opposite limiting case, at $v_g \gg v_{Ti}$, the damping is exponentially small. Rudakov[1] obtained an analogous equation as the equation of induced scattering in the limiting case of strong correlation. He took account the soliton damping only by the untrapped particles—both effects consequently produce equal distributions. We note that the characteristic reciprocal damping time of the soliton differs by a coefficient on the order of $kk_0r_D^2$ from the nonlinear increment known from the weak-turbulence theory. The correct limiting transition to the weakly-turbulent state is obtained if account is taken of the fact that in this case the nonlinear scattering conserves the number of plasmons, and only the energy of the plasmon can change by an amount on the order of $\delta\omega/\omega_{pe} \sim k^2r_D^2$. It is interesting to use the analogy between soliton damping and nonlinear scattering for the solution of Eq. (23).

Langmuir oscillations in a strongly turbulent state can be regarded as plasmons that are "trapped" in a well. We define the total number of plasmons[2,3] in the following manner

$$\sum_k N_k=\int d\xi\frac{\overline{E^2(\xi)}}{4\pi\omega_{pe}}. \tag{24}$$

In the case of weak turbulence, the right-hand side of (24) is actually equal to $\int dkW_k/\omega_k$, accurate at least to $(kr_0)^4$. The same quantity, as shown by Zakharov and Shabat[2,3], is the integral of the strong-turbulence dynamic equations. The number of quanta trapped in the Langmuir soliton (1) is

$$\sum_k N_k=\sqrt{6}E_0T/2\pi e\omega_{pe}, \tag{24'}$$

and the total soliton energy is

$$\mathscr{E}=\int d\xi\frac{\overline{E}^2}{8\pi}\left(1+\frac{\omega_{pe}^2}{\omega^2}\right)+\frac{3r_D^2}{4\pi}\int d\xi\overline{\left|\frac{\partial E}{\partial x}\right|^2}+O(E^4), \tag{25}$$

If the number of quanta is conserved in the case of nonlinear soliton damping, just as induced scattering of the waves in weak-turbulence theory, then the wave amplitude E_0 is conserved, as follows from (24'). The total energy (25) can change only as a result of a change in the wave number k, which determines the propagation velocity. The corresponding term in (25) is equal to

$$\frac{3}{2}k^2r_D^2\int d\xi\frac{\overline{E^2(\xi)}}{4\pi}.$$

Substituting it in (23) we get

$$\frac{d}{dt}(kr_D)^2=-\frac{3}{4\sqrt{3\pi}}\omega_{pe}\left[\frac{MT_i}{m(T_i+T_e)}\right]^{1/2}\frac{E_0^3}{(4\pi nT)^{3/2}}(kr_D)^2. \tag{26}$$

Equation (26) can be obtained also by direct calculation. To this end it is necessary to derive in similar fashion, in addition to (23), also the momentum balance equation

$$\frac{d}{dt}\mathbf{P}=-\frac{3}{4(2\pi)^{1/2}}\left[\frac{MT_i}{m(T_i+T_e)}\right]^{1/2}(\mathbf{k}r_D)\frac{E_0^4}{4\pi nT}, \tag{27}$$

where the soliton momentum is $\mathbf{P} \approx \sqrt{6}E_0T\mathbf{k}/2\pi e\omega_{pe}$.

Simultaneous solution of (24) and (27) leads to conservation of soliton amplitude and to Eq. (26). Thus, the law for the conservation of the number of quanta, if correctly defined, can be used not only in weak-turbulence theory, but also in the limiting case of a strong correlation between the harmonics. Substituting further the solution of (26) in the expression for the group velocity and integrating with respect to t, we obtain the path traversed by the soliton until it is completely stopped

$$L=8\sqrt{3\pi}r_D(kr_D)_{t=0}\left[\frac{m(T_e+T_i)}{MT_i}\right]^{1/2}\frac{(4\pi nT)^{3/2}}{E_0^3}. \tag{28}$$

We have thus shown that the interaction of solitons with thermal ions leads to an exponential decrease, in time, of the soliton velocity without a substantial change in its amplitude. We note that expressions (26) and (28) are valid only under the condition

$$\frac{E_0}{\sqrt{4\pi nT}} < \left[\frac{m(T_e+T_i)}{MT_i}\right]^{1/2} \tag{29}$$

In the opposite case, the characteristic stopping time, determined by the right-hand side of (26), turns out to be smaller than the time $\tau_s^{-1} \sim \omega_{pe}(k_0r_D)^2$ of soliton formation. This means apparently that such intense nonlinear waves exist with $v_g = 0$. Another possibility of the existence of solitons of this amplitude is $v_{Ti} < v_g < c_s$, If we now substitute the characteristic slowing-down time of the soliton and the right-hand side of the condition (22), then it turns out that our derivation of (23) is valid only if $k_0r_D < m/M$, meaning that at large amplitudes the soliton is slowed down more rapidly than during the time of particle reflection. The qualitative conclusion remains the same, namely, the conservation law for the number of quanta leads to a slowing down of the soliton without its appreciable damping. The lower bound of the quantitative slowing-down rate $\dot{k}/k$ can be determined from an equality that is the inverse of the condition (22), and accordingly the path of the soliton cannot exceed

$$L_{max} \sim r_D(kr_D)_{t=0}\left[\frac{M(T_e+T_i)}{mT_i}\right]^{1/2}\frac{4\pi nT}{E_0^2}. \tag{30}$$

The idea of the possibility of using the law of conservation of the number of quanta in the limiting case of strong turbulence, and in particular, the ensuing conclusion that the solitons are slowed down, belongs to L. I. Rudakov. The authors are sincerely grateful to him for interest in the work and for valuable discussions.

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