Quantum effects in white holes

Ya. B. Zel'dovich, I. D. Novikov, and A. A. Starobinskiĭ

Institute for Applied Mathematics, USSR Academy of Sciences
L. D. Landau Institute for Theoretical Physics, USSR Academy of Sciences
(Submitted January 21, 1974)
Zh. Eksp. Teor. Fiz. **66**, 1897–1910 (June 1974)

The influence of quantum effects (particle production near the singularity) on the external properties of a white hole is considered. It is shown that spontaneous particle production radically changes the properties of a white hole. In the big-bang model of the Universe, a white hole separates from the surrounding expanding matter of the Universe in the epoch $t \approx r_g/c$, where r_g is the gravitational radius measured by an external observer after the individualization of the white hole. Excluding extremely artificial choices of initial conditions, a white hole either blows up practically without delay at the epoch $t \approx r_g/c$ or never explodes at all. An appeal to the energy of explosions of white holes for an explanation of the energy production in astrophysical phenomena seems extremely unlikely.

INTRODUCTION

A white hole is a hypothetical object expanding from a singularity to the outside of its Schwarzschild horizon. One of the authors [1] and later also Ne'eman [2] have shown that the existence of such bodies (within the framework of classical general relativity) is possible in an expanding Universe. According to this hypothesis white holes are nuclei of matter of the Friedmann cosmological model which have been retarded in their cosmological expansion (owing to local inhomogeneities of the initial conditions). The delay (retardation) of the expansion in the time of an external observer can be prescribed arbitrarily. The magnitude of the delay and also the composition and the kinetic energy of the emitted matter are given as initial conditions at the singularity (or arbitrarily close to the singularity). The delayed core is surrounded by a cavity of lower density (a vacuole) or, in the limit, by vacuum.

It is tempting (but not necessary!) to compare the hypothesis of white holes with a series of not yet completely explained astrophysical phenomena, in particular quasars and explosions in galactic nuclei. In the present paper an attempt is made to take into account quantum phenomena which must occur in a white hole. The theory predicts spontaneous particle production inside the white hole near the singularity of spacetime. The produced particles have an extremely strong influence on the metric of spacetime and interact with the expanding matter exterior to the white hole, as well as with the matter of the retarded nucleus, and thus radically change the whole phenomenon. As will be shown below, it seems very unlikely that one can appeal to white hole explosions in order to explain energy production in astrophysical phenomena.

Let us consider the situation in more detail. The general theory of relativity introduces the concept of a noneuclidean spacetime continuum. After giving up the idea of euclidean space it is not hard to make the next step and introduce a change in the topology of space. Thus have appeared the ideas of Eddington and others on "channels" (wormholes) connecting "our" space with "other" spaces and of the possibility of "injection" of matter and energy through such a channel. We first stress that the presence of such channels does not allow one to change the mass of isolated structures by injection of energy from another space. There is a rigorous theorem asserting that the total mass of any object measured by a far away observer by means of the gravitational field can change only on account of an influx of mass (energy) through a remote sphere of <u>our</u> space, a sphere which surrounds the object in question.

The idea of white holes differs from Eddington's idea. A single space with a simply connected topology is considered. The evolution of the Universe as a whole is taken into account. The expansion starts from a singularity. Within the white hole, in the vacuum of the vacuole ($R_1 < R < R_2$) in the simplest case the singularity is of the Schwarzschild type and outside the vacuole ($R > R_2$) the singularity is of the Friedmann type (cf. Fig. 1).

The singularity is everywhere spacelike—both outside and inside the white hole—,i.e., there exists a frame in which this singularity is simultaneous. However, on account of the inhomogeneity of the singularity there appears a retardation (delay) (according to the clock of an external observer) of the expansion of part of the matter and this part expands later for an external observer. Depending on the magnitude of the length of the Schwarzschild singularity (the quantity $R_2 - R_1$) the retardation between the delayed matter and the external matter can be arbitrarily large for an external observer.

In the case of matter with vanishing pressure $p = 0$ an exact solution has been constructed in [1] for the case when in an isotropic homogeneous Universe some spherical mass M is replaced by a white hole of the same mass. In this case the unperturbed cosmological solution survives in the external region.

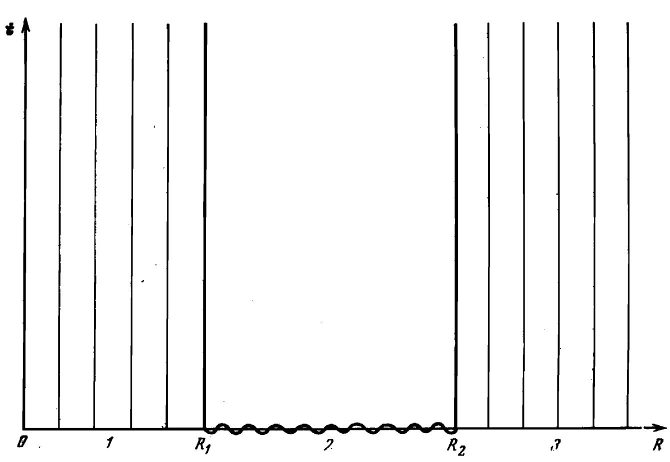

FIG. 1. The structure of space time near a white hole singularity without particle production: 1—the singularity of the delayed (retarded) core, 2—the Schwarzschild singularity, 3—the Friedmann singularity. The regions filled by dustlike matter have been shaded.

In the case when the retarded matter has a nonvanishing pressure p, in particular $p = \epsilon/3$, mathematical difficulties arise in the consideration of the interior region of the vacuole, difficulties which are not problems of principle and do not call into question the existence of solutions of the type of a "white hole." However, if $p \neq 0$ in the whole surrounding Universe, the external solution gets perturbed. Indeed, there is a discontinuity of the pressure in the solution at the boundary between the unperturbed Universe and the vacuum Schwarzschild solution. The discontinuity in pressure leads to a flow of plasma from the exterior into the interior region. The white hole leads to an accretion of plasma for any ratio of the proper mass of the hole to the mass torn off from the cosmological solution. So far there are only rough estimates of the effect, and there is no complete picture of the phenomenon. It remains to find out whether, taking accretion into account, one may use the hypothesis of white holes for the explanation of astrophysical phenomena.

In addition to the accretion problem there exists another side of the problem of the possible existence of white holes, related, first to the fact that in a white hole an extremely strong anisotropic expansion of space exists in the vacuole, near the Schwarzschild singularity, in distinction from the isotropic expansion outside the white hole in the isotropic Universe, and second, related to the fact that with the "white hole hypothesis" a remote observer will "see" the Schwarzschild singularity throughout all the delay time. In a white hole the so-called T_+-region is situated underneath the gravitational radius r_g, where all particles move only from the singularity towards r_g and can get out from under r_g to an external observer (in this connection, cf. [3] and the review in [4]).

The first circumstance—anisotropic expansion near the singularity—leads, as shown in [5,6], to a strong spontaneous particle production. The second circumstance—the fact that this occurs in an expanding T_+ region—gives rise to the possibility that the produced particles are ejected beyond r_g to an external observer, and consequently can lead to a spontaneous reduction of the mass of the white hole. Indeed, as will be shown in detail below, the produced particles very strongly modify the metric of space-time below r_g. As a result, in the real case of a big-bang Universe, the produced particles turn out to be "locked-in" in the white hole first, during the very early stages of the expansion of the Universe for $t \ll r_g/c$, on account of the pressure of the surrounding gas, and later, for t of the order of r_g/c and $t > r_g/c$, owing to gravitational self-closure: the gravitation of the produced particles is not allowed out beyond r_g. Owing to the outflow of particles a white hole can lose only an insignificant fraction of its mass.

It is very important that all the changes related to the produced particles lead to impeding the explosion of the delayed core if the retardation is larger than r_g/c. Thus, the main conclusion is that in a hot (big-bang) Universe a white hole consists of a mass of particles produced near a Schwarzschild singularity under which is "buried" a delayed core. The matter consisting of these particles expands and for a remote observer the outer boundary of the matter approached r_g from within asymptotically.

For times $t \gg r_g/c$ any emission of radiation or outflow of matter from the white hole decays exponentially very fast (in spite of the fact that an approach to r_g accompanies the expansion!). Thus, although the white hole can exist arbitrarily long, for an external observer it acquires the characteristics of a black hole: there occurs a freezing of processes and the boundary of matter tends to r_g. However, a qualitative distinction is the fact that such an object appears as a result of a (quantum) explosion from the interior and the singularity to r_g, and not as a result of gravitational collapse towards r_g of initially rarefied matter.

In the following sections we analyze in detail the quantum effects in white holes which have been enumerated above.

1. CALCULATION OF PARTICLE PRODUCTION

We first calculate the production of particles near the singularity in the vacuole. The vacuole which surrounds the delayed core is considered to be empty near the singularity. Near the Schwarzschild singularity there occurs particle production. The production of particles is a nonlocal effect. Since the Schwarzschild singularity is spacelike and homogeneous, the center of mass of each volume element of produced particles must be immobile in a space-time reference frame in which the singularity is simultaneous. Such a frame is the T system of Novikov [3] with the metric (we set $c = 1$)

$$ds^2 = dt^2 - e^\lambda dR^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{1}$$

where λ and r depend only on t.

As the singularity is approached we have $r \to 0$. We choose $t = 0$ at the singularity. Then asymptotically, as $t \to 0$, the metric (1) corresponds to Kasner's axially symmetric solution:

$$r \propto t^{2/3}, \quad e^{\lambda/2} \propto t^{-1/3}. \tag{2}$$

Particle production in such a metric has been calculated in [6]. The process occurs near the singularity $t = 0$. In order to obtain a finite expression for the energy density of the produced particles it is necessary to switch on the production at some $t_0 > 0$. It is likely that t_0 must be of the order of the Planck time $t_{Pl} = (G\hbar)^{1/2}$, $t_0 \sim t_{Pl}$. For the time being we leave the magnitude of t_0 arbitrary. The production process dies out very rapidly as t increases: $d\epsilon/dt \propto t^{-5}$. Therefore after $t \approx (3-5)t_0$ one may consider the particle production completed. At a time of the order t_0 the energy density of the produced particles is [5,6]

$$\varepsilon_0 \approx \hbar / t_0^4 = t_{Pl}^2 / G t_0^4. \tag{3}$$

After the production (occurring practically at the time t_0) there occurs a decrease of the energy density ϵ at subsequent times owing to the expansion of the system (1). The energy density decreases adiabatically and for a further concrete calculation one must know the equation of state of the produced particles.

2. A MODEL SOLUTION FOR $p = 0$

We start from the simplest assumption that the pressure vanishes identically ($p = 0$) in the delayed core, in the matter formed by the produced particles near the singularity, and in the external matter of the expanding Universe. These assumptions are blatantly false, but, as we shall see, they allow one to construct an exact solution on which one can clearly follow the important features surviving also in the more complicated case of nonvanishing pressure.

Thus, we shall assume that everywhere $p=0$. Fig. 1 represents space-time near the singularity of the white hole surrounded by the cold ($p=0$) matter of the expanding Universe, furthermore without taking into account the particles produced near the singularity, i.e., according to Novikov's original 1964 hypothesis [1]. According to the particle-production hypothesis [5,6], particles are produced near the Schwarzschild singularity (where the deformation of spacetime is highly anisotropic) and are not produced either near the singularity in the delayed core, nor near the Friedmann singularity (where the deformation of space-time is isotropic) [1)]. The process of matter production near the Schwarzschild singularity has been discussed in the preceding section. Let us follow through the further expansion of the produced matter. We first recall that at $p=0$ the external matter of the Universe to the right of the Lagrangian coordinate R_2 does not at all affect the internal solution [4]. The influence of the boundary R_1 can be essential, since, depending on the initial conditions, this boundary may have a velocity relative to the produced matter. Moreover, the motion of this boundary and its influence on the whole solution determines the explosion of the retarded nucleus. However, we shall take this influence into account later, and for the moment we consider the Schwarzschild singularity sufficiently extended in the Lagrangian coordinate R and shall consider produced matter which has not yet been reached by the influence of the boundary R_1.

In order to describe the expansion of the produced matter it is necessary to solve the Einstein equation for the metric (1) (cf. Appendix I, Eq. (I.1)) with the initial conditions

$$r/r_g=e^{-\lambda}=(3t/2r_g)^{2/3} \quad \text{for} \quad t=t_0\ll r_g, \tag{4}$$

where $r_g=\text{const.}$[2)] This solution is written in the parametric form:

$$\begin{gathered} r={}^1/_2 r_g(1-\cos\xi), \quad t={}^1/_2 r_g(\xi-\sin\xi), \\ e^{\lambda/2}=\operatorname{ctg}(\xi/2)+\alpha[1-{}^1/_2\operatorname{ctg}(\xi/2)], \\ 8\pi G\varepsilon=\alpha r^{-2}e^{-\lambda/2}, \quad 0\leqslant\xi\leqslant 2\pi. \end{gathered} \tag{5}$$

The constant α is determined from the condition (3):

$$\alpha\approx r_g t_{Pl}^{-1}(t_{Pl}/t_0)^2\sim r_g/t_{Pl}\gg 1. \tag{6}$$

The solution (5) describes the expansion of the whole produced matter from the instant $t=t_0$ to the size of the gravitational radius r_g and its subsequent compression to the singularity. The whole solution, including the region to the right of the Lagrangian coordinate R_2 is schematically represented in Fig. 2. For an external observer the boundary of the expanding matter approaches the gravitational radius only after an infinite time, asymptotically. The phase of matter compression ($\pi\leq\xi\leq 2\pi$) is not observed by an external observer at all.

Thus, in the indicated approximation ($p=0$) there is no flux of produced matter beyond the gravitational radius. Although the singularity produces particles via quantum effects, the matter never gets out from under r_g.

We stress two important properties of the solution (5) which persist also for solutions with other equations of state of matter. The first property: at the maximum of expansion of r, when $dr/dt=0$, the matter energy density ϵ is finite, independent of the initial conditions and equals

$$\varepsilon=(8\pi G r_{max}^2)^{-1}=(8\pi G r_g^2)^{-1}. \tag{7}$$

The second property is: the path traversed by a signal with the velocity of light along the radial coordinate R from the instant of beginning of expansion to the instant t_1 when $dr/dt=0$ is finite and is of the order of magnitude r_g, independent of the initial conditions.

The second property leads directly to an important result. If a portion of the Schwarzschild singularity is sufficiently long in the Lagrangian coordinate R, i.e. if in the frame (5)

$$R_2-R_1>R_{hor}=\int_0^{t_1} dt\, e^{-\lambda/2}=\frac{r_g}{\alpha}\cdot 2\int_0^{\pi/2}\frac{\sin^2 z}{1-z\operatorname{ctg} z}\,dz\approx 7\frac{r_g}{\alpha} \tag{8}$$

for $\alpha\gg 1$, then a signal emitted at $t=0$ at the left boundary R_1 is manifestly unable to reach the right boundary R_2 over the whole time of expansion of this boundary (up to the time when R_2 reaches the point A in Fig. 2; then $\xi=\pi$) no matter what velocity $v\leq 1$ it had. This means that a signal can never get out from under r_g to an external observer.

Thus, if $R_2-R_1>R_{hor}$, then no matter what conditions are imposed on the expanding nucleus, "matching" it at the initial instant t_0 to the boundary R_1, this nucleus can never expand beyond r_g to an external observer, i.e., for such an observer the white hole does not explode.

It is further clear that in order that a signal from R_1 should reach an external observer with a large delay time τ with respect to the clock of the external observer started at the instant of beginning of expansion of the external matter of the Universe, it is necessary that the length R_2-R_1 be only slightly smaller than the critical length R_{cr} corresponding to an infinite retardation time (the reaching of the boundary R_2 at A by a signal

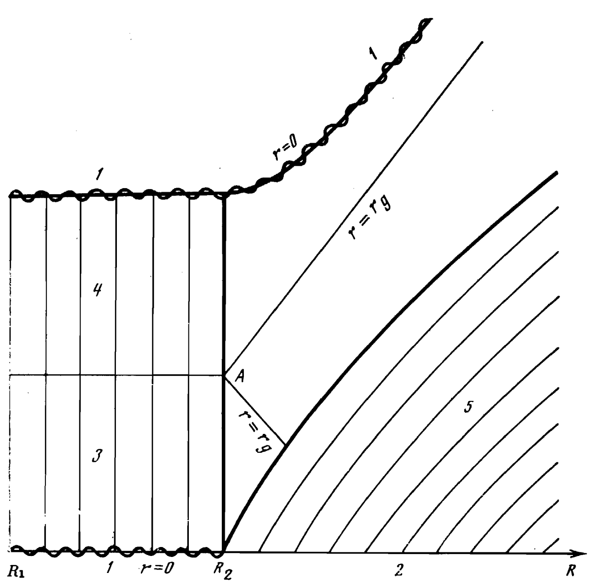

FIG. 2. The complete space-time for a white hole in the Friedmann model with p = 0, including the phase of compression of the dustlike matter of the white hole: 1—the Schwarzschild singularity, 2—the Friedmann singularity, 3—the phase of expansion of the white hole matter, 4—the phase of its compression, 5—the expanding matter of the Friedmann Universe.

from R_1).[3] The approximate formula has the form

$$R_{cr}-(R_2-R_1)\approx R_{cr}\exp(-\tau/2r_g), \tag{9}$$

where the quantity R_{cr} depends on the speed of the signal. If the speed of the signal tends to the speed of light, then $R_{cr}-R_{hor}$.

Thus, the enumerated properties make extremely unlikely a prolonged delay of the expansion of the matter of the white hole with subsequent explosion, since this would require an artificial readjustment of the initial data. The white hole either explodes practically instantaneously, during a time $t<r_g$ (for $R_2-R_1<R_{cr}$)[4], or it never explodes (for $R_2-R_1>R_{cr}$).

3. MASSLESS PARTICLES IN A WHITE HOLE

Let us consider a more realistic equation of state for particles produced near the Schwarzschild singularity. We assume that the produced particles do not interact with each other and move with the speed of light along the radial coordinate. Then $T_0^0=-T_1^1=\epsilon$, and all other $T_i^k=0$. In order not to consider accretion problems we shall assume, as before, that the white hole is placed in a cold Universe, i.e., that outside the white hole, at $R>R_2$, the pressure of matter vanishes. As in the case of dust, we shall not take into account the influence of the boundary R_1 for the time being (we consider first that it is sufficiently remote).

The problem reduces to the calculation of the motion of free massless particles produced at $t=t_0$ with a density (3) on the segment $R_1<R<R_2$. One can solve this problem exactly and the solution is given in Appendices I and II. At the time t_0 the produced particles form two noninteracting fluxes of equal density moving against each other with the speed of light. In the region of produced particles to which the decompression wave has not arrived yet the solution describes an expansion followed by a contraction of the system. These particles never get out from under r_g. The particles flying to the right which have been reached by the decompression wave before r reaches its maximal value in the expansion are emitted from under r_g towards an external observer. The flow of these particles carries off mass. Thus, in this model the white hole inevitably decreases its mass immediately owing to quantum effects. The calculation carried out in Appendices I and II shows that the ratio of the initial mass M_0 of the white hole (determined by the conditions near the singularity) to the final mass M_1 after emission, is

$$\frac{M_1}{M_0}=\frac{r_{max}}{r_g}=\left(\frac{\ln\gamma}{\gamma}\right)^{1/2},\quad \gamma\approx\left(\frac{r_g}{t_{Pl}}\right)^{4/3}\left(\frac{t_{Pl}}{t_0}\right)^{10/3}\gg 1. \tag{10}$$

We note that M_0 is determined by the properties of the "vacuum" solution near the Schwarzschild singularity (the magnitude of r at the time t_0) and M_1 is determined by r at the time of maximum expansion of the homogeneous metric ($2GM_1=r_{max}$). As we shall see in this model problem, the mass decreases quite substantially.

If the white hole explodes after a long delay, the energy released will be much smaller than the energy "introduced" into the singularity, which is carried away by the produced particles. One may not, however, apply this conclusion directly to astrophysics, since in this model the white hole is placed in a cold Universe. As we shall see in the following section, placing the white hole in a "hot Universe" is quite essential for the interpretation.

We finally note that a second property of the solution of the preceding section holds for the solution under consideration, namely: the path traversed along the radial coordinate in the matter of the produced particles is finite. It follows immediately (as in the preceding section) that if the left boundary is sufficiently remote

$$R_2-R_1>R_{hor}={}^1/_2 r_g\gamma^{-1}\ln\gamma, \tag{11}$$

the white hole will never explode. The condition (9) is necessary for a long delay of the explosion.

We consider now another assumption on the equation of state of the produced particles. We assume that owing to the interaction of the particles with each other the produced particles exhibit Pascal pressure $p=\epsilon/3$, i.e., $T_1^1=T_2^2=T_3^3=-T_0^0/3$. In the rest we maintain the conditions of the problem the same as in the preceding case. A simple exact solution for all space-time does not exist in this case, since from the boundary R_2 matter diverges to the right under the influence of hydrodynamic forces, and a decompression wave travels along the matter to the left with a speed $3^{-1/2}\leq v<1$.

However, an exact solution can be found for the matter which has not yet been reached by the decompression wave. This solution is listed in Appendix I (cf. Eq. (I.4)). Qualitatively it coincides with (5). The energy carried away by the divergent matter can be estimated in the following manner. We shall assume that from the surface which was initially attained by the decompression wave matter is emitted with the speed of light and with a density equal to the density of particles moving along the radius from R_1 to R_2 at that instant. Under these assumptions

$$\frac{M_1}{M_0}=\frac{r_{max}}{r_g}=2\beta^{-1/4},\quad \beta\approx\left(\frac{r_g}{t_{Pl}}\right)^{4/3}\left(\frac{t_{Pl}}{t_0}\right)^{8/3}\gg 1. \tag{12}$$

In this case the mass loss is also quite insignificant.

Other properties of this solution are analogous to the case of massless free particles which has been discussed above.

4. A WHITE HOLE IN THE HOT UNIVERSE

Finally, let us consider a white hole situated in a hot Universe. We shall assume that the equation of state in the matter outside the white hole is $p=\epsilon/3$. In addition we shall consider, as before, the apparently most realistic case when the particle production instant $t_0\approx t_{Pl}$. In this case the most acceptable approximation for the equation of state of the produced particles will be the isotropic pressure $p=\epsilon/3$. The expansion of the matter of produced particles (cf. Appendix I) becomes isotropic practically right after t_0 and remains isotropic up to a time slightly smaller than t_1, when r attains its maximum. The expansion of the matter of produced particles during the period $t_0<t<t_1$ occurs isotropically according to a law coinciding with the expansion law of the external matter of the hot (big-bang) Universe.

Thus, in the interval $t_0<t<t_1$ the presence of the boundary R_2 has practically no influence on the solution, since the conditions on both sides of the boundary R_2 are practically the same.

The expansion of the produced matter occurs up to dimensions of the gravitational radius r_{max} which is much smaller than r_g, the one "put into" the singularity (determined from r at the time t_0 by means of the initial condition (4)). Thus, the mass of the white hole decreases in the course of expansion. But this does not signify an outflow of mass from the white hole (no mass flows through the boundary R_2 in the given case, up to $t \sim t_1$ the boundary R_2 is unimportant). The decrease of mass is caused by the work done by pressure forces at the surface $R = R_2$ during the expansion. In the same manner the mass of any separated comoving volume of the Friedmann hot Universe decreases in the course of expansion. Indeed, the mass of the comoving volume of the hot model varies according to the law (cf.[7]):

$$M = M_0 a_0/a, \tag{13}$$

where $a(t)$ is a scale factor. If one uses for the white hole the relations

$$r_{max}^2 = r_g r_{Pl}, \quad r_{Pl} = r_g^{1/3} t_{Pl}^{2/3}, \tag{14}$$

one obtains

$$r_{max}/r_{Pl} = (r_g/t_{Pl})^{1/3}.$$

Substituting into (13) r_{max}/r_{Pl} in place of a/a_0 we obtain (12). Thus, in the hot Universe the mass of the white hole does not flow out of the hole throughout the period of expansion, but decreases only on account of the work done by the pressure forces corresponding to the decrease of the mass of any comoving volumes during expansion of the hot Universe.

This can be shown in two ways. Indeed, the second equation of the system (I.1) of Einstein equations for the metric (1) has the integral (taking into account the initial condition (4))

$$r\left(\frac{dr}{dt}\right)^2 = r_g - r + 8\pi G \int\limits_{r(t_0)}^{r} r^2 T_1^1\, dr. \tag{15}$$

It follows that

$$r_{max} = r_g + 8\pi G \int\limits_{r(t_0)}^{r_{max}} r^2 T_1^1\, dr \tag{16}$$

(We note that $r_{max} = r_g$ for $T_1^1 = 0$). If the pressure in the hot Universe is equal to the pressure of the produced particles in the white hole then the integral in the right-hand side of Eq. (16) equals exactly the work done by the pressure forces on the white hole in its expansion, $T_1^1 \leq 0$, so that $r_{max} \leq r_g$.

Only at a late stage, when $t \sim t_1$ the expansion of matter to the left of the boundary R_2 begins to differ markedly from the expansion of the external matter. This is the process of individualization of the white hole. In order to determine exactly the outflow or influx of mass, hydrodynamic calculations become necessary here. However, it seems that this process does not modify substantially the final mass of the white hole (cf. in this connection [4]). All conclusions of the preceding sections on the necessity of extremely specific initial conditions for a long retardation of the explosion of the white hole (Eq. (9)) remain valid also for a white hole in a hot Universe.

The question whether a white hole could explode after a long delay with the production of energy of the order $E = Mc^2$ requires separate consideration.

CONCLUSION

Spontaneous particle production in the vicinity of a singularity in a white hole has an importance of principle. In the big-band model of the Universe the white hole "individualizes" from the surrounding space at the epoch $t \sim t_1 \approx r_{max}/c$. The white hole either explodes before t_1 or (excluding degenerate initial conditions given by Eq. (9)) it never explodes.

In the light of what was said, it seems unlikely that one can explain in terms of explosions of white holes astrophysical phenomena of the type of quasar explosions or explosions of galactic nuclei. In addition it seems that the existence in nature of so-called "bare" singularities [8] is impossible, since they will immediately wrap themselves in a "fur" of produced particles.

APPENDIX I

The Einstein equations for the metric (1) have the form

$$\begin{aligned} 8\pi G T_0^0 &= 2\frac{\dot a}{a}\frac{\dot r}{r} + \left(\frac{\dot r}{r}\right)^2 + \frac{1}{r^2}, \\ 8\pi G T_1^1 &= 2\frac{\ddot r}{r} + \left(\frac{\dot r}{r}\right)^2 + \frac{1}{r^2}, \\ 8\pi G T_2^2 = 8\pi G T_3^3 &= \frac{\ddot a}{a} + \frac{\ddot r}{r} + \frac{\dot a}{a}\frac{\dot r}{r}, \end{aligned} \tag{I.1}$$

where $a = e^{\lambda/2}$ and the dot denotes differentiation with respect to t.

For dustlike matter ($T_1^1 = T_2^2 = T_3^3 = 0$) an exact solution of the system (I.1) is given by Eq. (5); then $r = r_{max} = r_g$ for $t = t_1 = r_g/2$ ($\xi = \pi$). If $\alpha > \pi^{-1}$, and in the particle production problem $\alpha \gg 1$, then for $t = \pi r_g$ ($\xi = 2\pi$), as well as for $t = 0$ ($\xi = 0$) the metric has a singularity of the Kasner type with exponent $(-1/3, 2/3, 2/3)$.

The exact solution of the system (I.1) for matter with an equation of state $p = \epsilon/3$ ($T_1^1 = T_2^2 = T_3^3 = -T_0^0/3$) is of the form [5)]

$$\begin{aligned} &dt = {}^3/_2 (r/a)^{1/3} d\zeta, \quad z = a^{2/3} r^{1/3}, \\ &|B - z^2| = C|A - \zeta|, \quad 8\pi G\varepsilon = B(ar^2)^{-4/3}, \\ &r = r_0 + 3C^{-2}(Bz - {}^1/_3 z^3), \end{aligned} \tag{I.2}$$

where A, B, C are arbitrary constants ($B, C > 0$). If $C = 0$ the last equation of the system (I.2) must be replaced by

$$r = {}^3/_2 B^{-1/2}(A\zeta - {}^1/_2 \zeta^2).$$

Taking into account the initial conditions (4) we select the arbitrary constants in the following manner:

$$A = r_g^{2/3}, \quad B = \beta r_g^{4/3}, \quad C = |\beta - 1|, \quad r_0 = -r_g \frac{3\beta - 1}{(\beta - 1)^2}, \tag{I.3}$$

where $\beta > 0$. Then (I.2) can be rewritten in the form (for $\beta \neq 1$)

$$\begin{aligned} &a^{2/3} r^{1/3} = [r_g^{2/3} + (\beta - 1)\zeta]^{1/2}, \\ &r = (\beta - 1)^{-2}\{[r_g^{2/3} + (\beta - 1)\zeta]^{1/2}[-(\beta - 1)\zeta + r_g^{2/3}(3\beta - 1)] - r_g(3\beta - 1)\}, \\ &dt = {}^3/_2 (r/a)^{1/3} d\zeta, \quad 8\pi G\varepsilon = \beta r_g^{2/3}(ar^2)^{-4/3}. \end{aligned} \tag{I.4}$$

From the equation (3) it follows that in the problem of expansion of matter consisting of the produced particles

$$\beta \approx \left(\frac{r_g}{t_{Pl}}\right)^{2/3}\left(\frac{t_{Pl}}{t_0}\right)^{8/3} \sim \left(\frac{r_g}{t_{Pl}}\right)^{2/3} \gg 1, \tag{I.5}$$

therefore we shall consider in the sequel just this case.

For $t \ll r_g\beta^{-1/2}$ the spatial curvature of the metric (1) is unimportant and the substitution $\zeta = r_g^{2/3}\beta^{-1}(\cosh^4\lambda - 1)$ the solution (I.4) transforms into an exact solution for the spatially flat model of Bianchi type I, filled with matter, with the equation of state $p = \epsilon/3$, which was found in the paper of Doroshkevich [10]. Then, for $t \sim r_g\beta^{-3/2}$ the expansion is Kasnerian with exponent $(-1/3, 2/3, 2/3)$, and for $r_g\beta^{-3/2} \ll t \ll r_g\beta^{-1/2}$ it is Friedmannian ($a \sim r \sim t^{1/2}$). For $t \sim r_g\beta^{-1/2}$ the spatial curvature of the metric (1) becomes essential.

For $t \gg r_g\beta^{-3/2}$ and $\beta \gg 1$ the solution (I.4) can be written in the form

$$e^{\lambda} \equiv a^2 = \frac{\beta^2\zeta}{3r_g^{2/3} - \zeta}, \quad r = \left(\frac{\zeta}{\beta}\right)^{1/2}(3r_g^{2/3} - \zeta), \qquad \text{(I.6)}$$
$$t = 3\left(\frac{3}{\beta}\right)^{1/2} r_g\left[1 - \left(1 - \frac{\zeta}{3r_g^{2/3}}\right)^{3/2}\right].$$

at the instant of maximal expansion

$$\zeta = r_g^{2/3}, \quad r = r_{max} = 2r_g\beta^{-1/2}, \quad t = t_1 = r_g\beta^{-1/2}(3^{3/2} - 2^{3/2}).$$

For $\zeta = 3r_g^{2/3}$, $t = 3^{3/2}r_g\beta^{-1/2}$ the metric has a singularity of the Kasner type with exponents $(-1/3, 2/3, 2/3)$. The size of the horizon along the R axis at the instant of maximal expansion is

$$R_{hor} = \int_0^{t_1} \frac{dt}{a(t)} = 8r_g\beta^{-3/2}. \qquad \text{(I.7)}$$

We now construct a solution of the system (I.1) for the case when the produced particles are free and move only along the radius, i.e., $T_2^2 = T_3^3 = 0$ and $T_1^1 = -T_0^0 = -\epsilon$. Then $8\pi G\epsilon = \gamma a^{-2} r^{-2}$, where

$$\gamma = \left(\frac{r_g}{t_{Pl}}\right)^{4/3}\left(\frac{t_{Pl}}{t_0}\right)^{10/3} \sim \left(\frac{r_g}{t_{Pl}}\right)^{4/3} \gg 1. \qquad \text{(I.8)}$$

For $t \ll r_g\gamma^{-1/2}$ one may neglect the spatial curvature, i.e., the term r^{-2} in the system (I.1), and then the solution corresponding to the initial conditions (4) is of the form

$$r = (2r_g\eta)^{1/2}, \qquad a = (r_g/2\eta)^{1/4}\exp(\gamma\eta/2r_g), \qquad \text{(I.9)}$$
$$dt = a(\eta)\, d\eta.$$

The solution (I.9) coincides with a solution describing a flux of free particles in a Bianchi type I metric, solution which was found in [11]. For $t << r_g\gamma^{-3/4}$ the solution is Kasnerian with exponents $(-1/3, 2/3, 2/3)$ and for $r_g\gamma^{-3/4} \ll t \ll r_g\gamma^{-1/2}$ it is of the form (up to terms of the form $\ln\ln t$ and $\ln\ln\gamma$)

$$a = \frac{\gamma}{2r_g}t, \quad r = \frac{2r_g}{\gamma^{1/2}}\ln^{1/2}\left(\frac{t}{r_g}\gamma^{3/4}\right). \qquad \text{(I.10)}$$

In order to investigate the subsequent evolution of the system it is convenient to take the first equation and the sum of the first and second equations of the system (I.1) and reduce them to the form

$$\frac{dr^2}{d\xi} = \frac{2}{a^2}(r_g - \xi), \quad \frac{da^2}{d\xi} = \frac{\gamma - a^2}{r_g - \xi} - \frac{r_g - \xi}{r^2}, \qquad \text{(I.11)}$$

where

$$\xi = \int_0^t a\, dt.$$

then in the region $r_g\gamma^{-3/4} \ll t \ll r_g(\gamma^{-1}\ln\gamma)^{1/2}$ one can neglect the term $(r_g - \xi)/r^2$ and the solution is of the form

$$a = \frac{\gamma}{2r_g}t, \quad r^2 = \frac{4r_g^2}{\gamma}\ln\left(\frac{t}{r_g}\gamma^{3/4}\right) - \frac{t^2}{2}. \qquad \text{(I.12)}$$

At the instant of maximal expansion

$$\xi = r_g, \quad r = r_{max} = r_g\left(\frac{\ln\gamma}{\gamma}\right)^{1/2}, \quad t = t_1 = \frac{2r_g}{\gamma^{1/2}}.$$

In the region $r_g\gamma^{1/2} \ll t \ll \frac{1}{2}\pi r_g(\gamma^{-1}\ln\gamma)^{1/2}$ we have $a \gg \gamma^{1/2}$, $\xi \ll r_g$, therefore in the system (I.11) one can omit all terms containing γ and r_g. Then the solution is of the form

$$r^2 = (r_g\ln\gamma - \xi)^2(\gamma\ln\gamma)^{-1}, \quad a^2 = \xi(r_g\ln\gamma - \xi)^{-1}\gamma\ln\gamma, \qquad \text{(I.13)}$$
$$dt = a^{-1}d\xi$$

For $t = \frac{1}{2}\pi r_g(\gamma^{-1}\ln\gamma)^{1/2}$ ($\xi = r_g\ln\gamma$) the metric has a singularity of the Kasner type with exponents $(-1/3, 2/3, 2/3)$.

The metric (I.12) joins smoothly to the metrics (I.9), (I.10) and (I.13) respectively from the left and from the right (up to terms of the form $\ln\ln(tt_1^{-1})$). Thus, a complete solution of the system (I.1) for the case of a flux of free particles has been constructed.

APPENDIX II

Let us construct a solution which describes a centrally-symmetric flying-apart of the matter of the white hole with speed of light into the vacuum (or into the cold Universe, filled with matter at zero pressure). This solution has the form

$$ds^2 = 2\,dr\,du - \left(\frac{f(u)}{r} - 1\right)du^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \qquad \text{(II.1)}$$

where $f(u)$ is some function which can be determined from the equation

$$R_u^{\ r} = -\frac{1}{r^2}\frac{df}{du} = 8\pi G T_u^{\ r}. \qquad \text{(II.2)}$$

The other Einstein equations for the metric (II.1) are identically satisfied for any choice of the function $f(u)$ if all the components of the energy-momentum tensor except T_u^r vanish. The metric (II.1) is a generalization of the solution of Vaidya [12] to the case when there exist both R and T regions. For $r > f(u)$ the coordinate r has a spacelike character and for $r < f(u)$ it has a timelike character. The coordinate u is isotropic (lightlike) and has the meaning of a retarded coordinate. For $f(u) = r_g = \text{const.}$ and $r > r_g$ the metric (II.1) reduces to the Schwarzschild metric by means of the transformation

$$u = t - r - r_g\ln(r - r_g). \qquad \text{(II.3)}$$

Thus, the function $f(u)$ has the meaning of a variable gravitational radius and shows how the mass enclosed in a certain volume changes with time on account of the flying-apart of the matter. It follows from (II.2) that if $T_u^r \geq 0$, $f(u)$ is a nonincreasing function.

Let the lightlike geodesic $u = 0$ pass through the point $t = t_0$, $R = R_2$. Since the production of particles takes place in the region $R < R_2$ one must assume that $f(u) = r_g$ for $u \leq 0$. It follows from the conservation law $T_{i;k}^k = 0$ that

$$T_u^{\ r} = r^{-2}g(u),$$

hence

$$f(u) = r_g - 8\pi G\int_0^u g(u)\, du. \qquad \text{(II.4)}$$

In order to determine the function $f(u)$ and $g(u)$ it is necessary to match the metric (II.1) with the homogeneous metric (1) (the solution (I.9)–(I.13)). Since in the case under consideration the matter consists of free massless particles, the decompression wave moves to the left with the speed of light according to the metric (1). Therefore the matching must be done along a light-like geodesic, the equation of which in the metric (II.1) has the form

$$2dr=[f(u)/r-1]\,du, \quad r=R_2 \text{ for } u=0, \tag{II.5}$$

and in the metric (1):

$$R=R_2-\int_{t_0}^{t} dt\, \exp[-{}^1/_2\lambda(t)]. \tag{II.6}$$

The continuity condition for the energy flux of the produced particles through the moving boundary (II.5), (II.6) yields

$$g(u(t))\,du/dt=r^2\varepsilon(t)\,dr/dt, \tag{II.7}$$

where $\epsilon(t)$ is the energy density of the matter consisting of the produced particles in the homogeneous system (1), $r(t)$ is determined from the solution (I.9)–(I.13) and the function $u(r(t))$ from the condition (II.5). Hence

$$f(u)=r_g-8\pi G\int_{t_0}^{t(u)} r^2\frac{dr}{dt}\varepsilon\, dt=r_g-8\pi G\int_{r(t_0)}^{r(u)} r^2\varepsilon\, dr, \tag{II.8}$$

where $t(u)$ is determined from the equation

$$du=2r\left(r_g-r-8\pi G\int_{r(t_0)}^{r} r^2\varepsilon\, dr\right)^{-1} dr=2\left(\frac{dr}{dt}\right)^{-1} dt \tag{II.9}$$

(in the derivation of which use has been made of the relation (15)).

It follows from (II.8) and (II.9) that for $t\to t_1$, when $dr/dt\sim t-t_1\to 0$, $u\to\infty$ and

$$2GM_1\equiv f(\infty)=r_{max}.$$

A similar relation between M_1 and r_{max} can be obtained also in the case when the matter consisting of the produced particles has in the homogeneous metric (1) the equation of state $p=\epsilon/3$, assuming that 1/6 of all the particles moves along the R axis to the right with the speed of light.

1)Strictly speaking this refers only to zero mass particles $m = 0$; however, the production of particles with $m \neq 0$ near the Friedmann singularity does not affect the subsequent expansion, if $Gm^2/\hbar$ 1, and can therefore be neglected.

2)The function e^λ is defined up to a scale transformation; the coefficient of e^λ in Eq. (4) has been selected from convenience considerations; r_g is the initial gravitational radius of the white hole.

3)The delay (retardation) is determined by an observer situated at a constant distance from the white hole.

4)But not in particular $R_{cr}-(R_2-R_1) < R_{cr}$, when the equality (9) holds.

5)In a slightly different form this solution has been first found by Shikin [9].

1. I. D. Novikov, Astron zh. **41**, 1075 (1964) [Sov. Astron.-A. J. **8**, 857 (1964)].
2. Y. Ne'eman, Astrophys. J. **141**, 1303 (1965).
3. I. D. Novikov, Soobshcheniya GAISh **132**, 3, 43 (1964).
4. Ya. B. Zel'dovich and I. D. Novikov, Teoriya tyagoteniya i evolyutsiya zvezd (Theory of gravitation and stellar evolution) Nauka, 1971 (cf. translation of [7], infra).
5. Ya. B. Zel'dovich, ZhETF Pis. Red. **12**, 443 (1970) [JETP Lett. **12**, 307 (1970)].
6. Ya. B. Zel'dovich and A. A. Starobinskiĭ, Zh. Eksp. Teor. Fiz. **61**, 2161 (1971) [Sov. Phys. JETP **34**, 1159 (1972)].
7. Ya. B. Zel'dovich and I. D. Novikov, Relyativistskaya astrofizika (Relativistic astrophysics), Nauka 1967 [Engl. Transl. of this + ref. [4], 2 vols., U. of Chicago Press, 1971, 1974].
8. S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space-Time, Cambridge Univ. Press, 1973.
9. I. S. Shikin, Dokl. Akad. Nauk SSSR **176**, 1048 (1967) [Sov. Phys. Doklady **12**, 950 (1968)].
10. A. G. Doroshkevich, Astrofizika **1**, 255 (1965).
11. A. G. Doroshkevich, Ya. B. Zel'dovich and I. D. Novikov, Zh. Eksp. Teor. Fiz. **53**, 644 (1967) [Sov. Phys. JETP **26**, 408 (1968)].
12. P. C. Vaidya, Nature **171**, 260 (1953).

Translated by Meinhard E. Mayer
195