Concerning one gyromagnetic effect in a liquid paramagnet

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The moment of the forces acting in a magnetic field on a heated cylinder suspended in a paramagnetic liquid is calculated (the arrangement is illustrated in Fig. 1). The cause of the effect is the gyromangetic interaction between the applied field and the velocity vortices of the convective flow in the boundary layer at the cylinder surface.

1. INTRODUCTION

The determination of the gyromagnetic ratios from the Einstein-de Haas effect calls for the measurement of very small torques acting on a suspended cylindrical sample. To increase the measurement accuracy, the torsion pendulum is placed in a vacuum chamber[1], but it is precisely in this case, in the gas pressure interval from 0.01 to 1 mm Hg, that one observes a small anomalous torque whose value depends on the nature of the gas and whose sign reverses with changing direction of the magnetic field parallel to the cylinder axis. Scott et al.[2,3] investigated this effect, replacing the ferromagnetic sample used in the gyromagnetic experiments with a glass rod. It was established that the anomalous torque is proportional to the temperature difference between the cylinder and the vacuum-chamber wall (in the Einstein-de Haas experiment, the ferromagnetic rod becomes heated during the course of magnetization reversal).

The phenomenon of rotation of a heated cylinder in a magnetic field was observed[3] in more than 20 molecular gases (NO, NO_2, O_2, N_2, Cl_2, CO etc.). In almost all the investigated gases, the torque goes through a maximum at a pressure near 0.03 mm Hg, i.e., at an average mean free path of the gas molecules $\lambda \approx 0.2$ cm.

The corresponding Knudsen numbers are not too small ($\lambda/l \sim 0.1$, where l is the characteristic macroscopic dimension), thus indicating that the Scott effect is kinetic in nature. Its cause is, as shown by Waldmann[4], the nonspherical character of the reflection of the field-polarized rotating molecules from the cylinder surface (the "cut tennis ball" model). The connection between the Scott effect and the kinetic Senftleben-Binakker effect, which consists of a change in the transport coefficients of a rarefied polyatomic gas under the influence of an applied field, is discussed in the paper by Hess and Waldmann[5].

We show in this paper that an effect similar to the Scott effect, but of pure hydrodynamic nature, should occur in liquid paramagnets. A heated cylinder immersed, e.g., in liquid oxygen is acted upon in a constant vertical magnetic field by a torque due in final analysis to gyromagnetic effects.

Since the molecules of the liquid or of the gas have rotational degrees of freedom, the angular momentum per unit volume of the medium does not reduce to the hydrodynamic ("orbital") angular momentum $\mathbf{L} = \rho \mathbf{r} \times \mathbf{v}$, but contains also an internal ("spin") angular momentum **S**. The latter is connected with the volume density of the magnetic moment **M** by the relation

$$\mathbf{M}=\gamma\mathbf{S}, \quad (1)$$

where γ depends, of course, on the material. In addition, a connection exists between the proper rotation of the molecules and the translational motion (with hydrodynamic velocity **v**) of their mass centers, which is macroscopically manifest in the law of conservation of the total angular momentum of the medium **L** + **S**. Owing to this relation, the vertical motion of the liquid goes over partially into the "latent" form **S**, and the liquid, according to (1), becomes magnetized (the analog of the Barnett effect). The inverse (Einstein-de Haas) effect is also possible: the magnetic field, which forms **M** directly, generates by the same token an internal angular momentum, which can then be partially transformed into vortical motion of the liquid.

The gyromagnetic torque discussed in this paper is, however, not a direct analog of the Einstein-de Haas effect. As indicated above, in a liquid ferromagnet, the torque is expected to appear under the same experimental conditions in which the Scott effect is observed in a rarefied gas. Under the conditions of such an experiment, the two gyromagnetic effects become intertwined, since a vertical motion of the liquid, due to convection near the heated cylinder, is present along with the magnetic field. At the cylinder surface, a convective boundary layer is produced (see Fig. 1), with velocity components v_r and v_z that depend on the coordinates r and z. The curl of the velocity of this flow, curl $\mathbf{v} = 2\Omega$, is equivalent in its action on the magnetic properties to an external field $H_\varphi = \Omega_\varphi/\gamma$. Under the influence of the "gyromagnetic field" H_φ, a magnetization component $M_\varphi = (\chi/\gamma)\Omega_\varphi$ is produced, where χ is the magnetic susceptibility of the liquid. By producing M_φ, a certain angle between the vectors **M** and **H** the stationary motion of the liquid is maintained by the same token in the applied field $H_z = H$. It is the torque due to the fact that these vectors are not parallel which leads to the rotation of the liquid around the cylinder. The purpose of this article is to calculate the velocity component v_φ and the associated friction moment acting on the cylinder. The equations used in the article for the motion of a liquid with the gyromagnetic properties were derived phenomenologically by the author in[6], and from the kinetic equations by Kagan and Maksimov[7].

2. FUNDAMENTAL EQUATIONS

In a liquid with internal rotation, the momentum and angular-momentum conservation laws are given by the equations[8]

$$\rho(\partial v_i/\partial t+v_k\partial v_i/\partial x_k)=\partial\sigma_{ik}/\partial x_k, \quad (2)$$

$$\partial S_{ik}/\partial t+v_l\partial S_{ik}/\partial x_l=\sigma_{ki}-\sigma_{ik}. \quad (3)$$

Here $S_{ik} = e_{ikl}S_l$, where S_l is the pseudovector of the internal angular momentum, and σ_{ik} is the stress tensor and consists of a symmetrical part and an antisymmetrical part:

$$\sigma_{ik}=\sigma_{ik}^s+e_{ikl}\sigma_l$$

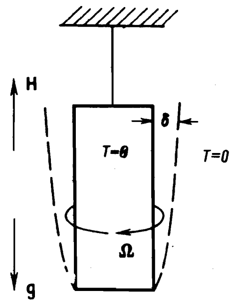

FIG. 1

From (1) and (3) we obtain the equation of motion of the magnetic moment of the liquid

$$\partial \mathbf{M}/\partial t + (\mathbf{v}\nabla)\mathbf{M} = -2\gamma\boldsymbol{\sigma}. \tag{4}$$

The irreducible parts of the tensor σ_{ik} were obtained in[6]. For an incompressible liquid we have

$$\boldsymbol{\sigma} = -\frac{1}{2}[\mathbf{MH}] + \frac{1}{2\gamma\tau}(\mathbf{M} - \chi\mathbf{H}_e), \tag{5}^*$$

$$\begin{aligned}\sigma_{ik}' &= -p_e\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) + \frac{1}{2}(M_iH_k + M_kH_i) + \frac{H_iH_k}{4\pi} \\ &= -p_e\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) + \frac{1}{2}(M_iH_k - M_kH_i) + \frac{H_iB_k}{4\pi}, \\ &\mathbf{H}_e = \mathbf{H} + \boldsymbol{\Omega}/\gamma, \quad \mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}.\end{aligned} \tag{6}$$

All the isotropic terms of the stress vector, in addition to the pressure p, are included in this case in p_e.

In the equation for the magnetization

$$\frac{\partial \mathbf{M}}{\partial t} + (\mathbf{v}\nabla)\mathbf{M} = \gamma[\mathbf{MH}] - \frac{1}{\tau}(\mathbf{M} - \chi\mathbf{H}_e), \tag{7}$$

which is obtained from (4) and (5), the term $(\mathbf{v}\cdot\nabla)\mathbf{M}$ is always small in comparison with M/τ. For example, in the convective boundary layer which will be discussed below, we have $v_r/v_z \sim \delta/l$, and therefore

$$(\mathbf{v}\nabla)\mathbf{M} \sim M\sqrt{g\beta\Theta/l}$$

(δ is the thickness of the boundary layer, l is the height the cylinder, β is the coefficient of thermal expansion of the liquid, and Θ is the difference between the surface temperature of the cylinder and the temperature of the liquid far from it). At arbitrary reasonable values of the parameters l and Θ, the characteristic "hydrodynamic" time $(l/g\beta\Theta)^{1/2}$ is not less than 0.1 sec. This is larger by many orders of magnitude than the magnetization-relaxation time ($\tau \lesssim 10^{-10}$ sec for liquid paramagnets with atomic paramagnetism), so that the term $(\mathbf{v}\cdot\nabla)\mathbf{M}$ in (7) can be neglected. Under stationary conditions $\partial \mathbf{M}/\partial t = 0$ and Eq. (7) takes the form $\sigma = 0$, i.e.,

$$\mathbf{M} - \chi(\mathbf{H} + \boldsymbol{\Omega}/\gamma) = \tau\gamma[\mathbf{MH}]. \tag{8}$$

For the stationary motion of the liquid we obtain, by substituting (6) in (2), the equation

$$\rho(\mathbf{v}\nabla)\mathbf{v} = -\nabla p_e + \eta\Delta\mathbf{v} - \rho g\beta T + {}^1/_2\,\mathrm{rot}\,[\mathbf{MH}]. \tag{9}$$

When calculating the divergence of the symmetrical part of the stress tensor (6) we used the equation div B = 0 and took into account the homogeneity of the applied field. In addition, the right-hand side of (9) includes the Archimedean bouyancy force, which causes the convective motion.

From (8) we easily obtain

$$[\mathbf{MH}] = (\chi/\gamma)\,[1 + (\tau\gamma H)^2]^{-1}\{[\boldsymbol{\Omega}\mathbf{H}] + \tau\gamma[\mathbf{H}[\boldsymbol{\Omega}\mathbf{H}]]\}. \tag{10}$$

Substituting (10) in (9), we obtain an equation of motion that does not contain M.

It is necessary to add to (9) and (10) also the stationary equation of the thermal conductivity in the moving liquid and the incompressibility condition

$$\mathbf{v}\nabla T = \varkappa\Delta T, \quad \mathrm{div}\,\mathbf{v} = 0 \tag{11}$$

(κ is the temperature diffusivity coefficient).

3. SOLUTION OF PROBLEM

The velocity and temperature determined by Eqs. (9)–(11) can be expanded in powers of the magnetic susceptibility χ. In the zeroth order in χ, the term $\mathrm{curl}[\mathbf{M}\times\mathbf{H}]$ in (9) vanishes, and the equations

$$\begin{aligned}(\mathbf{v}\nabla)\mathbf{v} &= -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} - g\beta T \quad (\nu = \eta/\rho), \\ \mathbf{v}\nabla T &= \varkappa\Delta T, \quad \mathrm{div}\,\mathbf{v} = 0\end{aligned} \tag{12}$$

describe the usually free convection near a heated cylinder. At sufficiently large temperature difference Θ between the cylinder and the liquid, the boundary layer thickness is $\delta \ll R$, where R is the radius of the cylinder. In this case the curvature of the cylinder surface can be neglected, and Eqs. (12) lead to the well known Polhausen problem of the convective boundary layer on a flat vertical plate (see, e.g.,[9], p. 263). We choose the origin of a rectangular coordinate system on the lower edge and direct the x axis horizontally, along the cylindrical coordinate r, and the z axis vertically upward. From the symmetry of the problem it is clear that the distributions of the velocity and of the temperature do not depend on the coordinate y, and the velocity has no y component. With the customary accuracy for the boundary layer[9], Eqs. (12) take the form

$$\begin{aligned}v_x\frac{\partial v_z}{\partial x} + v_z\frac{\partial v_z}{\partial z} &= \nu\frac{\partial^2 v_z}{\partial x^2} + g\beta T, \\ v_x\frac{\partial T}{\partial x} + v_z\frac{\partial T}{\partial z} &= \varkappa\frac{\partial^2 T}{\partial x^2}, \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0\end{aligned} \tag{13}$$

with the boundary conditions

$$\begin{aligned}&v_x = v_z = 0, \quad T = \Theta, \quad \text{at} \quad x = 0 \\ &v_z \to 0, \quad T \to 0 \quad \text{as} \quad x \to \infty\end{aligned} \tag{14}$$

(the zero point of the temperature is taken to be the temperature of the liquid far from the cylinder).

The problem (13)–(14) determines the temperature T and the velocity components v_x and v_z in the zero-order approximation in χ. In the first-order approximation the quantities T, v_x, and v_z acquire corrections which, however, are of no interest because χ is small. What is qualitatively new in this approximation is the appearance of the azimuthal velocity component v_y (v_φ in cylindrical coordinates). An equation for v_y is obtained from the general equation (9) by substituting in it $\mathbf{M}\times\mathbf{H}$ from (10). We note that inasmuch as the right-hand side of (10) already contains the first power of χ in the coefficient, the value of $\boldsymbol{\Omega}$ in this formula should be determined by the zeroth-approximation velocity:

$$\Omega_x = \Omega_z = 0, \quad \Omega_y = {}^1/_2(\partial v_x/\partial z - \partial v_z/\partial x).$$

Recognizing also the character of the motion in the boundary layer ($v_x \ll v_z$, $\partial/\partial x \gg \partial/\partial z$), we obtain for v_y the equation

$$v_x\frac{\partial v_y}{\partial x} + v_z\frac{\partial v_y}{\partial z} = \nu\frac{\partial^2 v_y}{\partial x^2} - \varepsilon\frac{\partial^2 v_z}{\partial x\,\partial z}, \tag{15}$$

where

$$\varepsilon=\frac{\chi}{4\rho\gamma^2\tau}\frac{\omega_H\tau}{1+(\omega_H\tau)^2},\quad \omega_H=\gamma H. \tag{16}$$

The condition $v_y = 0$ should be satisfied on the surface of the cylinder at rest $(x = 0)$ and far from this surface $(x \to \infty)$.

We emphasize that the velocity component v_x and v_z in (15) are determined by the zeroth-approximation solution of the problem. Thus, Eq. (15) is a linear inhomogeneous equation with respect to v_y, in which the term with ϵ plays the role of the mass density of the extraneous forces.

There exist, as is well known[9], a similarity transformation that makes it possible to reduce the zeroth-approximation problem (13)—(14) to a system of ordinary differential equations. This is accomplished by introducing the dimensionless independent variable

$$\xi=C\frac{x}{z^{1/4}},\quad C=\left(\frac{g\beta\Theta}{4\nu^2}\right)^{1/4}. \tag{17}$$

We seek the stream function Ψ of the stationary motion and the temperature T in the form

$$\Psi(x,z)=4\nu Cz^{3/4}\psi(\xi),\quad T(x,z)=\Theta\zeta(\xi). \tag{18}$$

The velocity components are in this case

$$v_x=-\frac{\partial\Psi}{\partial z}=\nu Cz^{-1/4}(\xi\psi'-3\psi),$$
$$v_z=\frac{\partial\Psi}{\partial x}=4\nu C^2z^{1/2}\psi'. \tag{19}$$

From (13) and (14) we obtain for ψ and ζ the equations

$$\psi'''+3\psi\psi''-2\psi'^2+\zeta=0,\quad \zeta''+3P\psi\zeta'=0 \tag{20}$$

(here $P = \nu/\kappa$ is the Prandtl number) with boundary conditions

$$\psi(0)=\psi'(0)=0,\quad \zeta(0)=1,\quad \psi'(\infty)=\zeta(\infty)=0. \tag{21}$$

The results of a numerical integration of the problem (20)—(21) can be found, e.g., in Schlichting's book[10] The profile of the vertical velocity ψ' at P = 1 is shown in Fig. 2.

The condition for the applicability of the solution of the Polhausen problem to the boundary layer on a vertical round cylinder is $\delta/R \ll 1$. We present an estimate. As seen from (17), the thickness of the boundary layer $\delta \sim z^{1/4}/C$ reaches its largest value $\delta_m \sim (l\nu^2/g\beta\Theta)^{1/4}$ at the upper end of the cylinder. For liquid oxygen ($\nu = 2\times 10^{-3}$ cm^2/sec, $\beta = 4\times 10^{-3}$ deg^{-1}) at $R \sim 1$ cm, $l \sim 10$ cm, and $\Theta \sim 10°$ we have $\delta_m/R \sim 0.1$.

We proceed to find the y component of the velocity. Substituting in (15) v_x and v_z from (19), and assuming

$$v_y=\varepsilon Cz^{-1/4}f(\xi), \tag{22}$$

we obtain for f the equation

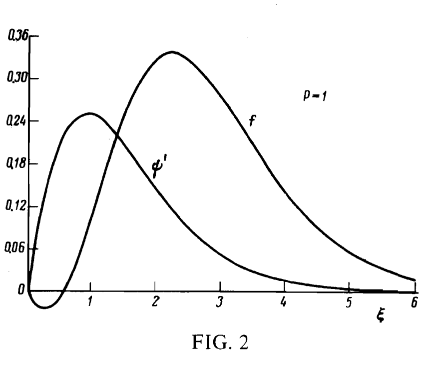

FIG. 2

$$f''+3\psi f'+\psi' f=(3\psi-\xi\psi')'', \tag{23}$$

which should be solved with the boundary conditions

$$f(0)=0,\quad f(\infty)=0. \tag{24}$$

The problem (23) and (24) was integrated numerically by the Runge-Kutta method (the solution ψ of the zeroth-approximation problem was in this case also obtained numerically). A plot of the function f for P = 1 is shown in Fig. 2. We note the presence of a node in f: the liquid rotates in opposite directions near the cylinder and far from it.

4. DISCUSSION OF RESULTS

Let us determine the friction torque acting on the cylinder. In the azimuthal direction, a unit cylinder surface is acted upon by a friction force $\eta\partial v_y/\partial x|_{x=0}$. From (22) we obtain with the aid of (17)

$$\eta\frac{\partial v_y}{\partial x}\Big|_{x=0}=\eta\varepsilon C^2z^{-1/2}f'(0).$$

The entire cylinder is acted upon by the friction torque

$$K=2\pi R^2\eta\int_0^l\frac{\partial v_y}{\partial x}\Big|_{x=0}dz=4\pi R^2l^{1/2}\eta\varepsilon C^2f'(0).$$

Substituting ϵ from (16) and C from (17), we obtain ultimately

$$K=\pi R^2\frac{\chi}{2\gamma^2\tau}\frac{\omega_H\tau}{1+(\omega_H\tau)^2}f'(0)\sqrt{g\beta\Theta l}. \tag{25}$$

In molecular gases, the relaxation of the magnetic moment is ensured by the collisions of the molecule, whereas the relaxation of the magnetization τ is determined by the mean free path time, i.e., $\tau \approx \eta/p$. Assuming the magnetic susceptibility of the gas to be equal to $\chi = n\mu^2/3kT$, where μ is the effective magnetic moment of the molecule, and using the equation of state $p = nkT$, we obtain from (25)

$$K=\frac{\pi R^2}{6\gamma}\left(\frac{\mu}{kT}\right)^2\frac{pH}{1+(\eta\gamma H/p)^2}f'(0)\sqrt{g\beta\Theta l}. \tag{26}$$

Comparing K with the torque K^S observed in rarefied gases (the Scott effect), we note the similarity of the dependence on H and the difference in their dependence on p and Θ. In both cases the torque as a function H has a maximum at a certain volume $H/p = \text{const}$. It is seen from (26), however, that the maximum value K reached in a field $H = p/\eta\gamma$ and equal to

$$K_m=\frac{\pi R^2}{12\eta}\left(\frac{\mu p}{\gamma kT}\right)^2f'(0)\sqrt{g\beta\Theta l}$$

increases with pressure monotonically ($\sim p^2$), whereas $K^S_m(p)$ goes through a maximum at $p \sim 10^{-5}$ atm. With increasing temperature difference between the cylinder and the gas, K increases like $\Theta^{1/2}$ and K^S like Θ.

In a sufficiently dense paramagnetic gas ($p \sim 1$ atm), and all the more in a liquid, the magnetization relaxation time is so small, that the condition $\omega_H\tau \ll 1$ is satisfied for all reasonable fields. In this case τ drops out completely from (25), and the formula for the moment of the forces becomes

$$K=\pi R^2\frac{\chi H}{2\gamma}f'(0)\sqrt{g\beta\Theta l}. \tag{27}$$

It is interesting to note that the viscosity of the liquid enters in K only via the dependence of $f'(0)$ on the Prandtl number P. This dependence, obtained as a result of numerical integration of Eqs. (20) and (23), is plotted in Fig. 3.

The moment of the forces K should reach in liquids

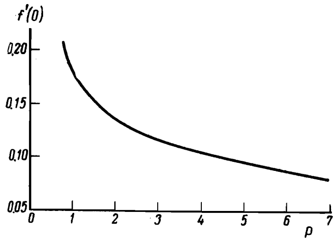

FIG. 3

much larger values than in gases, since the magnetic susceptibility χ is proportional to the density of the number of particles—the carriers of the magnetic moment. Further, as seen from (27), K is proportional to the ratio χ/γ. The most promising for the observation of the effect is therefore oxygen, in which owing to its special properties[11], a large χ is combined with a small γ. The magnetic susceptibility of liquid oxygen is anomalously large ($\chi = 2.8 \times 10^{-4}$ at 80°K). The electron spin of the O_2 molecule is coupled with its angular momentum by the "spin-axis" interaction, and this coupling is such that the gyromagnetic ratio γ, defined by formula (1), is smaller by three orders of magnitude than for free electrons, i.e., it is of the same order as for nuclei. Scott et al.[3] obtained for the rotational g-factor of the O_2 molecule a value $g_{rot} = -9$ (the effective magnetic moment is measured here in nuclear magnetons μ_p). This experimental value agrees in order of magnitude with the theoretical estimate of Maksimov[12]. Thus, we have $\gamma = g_{rot}\ (\mu_p/\hbar) = -4.3 \times 10^4$.

Let us estimate the effect numerically. In the case of liquid oxygen it is necessary to substitute in (27), in addition to the indicated values of β, χ, and γ, also $f'(0) = -0.12$ ($P = 2.8$). For a cylinder of radius 2 cm and height $l = 10$ cm, at a temperature difference $\Theta = 10°$, we obtain $K = \alpha H$, where $\alpha = 10^{-7}$ dyne-cm/Oe. The smallest torque that could be measured in the experiments[2] was 2×10^{-5} dyne-cm. According to the estimate, this value of K is reached already in a field H = 200 Oe. Thus, the effect under discussion can be observed.

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*[MH] ≡ M × H.

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