Nonstationary turbulence of parametrically unstable plasma

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(Submitted August 3, 1973)
Zh. Eksp. Teor. Fiz. **66**, 930–944 (March 1974)

A nonlinear theory of nonstationary parametric turbulence of a homogeneous isotropic plasma is developed. The perturbations grow in time as a result of the parametric instability of the plasma located in the field of strong monochromatic exciting radiation. On the other hand, the saturation of the amplitude of the initially unstable plasma fluctuations is due to the nonlinear shift of their frequency. The basic assumptions and the theoretical results are illustrated by two examples of parametric instability, namely, aperiodic instability and instability resulting in the decay of the exciting wave into two plasmons. An exact solution of the nonlinear equations is found for the spectral energy densities of turbulent noise as functions of time and wave vector. The time for the noise to reach the stationary state and the turbulence energy per unit volume of the plasma are determined as functions of the excess of the exciting flux over the threshold and the plasma parameters. The role of spontaneous noise in the relaxation of nonstationary turbulence is elucidated. The resulting formulas are illustrated by numerical estimates for hot laser plasma.

Plasma exposed to high-intensity radiation is known to become unstable.[1] The development of this (parametric) instability takes the plasma to a new stationary state which is turbulent. The theory of stationary turbulence of parametrically excited plasma has been given by a number of workers[2-5] (see also the review in[6]), who used stimulated scattering of plasma noise by ions as a nonlinear mechanism ensuring saturation. It was shown in[5,7] that a high level of stationary turbulent noise could also be reached in parametrically excited plasma as a result of a nonlinear shift in the frequency of plasma oscillations.

The theory of nonstationary plasma turbulence, which should be able to predict whether the stationary state can be reached and to establish the character of the process whereby this may occur, is given in[8-12]. These papers are concerned with the joint evolution of the spectral energy density of plasma noise and of the plasma particle distribution function[8-10] in the quasi-linear approximation[12] (see also[13,14]) and show that the noise reaches the stationary level as a result of the increase in number of epithermal electrons (acceleration),[8] i.e., the distortion of the initially Maxwellian electron distribution. On the other hand, numerical solutions of a nonlinear integro-differential equation was used by Kruer and Valeo[11] to investigate the wavelength dependence of the nonstationary spectral energy density of electron Langmuir oscillations scattered by ions. In particular, it is shown in[11] that the approach to the stationary state of full-spectrum plasma noise does not occur monotonically but oscillates in time.

The present paper is concerned with a theory of nonstationary turbulence of parametrically unstable plasma, based on nonlinear nonstationary equations describing the evolution of the noise in time and in wave-number space as a result of the nonlinear shift of plasma wave frequency for a fixed (Maxwellian) plasma particle distribution. The analysis has resulted in simple analytic solutions for the turbulent noise energy which is a function of time. Examples of a periodic parametric instability and of instability due to the decay of high-intensity exciting waves into two plasmons are used to show that the nonlinear plasma wave frequency shift leads to the eventual approach of the noise to the stationary state. The spectral energy density of the plasma noise (cf.[5,7]) and the characteristic time taken by the noise to reach the steady state are determined. The results include the prediction that the growth of plasma fluctuations to a level considerably in excess of the thermal level is possible only for a sufficient excess over and above the instability threshold. A prediction which substantially simplifies the theory of nonstationary turbulence is that the level of turbulent fluctuations has a weak dependence on the spontaneous emission of plasma waves by the plasma particles. Finally, there is the important result that the spectral interval of wave vectors in which turbulent noise grows must contract as time increases.

The analysis given below is divided into two sections. In Sec. 1 we consider the evolution in time of plasma noise energy when the instability is due to the decay of the exciting wave into two plasma waves. In the second section, we find the nonstationary distribution of plasmons for an aperiodic parametric instability. In contrast to the instability which involves decay into two plasmons, the aperiodic parametric instability leads to the development in plasma of not only the high-frequency (Langmuir) but also the low-frequency turbulence which is an aperiodic function of time and is similar to hydrodynamic turbulence in a liquid. The common feature of both instabilities is that the noise reaches the stationary state in a monotonic fashion.

1. DECAY OF EXCITING WAVE INTO TWO PLASMONS. SATURATION OF PLASMA-WAVE ENERGY DENSITY DUE TO NONLINEAR FREQUENCY SHIFT

Homogeneous isotropic plasma placed in a plane monochromatic wave of wave number ω_0, wave vector $\mathbf{k}_0$, and electric field given by

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\omega_0 t - \mathbf{k}_0 \mathbf{r}), \tag{1.1}$$

can exhibit parametric instability corresponding to the decay of the exciting wave (1.1) into two electron plasma waves[15,16] if the frequency ω_0 approaches twice the Langmuir electron frequency $2\omega_{Le}$ [$\omega_{Le} = (4\pi e^2 n_e/m)^{1/2}$, n_e is the electron density, e is the electron charge, and m the electron mass]. The plasma waves excited in this way have wave vector k and frequency close to the Langmuir electron frequency. In the linear stage of the instability, the plasma waves increase exponentially in time with a growth rate

$$\gamma(\mathbf{k}) = -\tilde{\gamma}(k) + \omega_{Le}\left\{ \frac{(\mathbf{k}\mathbf{r}_E)^2}{4k^2} \frac{(k^2 - 2\mathbf{k}\mathbf{k}_0)^2}{(\mathbf{k}-\mathbf{k}_0)^2}\right.$$

$$-\left[\frac{\omega_0-2\omega_{Le}}{2\omega_{Le}}-\frac{3}{4}k^2r_{De}^2-\frac{3}{4}(\mathbf{k}-\mathbf{k}_0)^2r_{De}^2\right]^2\Bigg\}^{1/2}. \quad (1.2)$$

In this expression, $\tilde{\gamma}(\mathbf{k})$ is the plasmon decay rate in the absence of the exciting wave

$$\tilde{\gamma}(k)=\frac{\nu_{ei}}{2}+\left(\frac{\pi}{8}\right)^{1/2}\frac{\omega_{Le}}{(kr_{De})^3}\exp\left(-\frac{1}{2k^2r_{De}^2}\right),$$

and is determined by the frequency of the Coulomb collisions between electrons and ions (ν_{ei}) and the Landau damping; $r_{De} = (\kappa T_e/4\pi e^2 n_e)^{1/2}$ is the Debye length for electrons at temperature T_e, κ is Boltzmann's constant, and $\mathbf{r}_E = eE_0/m\omega_0^2$ is the amplitude of electron oscillations in the field of the exciting wave.

In a previous paper,[5] we found the stationary energy level of plasma waves determined both by their stimulated scattering by ions and the nonlinear frequency shift $\delta\omega$ (see[7]) which is given by

$$\delta\omega=-\frac{\omega_{Le}}{4}\frac{1}{n_e\kappa T_e}\frac{r_{De}^2}{r_{De}^2+r_{Di}^2}\int\frac{d\mathbf{k}}{(2\pi)^3}W(\mathbf{k}')\left(\frac{\mathbf{k}\mathbf{k}'}{kk'}\right)^2, \quad (1.3)$$

where $W(\mathbf{k})$ is the spectral energy density of the plasmons, $r_{Di} = (\kappa T_i/4\pi e_i^2 n_i)^{1/2}$ is the Debye length of the ions at temperature T_i, e_i is the ion charge, and n_i the ion density. In this paper, we consider the nonstationary saturation of parametric instability due to the frequency shift (1.3). In addition to the previous results reported in[5], we now determine the nonstationary spectral energy density of plasma waves and the characteristic time of approach to the stationary state. The physical cause of the stabilization of the oscillations is the reduction in the growth rate (for all wave vectors k) with increasing plasma-noise energy, due to the reduction in the nonlinear frequency shift given by (1.3).

Consider the nonlinear evolution of plasma noise in the simple case of sufficiently hot plasma and small detuning:

$$(v_{Te}/c)^2\gg\max\{\nu_{ei}/\omega_{Le},\ |\omega_0-2\omega_{Le}|/\omega_{Le}\}, \quad (1.4)$$

where $v_{Te} = \omega_{Le}r_{De}$ is the thermal velocity of electrons and c is the velocity of light in vacuo. It was shown earlier[5] that it was precisely in this case that stabilization by the nonlinear frequency shift successfully competed with the spectral transformation, giving the minimum level of stationary turbulent noise. The growth rate given by (1.2) subject to (1.4) has two identical maxima along and across the direction of propagation $\mathbf{k}_0$ of the exciting wave (1.1) which corresponds to the excitation of two plasma waves with wave vectors k and $\mathbf{k}-\mathbf{k}_0$ and identical energy densities $W(\mathbf{k}, t) = W(\mathbf{k}-\mathbf{k}_0, t)$.

The minimum threshold field when (1.4) is satisfied is given by

$$\frac{E_{min}^2}{8\pi n_e\kappa T_e}=\frac{16}{3}\frac{c^2}{v_{Te}^2}\frac{\nu_{ei}^2}{\omega_{Le}^2}+108\frac{v_{Te}^2}{c^2}.$$

When plasma waves are excited at 90° to the wave vector of the exciting wave, $\mathbf{k}\cdot\mathbf{k}_0 = 0$, the extremal lengths of the plasma waves are large in comparison with the length of the exciting wave ($k \ll k_0$), so that when the frequency shift (1.3) is taken into account, the growth rate (1.2) for a small excess over the threshold

$$\frac{E_0}{E_{min}}-1\equiv p-1\ll\left(\frac{c}{v_{Te}}\right)^4\left(\frac{\nu_{ei}}{\omega_{Le}}\right)^2$$

can be written in the form (for naturally polarized exciting waves):

$$\gamma(k,\theta)=\frac{81}{16}\frac{\omega_{Le}^2}{\nu_{ei}}\left(\frac{v_{Te}}{c}\right)^4\left[2(p-1)-\cos^2\theta+2\frac{k}{k_0}\cos\theta\right.$$
$$\left.-5\left(\frac{k}{k_0}\right)^2+\frac{8}{9}\frac{c^2}{v_{Te}^2}\frac{\delta\omega}{\omega_{Le}}\right]. \quad (1.5)$$

In this expression θ is the angle between the wave vector k of one of the parametrically excited plasma waves and the wave vector $\mathbf{k}_0$ of the exciting wave (this angle is close to $\pi/2$), and the ratio $p \equiv E_0/E_{min}$ characterizes the excess of the exciting field above the threshold ($p > 1$).

The temporal evolution of noise in homogeneous plasma, assuming there is no transformation along the spectrum, is described by

$$\partial W(\mathbf{k}, t)/\partial t-2\gamma(\mathbf{k}, t)W(\mathbf{k}, t)=\nu_{ei}\kappa T_e(1+p^2), \quad (1.6)$$

where the right-hand side represents the effect of spontaneous noise. Substituting

$$\tau=\frac{81}{4}\frac{\omega_{Le}^2}{\nu_{ei}}\left(\frac{v_{Te}}{c}\right)^4(p-1)t,\quad x=\frac{k}{k_0}\left(\frac{2}{p-1}\right)^{1/2},\quad \mu=\frac{\cos\theta-k/k_0}{[2(p-1)]^{1/2}}, \quad (1.7)$$

we find that the function

$$s(x,\mu,\tau)=\frac{p-1}{144\pi^2}\frac{c^2}{v_{Te}^2}\left(1+\frac{r_{Di}^2}{r_{De}^2}\right)^{-1}\frac{k_0^3W(\mathbf{k},t)}{n_e\kappa T_e} \quad (1.8)$$

is the solution of the nonlinear equation

$$\partial s(x,\mu,\tau)/\partial\tau=\Gamma(x,\mu,\tau)s(x,\mu,\tau)+s_0. \quad (1.9)$$

In these expressions, $\Gamma(x, \mu, \tau)$ is the dimensionless growth rate

$$\Gamma(x,\mu,\tau)=\begin{cases}1-s(\tau)-x^2-\mu^2, & x^2+\mu^2\leqslant[g(p-1)]^{-1},\\ -[g(p-1)]^{-1}, & x^2+\mu^2>[g(p-1)]^{-1},\end{cases} \quad (1.10)$$

which for large x and μ $[g(p-1) \ll 1]$ is defined in such a way that the damping of the plasma waves occurs only as a result of electron-ion collisions[1)]

$$g={}^{81}/_4(\omega_{Le}/\nu_{ei})^2(v_{Te}/c)^4,$$

and the quantity s_0 is determined by spontaneous noise:

$$s_0=\frac{1}{1458\pi^2}\left(\frac{\nu_{ei}}{\omega_{Le}}\right)^2\left(\frac{c}{v_{Te}}\right)^6\left(1+\frac{r_{Di}^2}{r_{De}^2}\right)^{-1}\frac{k_0^3}{n_e}. \quad (1.11)$$

The total energy density $s(\tau)$ of the turbulent noise integrated over wave numbers $kr_{De} \lesssim 1$ and over the angles is given by

$$s(\tau)=\int_0^{x_0}dx\,x^2\int_{\mu_1(x)}^{\mu_2(x)}d\mu\,s(x,\mu,\tau),\qquad x_0=\frac{1}{k_0r_{De}}\left(\frac{2}{p-1}\right)^{1/2},$$

$$\mu_1(x)=-[2(p-1)]^{-1}-x/2,\qquad \mu_2(x)=[2(p-1)]^{-1}-x/2.$$

The formal solution of (1.9), subject to the initial condition $s(x, \mu, 0) = s_1(x, \mu)$, is

$$s(x,\mu,\tau)=\exp\left[\int_0^\tau d\tau'\,\Gamma(x,\mu,\tau')\right]\Bigg\{s_1(x,\mu)$$
$$+s_0\int_0^\tau d\tau'\exp\left[-\int_0^{\tau'}d\tau''\,\Gamma(x,\mu,\tau'')\right]\Bigg\}. \quad (1.12)$$

Integrating (1.12) with respect to x and μ enables us to obtain a relatively simple integral differential equation for the function

$$u(\tau)=\exp\left[-\tau+\int_0^\tau d\tau'\,s(\tau')\right],$$

which determines the total noise $s(\tau) = -(d/d\tau)(\tau^{-1}\ln u)$

$$\frac{du(\tau)}{d\tau}+u(\tau)=u(\tau)\left\{\sigma_1\exp\left[-\frac{\tau}{g(p-1)}\right]+\sigma_0\left[1-\exp\left[-\frac{\tau}{g(p-1)}\right]\right]\right\}$$
$$+s_1F(\tau)+s_0\int_0^\tau d\tau'\,u(\tau')F(\tau-\tau'),\qquad s_1\equiv s_1(0,0). \quad (1.13)$$

It is assumed that the initial noise $s_1(x, \mu)$ is a sufficiently smooth function, so that over the scales x, $\mu \sim [g(p-1)]^{-1/2}$ it does not show a substantial change,

$$\sigma_1=\int\limits_0^\infty dx\,x^2\int\limits_{\mu_1}^{\mu_2} d\mu\, s_1(x,\mu),$$
$$\sigma_0=\frac{1}{3}s_0(\mu_2-\mu_1)x_0^3 g(p-1)=\frac{4}{3}\frac{gs_0}{p-1}(k_0 r_{De})^{-3},$$
$$F(\tau)=\frac{\pi}{4\tau^2}\left[1-\left(1+\frac{\tau}{g(p-1)}\right)\exp\left\{-\frac{\tau}{g(p-1)}\right\}\right].$$

The solution of (1.13) for times $\tau > g(p-1)$ can be found, for example, through the Laplace transformation. For the total noise $s(\tau)$ we then obtain the following expression ($\tau > 1$):

$$\begin{aligned}s(\tau)&=1-\frac{2}{\tau}-\left(1-\frac{2}{\tau}-\sigma_0\right)\left[1+\frac{\pi s_0}{4\tau^2}\frac{s_0+s_1}{3s_0+s_1}\exp\{\tau(1-\sigma_0)\}\right]\\&\approx 1-(1-\sigma_0)\left[1+C\frac{\pi}{4}\frac{s_0}{\tau^2}\exp\{\tau(1-\sigma_0)\}\right],\quad 1>C>{}^1/_3,\end{aligned}\tag{1.14}$$

from which it is clear that the turbulence arises only for sufficiently large excess over the threshold ($\sigma_0 < 1$):

$$p-1>\frac{1}{54\pi^2}\left(\frac{c}{v_{Te}}\right)^2\frac{1}{n_e r_{De}^3}\left(1+\frac{r_{Di}^2}{r_{De}^2}\right)^{-1}.\tag{1.15}$$

The approach to the stationary value $s(\infty) = 1$ under these conditions is reached in a time

$$\tau_\infty\simeq\ln(1/s_0),\tag{1.16}$$

which is of the same order as the product of the reciprocal of twice the maximum growth rate [see the notation in (1.7)] by the logarithm of the ratio of the steady-state stationary turbulent noise to the spontaneous (thermal) noise. If, on the other hand, the excess over the threshold is so small that (1.15) is not satisfied, then the instability does not develop and the noise remains at the spontaneous level.

It is important to note that the solution given by (1.14) for the total noise does not, in fact, depend on the detailed behavior of the growth rate (1.10) near the point $x^2 + \mu^2 \sim [g(p-1)]^{-1}$. In this sense, the choice of the point at which the two rates in (1.10) are joined is unimportant for the further solution of the problem. Moreover, it is celar from (1.14) that the evolution of noise in time is practically independent of the initial noise distribution $s_1(x, \mu)$ even when the total initial noise σ_1 is comparable with the stationary turbulent noise $s(\infty) = 1$ because the constant C in (1.14) varies only between 1 and 1/3. This is connected with the fact that, as will be shown below, the stationary spectral energy density $s(x, \mu, \infty)$ is a very rapidly varying function of x and μ, so that the main contribution to the energy of the plasma waves is provided by the region $x^2 + \mu^2 \lesssim 1$. Therefore, only the specification of a very rapidly varying initial distribution with a maximum at $x = \mu = 0$ can substantially reduce the time taken to reach the stationary state. This exceptional case of initial distribution $s_1(x, \mu)$ will not be discussed below.

Therefore, analysis of the solution of (1.9) shows that, when the behavior of turbulent noise is considered for times $\tau > 1$, spontaneous radiation can be neglected. As the initial condition for the solution of (1.9) we can take the spontaneous noise

$$s(x,\mu,0)=s_0.$$

For the total noise, we then have the very simple differential equation [see (1.13)]

$$du(\tau)/d\tau+u(\tau)=s_0F(\tau),$$

the exact solution of which

$$s(\tau)=s_0F(\tau)\left[1+s_0\int\limits_0^\tau d\tau'\,F(\tau')\right]^{-1}$$

becomes identical with (1.14) when $\tau > 1$ provided $\sigma_0 \ll 1$, which corresponds to a high turbulent noise level.

The above solution for the total noise $s(\tau)$ can be used with the aid of (1.12) to determine the spectral energy distribution of plasma waves $s(x, \mu, \tau)$. Since the effect of the spontaneous term s_0 in (1.9) in the region of turbulence is negligible, we have

$$s(x,\mu,\tau)=s_0\exp[\tau(1-x^2-\mu^2)]\left[1+s_0\int\limits_0^\tau d\tau' F(\tau')\right]^{-1}.\tag{1.17}$$

It follows from (1.17) that for times $\tau \ll \tau_\infty$ less than the characteristic time given by (1.16), the role of the nonlinear frequency shift is small, and the growth in noise in the instability region $x^2 + \mu^2 < 1$ occurs practically exponentially. On the other hand, for large times $\tau > \tau_\infty$, when the stationary value of the total noise has been established, the spectral energy density of the plasma waves

$$s(x,\mu,\tau)=\frac{4}{\pi}\tau^2\exp[-\tau(x^2+\mu^2)]\tag{1.18}$$

is large only in a narrow and contracting region

$$x^2+\mu^2\leqslant\tau^{-1}\ln\tau^2.$$

Outside this region, the noise attenuates to the spontaneous level. In fact, for large $\tau \gg \tau_\infty$, the spectral distribution given by (1.18) assumes the form of the δ-function

$$s(x,\mu,\infty)\approx(4/\pi x^2)\,\delta(x^2+\mu^2).\tag{1.19}$$

Under these conditions, the spectral energy density $W(k, \infty)$ [see (1.18)] of the parametrically excited plasma waves is given by

$$\begin{aligned}W(k,\theta,\infty)&\approx576\pi n_e\varkappa T_e\frac{v_{Te}^2}{c^2}\frac{k_0}{k^2}(p-1)\\&\times\left(1+\frac{r_{Di}^2}{r_{De}^2}\right)\delta(5k^2-2kk_0\cos\theta+k_0^2\cos^2\theta).\end{aligned}\tag{1.20}$$

The above form of the spectral distribution can be used to justify the methods put forward in [5] for finding the stationary noise, whereby one equates to zero the effective maximum growth rate due to parametric excitation

$$\gamma_{max}=\frac{81}{16}\frac{\omega_{Le}^2}{\nu_{ei}}\left(\frac{v_{Te}}{c}\right)^4\left[2(p-1)-\frac{1}{9}\frac{c^2}{v_{Te}^2}\left(1+\frac{r_{Di}^2}{r_{De}^2}\right)^{-1}\frac{E^2(\infty)}{8\pi n_e\varkappa T_e}\right],$$

in which $E(\infty)$ is the electric field of the Langmuir oscillations in the stationary state:

$$\frac{E^2(\infty)}{8\pi n_e\varkappa T_e}=18\frac{v_{Te}^2}{c^2}\left(1+\frac{r_{Di}^2}{r_{De}^2}\right)(p-1).$$

The nonstationary electric field found in this way for the parametrically excited plasma waves, i.e.,

$$E^2(t)=E^2(\infty)s(\tau)$$

enables us to investigate the evolution in time of the coefficient of nonlinear transformation of the exciting-wave energy into the plasma wave:

$$K(t)=\frac{1}{6}(p-1)\left(1+\frac{r_{Di}^2}{r_{De}^2}\right)s(\tau).\tag{1.21}$$

This time dependence of the coefficient of nonlinear transformation of light energy into the energy of plasma waves may be of interest in the analysis of the parametric decay instability in laser plasma when the length of the light pulses is sufficiently small. It is clear that, when the length of the laser pulse is greater than the characteristic time (1.16) taken by the plasma to reach the stationary state, i.e.,

$$t_\infty=\frac{4}{81}\frac{\nu_{ei}}{\omega_{Le}^2}\left(\frac{c}{v_{Te}}\right)^4\frac{1}{p-1}\ln\left[2\cdot3^{1/2}\pi^2 n_e r_{De}^3\left(\frac{\omega_{Le}}{\nu_{ei}}\right)^2\left(\frac{v_{Te}}{c}\right)^3\left(1+\frac{r_{Di}^2}{r_{De}^2}\right)\right], \tag{1.22}$$

the nonlinear transformation coefficient (1.21) is a maximum and reaches a stationary value.

This situation arises, for example, for light pulses of length corresponding to tenths of a nanosecond or more in laser plasma in which the electron temperature is of the order of 1 keV and the density is $n_e \approx 2.5 \times 10^{20}$ cm^{-3} (neodymium laser). Under these conditions, the characteristic time $t_\infty \approx 5\times10^{-11}(p-1)^{-1}$ sec decreases with increasing excess over the threshold, $p-1$, i.e., with increasing light flux. When the length of the light pulse is less than the value given by (1.22), the nonstationary character of the laser radiation tends to reduce the nonlinear transformation coefficient (1.21) which represents the transformation of the light energy into the plasma wave energy.

2. APERIODIC INSTABILITY. RELAXATION OF THE INTENSITY OF PLASMA WAVES STABILIZED BY THE NONLINEAR FREQUENCY SHIFT

It is well known (see, for example,[18]) that plasma placed in an electromagnetic wave (1.1) becomes parametrically unstable when the frequency ω_0 approaches the electron Langmuir frequency ω_{Le}. In particular, if the exciting frequency ω_0 is less than the frequency of natural Langmuir plasma oscillations ($\omega_0 \lesssim \omega$)

$$\omega^2(k)=\omega_{Le}^2+3k^2v_{Te}^2, \tag{2.1}$$

then we can have an aperiodic instability corresponding to the excitation of high-frequency Langmuir oscillations of frequency (2.1) and aperiodic perturbation which grow exponentially on the linear stage, with growth rate

$$\gamma(\mathbf{k})=-\tilde\gamma(k)+\omega_{Le}\left\{\frac{1}{4}\frac{(\mathbf{kr}_E)^2}{r_{De}^2+r_{Di}^2}(k^2+k_0^2)\left[(k^2+k_0^2)^2-4(\mathbf{kk}_0)^2\right]^{-1}\right.$$
$$\times\left[\frac{\omega_{Le}-\omega_0}{\omega_{Le}}+\frac{3}{2}(k^2+k_0^2)r_{De}^2\right]-\left[\frac{3}{2}(k^2+k_0^2)r_{De}^2+\frac{\omega_{Le}-\omega_0}{\omega_{Le}}\right]^2 \tag{2.2}$$
$$\left.-\left[\frac{1}{4}\frac{(\mathbf{kr}_E)^2}{r_{De}^2+r_{Di}^2}\right]^2(\mathbf{kk}_0)^2\left[(k^2+k_0^2)^2-4(\mathbf{kk}_0)^2\right]^{-2}\right\}^{1/2}.$$

The last term in the braces in (2.2) is due to the contribution of the low frequency of the parametrically growing aperiodic perturbation

$$3\omega_{Le}(\mathbf{kk}_0)r_{De}^2\left\{1+\frac{1}{12}\frac{(\mathbf{kr}_E)^2}{(r_{De}^2+r_{Di}^2)r_{De}^2}\left[(k^2+k_0^2)^2-4(\mathbf{kk}_0)^2\right]^{-1}\right\},$$

which is nonzero for finite exciting wavelength $k_0 \neq 0$.

The growth rate of the parametric excitation given by (2.2) is valid under the conditions of Debye shielding of the aperiodic perturbation $\gamma < kv_{Ti}$ (where v_{Ti} is the thermal velocity of the ions) and reaches its maximum value for plasma perturbations propagating in the direction of the electric field in the exciting wave, $\mathbf{k}\cdot\mathbf{r}_E = \pm k r_E$, $\mathbf{k}\cdot\mathbf{k}_0 = 0$. The minimum threshold for the exciting flux in the case of dissipation due to Coulomb collisions, when $\tilde\gamma(k) = \nu_{ei}/2$, is given by the ratio:[18,19]

$$\frac{r_{E\,\mathrm{th}}^2}{r_{De}^2+r_{Di}^2}=4\frac{\nu_{ei}}{\omega_{Le}}. \tag{2.3}$$

Since we are interested in the development in time of the instability near the threshold, we shall confine our attention to the expansion of the growth rate (2.2) around its maximum value ($\theta \equiv \widehat{\mathbf{kr}_E}$)

$$\gamma(k,\theta)=\frac{\nu_{ei}}{2}\left[p^2-1-6\frac{\omega_{Le}}{\nu_{ei}}\frac{p^2+2(\omega_0-\omega_{Le})/\nu_{ei}}{p^2}r_{De}^2(k-k_m)^2-p^2\sin^2\theta\right] \tag{2.4}$$

In this expression, $p^2 = E_0^2/E_{\mathrm{th}}^2$ is the ratio of the exciting flux to the threshold flux (2.3), and the extremal wave number k_m is given by (see[7])

$$k_m^2=r_{De}^{-2}\frac{p^2\nu_{ei}+2(\omega_0-\omega_{Le})}{3\omega_{Le}}. \tag{2.5}$$

It was shown earlier[7] that the rapid dependence of the growth rate (2.2) on the detuning of the exciting frequency can be used to stabilize the aperiodic parametric instability by the small nonlinear frequency shift (1.3) of the Langmuir oscillations. When the excess over the threshold is small, $p^2 - 1 \ll 1$, the frequency correction (1.3) is determined essentially by the electric field of the Langmuir wave (in the case of long waves $kr_{De} < v_{Ti}/v_{Te}$):

$$\delta\omega=-\frac{\omega_{Le}}{4}\left(1+\frac{r_{Di}^2}{r_{De}^2}\right)^{-1}\frac{E^2}{8\pi n_e\varkappa T_e},\quad \frac{E^2}{8\pi}=\int\frac{d\mathbf{k}}{(2\pi)^3}W(\mathbf{k}). \tag{2.6}$$

Therefore, allowance for the nonlinear frequency shift (2.6) of Langmuir oscillations does not modify the dependence of the growth rate (2.4) on the wave number k and angle θ, and reduces to the addition of the term $E^4/2p^2E_{\mathrm{th}}^4$ on the right-hand side of (2.4), which shows that there is an increase in the effective threshold value of the exciting field with increasing E.

In the preceding section, we showed that when the evolution of the nonstationary plasmon spectral energy density (unstable because of the decay of the exciting wave into two plasmons) is analyzed, there is no necessity to solve the inhomogeneous equation (1.6). In particular, the description of this evolution turns out to be sufficiently complete, and for times $\tau > 1$ it is even exhaustive, provided we restrict our attention to solving the homogeneous equation (without the spontaneous term) with spontaneous noise taken as the initial value of the plasmon spectral energy density. In this section, in which we are concerned with the nonstationary evolution of aperiodic parametric instability, we shall also evaluate the consequences which ensue from the homogeneous equation, remembering that the above conclusion with regard to the small effect of the spontaneous term on noise relaxation is valid also for the aperiodic instability. Therefore, for the dimensionless spectral energy density of the electron Langmuir wave

$$s(x,\psi,\tau)=\frac{1}{3\pi\sqrt3}\frac{p^2-1}{p^2}\left(\frac{\nu_{ei}}{\omega_{Le}}\right)^{3/2}\left[p^2+2(\omega_0-\omega_{Le})/\nu_{ei}\right]^{1/2}r_{De}^{-3}E_{\mathrm{th}}^{-2}W(k,\theta,t),$$
$$\tau=\nu_{ei}(p^2-1)t,\qquad \psi=\frac{p\theta}{(p^2-1)^{1/2}}, \tag{2.7}$$
$$x=(k-k_m)r_{De}\left[6\frac{\omega_{Le}}{\nu_{ei}}\frac{p^2+2(\omega_0-\omega_{Le})/\nu_{ei}}{p^2(p^2-1)}\right]^{1/2}$$

we have the following simple equation:

$$\partial s(x,\psi,\tau)/\partial\tau=\left[1-x^2-\psi^2-s^2(\tau)\right]s(x,\psi,\tau), \tag{2.8}$$

where $s(\tau)$ describes, when (2.7) and (2.8) are taken into account, the time dependence of the complete (integrated over all the wave numbers and angles) noise energy density

$$s(\tau)=2\int dx\int\psi\,d\psi\,s(x,\psi,\tau),\qquad E^2=s(\tau)\left[2p^2(p^2-1)\right]^{1/2}E_{\mathrm{th}}^2. \tag{2.9}$$

The integrals in (2.9) with respect to ψ and the wave numbers x are evaluated (in accordance with the conclusions of Sec. 1) over the region which, in general, exceeds the region of excitation at the initial time:

$$1\geqslant\psi^2+x^2\geqslant0,\quad -1\leqslant x\leqslant1,\quad 1\geqslant\psi\geqslant0.$$

The formal solution of (2.8), subject to the initial condition $s(x, \psi, 0) = s_0$ where s_0 is the dimensionless spontaneous noise

$$s_0=\frac{1}{8n_e r_{De}^3}(2\pi)^{-5/2}\left(1+\frac{r_{Di}^2}{r_{De}^2}\right)^{-2}\left(1+\frac{e_i}{|e|}\frac{v_{Te}}{v_{Ti}}\right), \quad (2.10)$$

is of the form

$$s(x,\psi,\tau)=s_0\exp\left[\tau(1-x^2-\psi^2)-\int_0^\tau d\tau'\, s^2(\tau')\right]. \quad (2.11)$$

Equation (2.11) shows that the distribution of turbulent noise over the angles and wave numbers is completely determined by the growth rate $\gamma(k, \theta)$. Integration of both sides of (2.11) with respect to angles and wave numbers gives an integral equation for the total noise energy:

$$s(\tau)=s_0F(\tau)\exp\left(-\int_0^\tau d\tau'\, s^2(\tau')\right);$$
$$F(\tau)=e^\tau\int_0^{x_0} dx\, x^{1/2}e^{-\tau x};\quad x_0\approx(p^2-1)^{-1}. \quad (2.12)$$

The upper limit in the integral with respect to x in the expression for the function $F(\tau)$ is set by the condition $x^2 + x^2 \sim (p^2-1)^{-1}$ for which the growth rate given by (2.4) turns out to be of the order of the plasmon attenuation rate $\nu_{ei}/2$.

The solution of (2.12), namely,

$$s(\tau)=s_0F(\tau)\left[1+2s_0^2\int_0^\tau d\tau'\, F^2(\tau')\right]^{-1/2} \quad (2.13)$$

determines the evolution of both the integrated noise and the spectral distribution given by (2.11):

$$s(x,\psi,\tau)=s_0\exp[\tau(1-x^2-\psi^2)]\left[1+2s_0^2\int_0^\tau d\tau'\, F^2(\tau')\right]^{-1/2}. \quad (2.14)$$

In accordance with the conclusions obtained in the preceding section, Equations (2.13) and (2.14) provide a correct description of the behavior of the turbulent noise for $\tau \gtrsim 1$ and, therefore, complete the solution of the problem of the nonstationary nonlinear stabilization of an aperiodic parametric instability by the shift of the Langmuir frequency.

To illustrate the above solution, let us consider the behavior of the noise for $\tau \gg 1$, when

$$F(\tau)\approx{}^1/_2\sqrt{\pi}e^\tau\tau^{-3/2}.$$

In this case, the integrated noise is saturated $[s(\tau) \to 1]$

$$s(\tau)\approx\left(1+\frac{4}{\pi}\frac{\tau^3}{s_0^2}e^{-2\tau}\right)^{-1/2} \quad (2.15)$$

and reaches the stationary value $s(\infty) = 1$ in a time

$$\tau_\infty\approx\ln\left(\frac{1}{s_0}\ln^{3/2}\frac{1}{s_0}\right), \quad (2.16)$$

whilst the spectral energy density of the Langmuir oscillations becomes localized in a very narrow neighborhood of the extremal wave number (2.5) and the direction collinear with the exciting electric field $\mathbf{E}_0$

$$s(x,\psi,\tau)\approx\frac{2\tau^{3/2}}{\sqrt{\pi}}\exp[-\tau(x^2+\psi^2)],$$

so that for $\tau \gg \tau_\infty$ the noise is large only at the point where the growth rate (2.4) is a maximum, $x = \psi = 0$,

$$s(x,\psi,\infty)\approx\frac{4}{\pi\psi}\delta(x^2+\psi^2). \quad (2.17)$$

Using (2.7) and (2.17), we obtain the following expression for the stationary plasmon energy density:

$$W(k,\theta,\infty)=192\sqrt{3}\pi p(p^2-1)^{1/2}n_e\varkappa T_e r_{De}^3\left(1+\frac{r_{Di}^2}{r_{De}^2}\right)\left(p^2\frac{\nu_{ei}}{\omega_{Le}}+2\frac{\omega_0-\omega_{Le}}{\omega_{Le}}\right)^{-1/2}\delta\left[p^2\theta^2+6\frac{\omega_{Le}}{\nu_{ei}}\frac{p^2\nu_{ei}+2(\omega_0-\omega_{Le})}{p^2\nu_{ei}}(k-k_m)^2r_{De}^2\right]. \quad (2.18)$$

The stationary solution given by (2.17) and (2.18) corresponds to the result obtained earlier [7] (near the threshold) for the stationary electric field (2.9) of the Langmuir oscillation

$$E^2=[2E_0^2(E_0^2-E_{th}^2)]^{1/2}.$$

Figure 1 illustrates the saturation of the turbulent noise level. It shows the effective (nonlinear) growth rate of the aperiodic instability as a function of wave number for four successive times $0 < \tau_1 < \tau_2 < \infty$. Figure 2 shows (2.13) as a function of time τ for different values of the spontaneous noise s_0. These plots provide a qualitatively correct representation of the saturation of parametric instability corresponding to the decay of the exciting wave into two plasmons (see Sec. 1).

The characteristic time (2.16) taken by the integrated noise $s(\tau)$ to reach the stationary value $s(\infty) = 1$ is

$$t_\infty\approx\frac{1}{\nu_{ei}(p^2-1)}\ln\left[8(2\pi)^{5/2}n_e r_{De}^3\left(1+\frac{r_{Di}^2}{r_{De}^2}\right)^2\left(1+\frac{e_i}{|e|}\frac{v_{Te}}{v_{Ti}}\right)^{-1}\right] \quad (2.19)$$

and is determined to within an order of magnitude by the time of linear parametric growth of oscillations with a logarithmic increase in the ratio of the stationary integrated noise to the spontaneous noise. For the spontaneous noise (2.10), this logarithmic factor reaches a value roughly equal to ten. In particular, in hydrogen laser plasma with $T_e \approx 1$ keV and $n_e \approx 10^{21}$ cm^{-3}, the characteristic time t_∞ (for the excitation of aperiodic parametric instability by neodymium radiation) is greater by a factor of ten than the time 10^{-12} sec between collisions of an electron and an ion for a light flux of 10^{13} W/cm^2. Therefore, the noise reaches its stationary turbulent level in a time of 10^{-11} sec, which is much less than the length of the nanosecond light pulse.

It is clear from (2.19) that the time taken by tur-

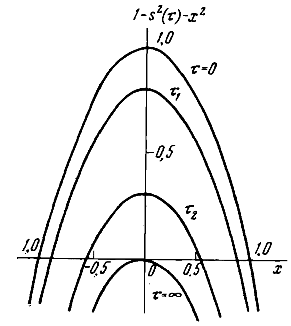

FIG. 1. Dependence of the growth rate $2\gamma(k,0)/\nu_{ei}(p^2-1) = 1-s^2(\tau)-x^2$ of the parametric excitation of an aperiodic perturbation along the exciting-wave field as a function of wave number for successive instants of time $0<\tau_1<\tau_2<\infty$ during the evolution of turbulent noise [see (2.7) for notation].

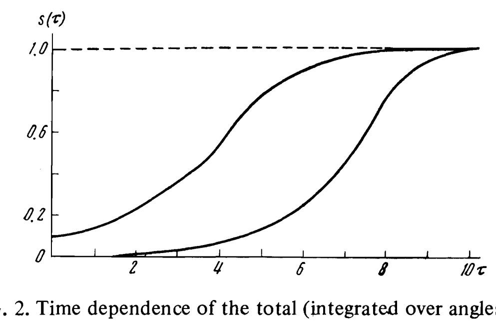

FIG. 2. Time dependence of the total (integrated over angles and wave numbers) energy density of high-frequency plasma waves excited parametrically in isotropic plasma as a result of the development of aperiodic instability [see (2.7), (2.9), (2.10), and (2.13)]. The upper curve corresponds to spontaneous noise $s_0 = 0.1$ and the lower curve to $s_0 = 0.01$.

bulent noise to reach its stationary level increases with decreasing excess over the threshold. On the other hand, when the exciting flux substantially exceeds the threshold, there is an increase in the growth rate of parametric excitation and a reduction in the duration of the relaxation of the turbulent noise. The above analysis of the evolution of noise near the threshold must then be replaced by the analysis of the more general equation for the spectral energy density ($\mu = \cos\theta$):

$$\frac{\partial W(k,\mu,t)}{\partial t}=\nu_{ei}\left\{\left[2p^2\mu^2\left(3\frac{\omega_{Le}}{\nu_{ei}}k^2r_{De}^2-\Delta\right)\right.\right.$$
$$\left.\left.-\left(3\frac{\omega_{Le}}{\nu_{ei}}k^2r_{De}^2-\Delta\right)^2\right]^{1/2}-1\right\}W(k,\mu,t);$$

$$\Delta=2\frac{\omega_0-\omega_{Le}}{\nu_{ei}}+\frac{(4\pi)^{-2}}{n_e\varkappa T_e}\frac{\omega_{Le}}{\nu_{ei}}\left(1+\frac{r_{Di}^2}{r_{De}^2}\right)^{-1}\int_0^\infty k'^2\,dk'$$
$$\times\int_{-1}^{1}d\mu' W(k',\mu',t)\left[1-\mu'^2+\mu^2(3\mu'^2-1)\right].$$

We shall write this equation in a form suitable for numerical solution by introducing the dimensionless wave number $x = kr_{De}(3\omega_{Le}/\nu_{ei})^{1/2}$ and spectral energy density y of Langmuir oscillations $[\Delta_0 \equiv 2(\omega - \omega_{Le})/\nu_{ei}]$

$$y(x,\mu,\tau)=\frac{1}{(4\pi)^2 3^{3/2}}\left(\frac{\nu_{ei}}{\omega_{Le}}\right)^{1/2}\left(1+\frac{r_{Di}^2}{r_{De}^2}\right)^{-1}\frac{1}{n_e r_{De}^3}\frac{W(k,\mu,t)}{\varkappa T_e}$$

$$\frac{\partial y(x,\mu,\tau)}{\partial\tau}=y(x,\mu,\tau)\{-1+[2p^2\mu^2(x^2-\Delta_0-y_0(\tau)-y_1(\tau)\mu^2)$$
$$-(x^2-\Delta_0-y_0(\tau)-y_1(\tau)\mu^2)^2]^{1/2}\}.$$

In these expressions,

$$y_0(\tau)=\int_{-1}^{1}d\mu(1-\mu^2)\int_{x_1}^{x_2}x^2\,dx\,y(x,\mu,\tau),$$

$$y_1(\tau)=\int_{-1}^{1}d\mu(3\mu^2-1)\int_{x_1}^{x_2}x^2\,dx\,y(x,\mu,\tau)$$

are the moments of the spectral density and determine its time dependence.

The wave number interval $x_1 \leq x \leq x_2$ is specified by the necessary condition for the existence of aperiodic instability (positive effective detuning $x^2 - \Delta \geq 0$ and positive expression under the root in the parametric growth rate)[2)]

$$x_1\equiv x_1(\mu,\tau)=(y_0+\Delta_0+y_1\mu^2)^{1/2}\eta(y_0+\Delta_0+y_1\mu^2),$$
$$x_2\equiv x_2(\mu,\tau)=[y_0+\Delta_0+\mu^2(2p^2+y_1)]^{1/2},$$

and the two functions of time y_0 and y_1 are related by the following set of two nonlinear equations:

$$y_0(\tau)=\int_{-1}^{1}d\mu(1-\mu^2)\int_{x_1}^{x_2}x^2\,dx\,y(x,\mu,0)\exp\left\{-\tau+\int_0^\tau d\tau'[2p^2\mu^2(x^2-\Delta_0-y_0(\tau')\right.$$
$$\left.-y_1(\tau')\mu^2)-(x^2-\Delta_0-y_0(\tau')-y_1(\tau')\mu^2)^2]^{1/2}\right\},$$

$$y_1(\tau)=\int_{-1}^{1}d\mu(3\mu^2-1)\int_{x_1}^{x_2}x^2\,dx\,y(x,\mu,0)\exp\left\{-\tau+\int_0^\tau d\tau'[2p^2\mu^2(x^2-\Delta_0-y_0(\tau')\right.$$
$$\left.-y_1(\tau')\mu^2)-(x^2-\Delta_0-y_0(\tau')-y_1(\tau')\mu^2)^2]^{1/2}\right\},$$

in which $y(x, \mu, 0)$ is the dimensionless spectral density of the turbulent noise at the initial time.

The above analysis of aperiodic instability in the near-threshold region corresponds to the conditions $p^2 - 1 < 1$, $y_0(\tau) \ll y_1(\tau)$, $\mu^2 \approx 1$ in the language of these equations. Numerical solution of the equations for $y_0(\tau)$ and $y_1(\tau)$ [or, directly, of the equation for $y(x, \mu, \tau)$] for an arbitrary excess above the parametric excitation threshold enables us to establish the effectiveness of the stabilization of turbulence by frequency shift under conditions approaching the experimental conditions.

In conclusion, we note that, because of the linear connection between the high- and low-frequency perturbations in the theory of parametric resonance,[18] the above spectral energy density and electric field of the Langmuir wave can be used to obtain similar quantities for the aperiodic perturbation of parametrically unstable plasma. In particular, the evolution of the electric field E_a of the aperiodic perturbation [with spectral potential density $\varphi(\mathbf{k})$]

$$\frac{E_a^2(t)}{8\pi}=\int\frac{d\mathbf{k}}{(2\pi)^3}k^2|\varphi(\mathbf{k},t)|^2$$

is described by the function $s(\tau)$ [see (2.13)]:

$$E_a^2(t)=\frac{4}{3}s(\tau)E_{th}^{3/2}(E_0-E_{th})^{1/2}\frac{e_i}{|e|}\frac{\nu_{ei}}{\omega_0}\left[\frac{\nu_{ei}}{\omega_0}+\frac{2(\omega_0-\omega_{Le})}{\omega_0}\right]$$
$$\times T_eT_i^2\left(T_i+\frac{e_i}{|e|}T_e\right)^{-3},$$

so that the stationary energy density $E_a^2(\infty)/8\pi$ coincides with that found previously in[7].

CONCLUSION

Summarizing the above discussion of nonstationary turbulence of parametrically unstable plasma, we list the most important conclusions and results. We note above all that the above nonstationary theory of generation of turbulent states in parametrically unstable plasma takes into account the effect of the intensity of growing perturbations on the frequency of plasma oscillations. We have obtained, for the first time, the explicit expressions [given by (2.14) and (1.17)] for the nonstationary spectral energy density of electron Langmuir oscillations growing in plasma as a result of the development of two parametric instabilities (aperiodic instability and instability involving the decay of the exciting wave into two plasmons), stabilized by the nonlinear frequency shift. The region of wave number space in which the turbulent noise is localized at the initial time has little effect on the region of parametric growth and contracts in the course of time. The energy density of nonstationary noise of this type of parametric turbulence, integrated over the spectrum, approaches the stationary state in a monotonic fashion. When the finite level of spontaneous noise is taken into account in the equation for the spectral density, this is found to have no effect on the behavior and characteristic time of relaxation averaged over the turbulent noise spectrum.

The results given above appear to be useful for the understanding of the physics of parametric interaction of high-intensity light, microwaves, and radio waves with hot plasma. The ideas developed here can be used not only to consider the near-threshold region of parametric excitation but also for arbitrary (sufficiently large) excesses above the threshold, which are realized under laboratory conditions.

1)It will be shown below that the particular choice of the joining point $x^2+\mu^2\sim[g(p-1)]^{-1}$ between the growth rate (1.5) and attenuation rate $\gamma_{ei}/2$ is unimportant for finding the nonstationary plasmon spectral energy density.

2)$\eta(z)$ is a unit step function so that $\eta(z) = 0$ when $z < 0$ and $\eta(z) = 1$ when $z > 0$.

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Translated by S. Chomet
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