Self-similar motions in general relativity for a spherically symmetric reference system in special coordinates

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The equations of the general theory of the relativity in the case of central symmetry are considered with the object of finding self-similar solutions. Instead of t and r, the special set of independent variables m and r is chosen which simplifies the equations and allows their solution. Some exact particular solutions of the "self-similar" equations are found.

It is always of great interest to find exact solutions in the general theory of relativity. In the present paper we present some exact solutions for self-similar adiabatic motions possessing central symmetry in the proper gravitational field with the interval

$$-ds^2=-c^2e^{\nu}dt^2+e^{\lambda}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2). \tag{1}$$

For the purpose of finding these solutions, we find it sensible to transform the basic system of equations, choosing r and $m=r\kappa^{-1}(1-e^{-\lambda})$, instead of r and t, as the independent variables. The system of equations then assumes the simplest form, and allows us to first find p, ϵ, and u as functions of r and m and then to determine $\nu=\nu(r, m)$ and $t=t(r, m)$. In the variables r and λ, the equations have a slightly more complicated form.

Taking the matter tensor in its usual form $T_i^k=(p+\epsilon)u_iu^k+\delta_i^kp$, and equating its covariant derivative $T_{i,k}^k$ to zero, we obtain the equations

$$\begin{gathered}\frac{1}{c^2\vartheta^2}[Au_t+uu_r]+\left(\frac{Au}{c^2}p_t+p_r\right)\frac{1}{p+\varepsilon}=\frac{1}{2u}(A\lambda_t+u\lambda_r),\\(A\varepsilon_t+u\varepsilon_r)\frac{1}{p+\varepsilon}+\frac{1}{\vartheta^2}\left(A\frac{u}{c^2}u_t+u_r\right)+\frac{2u}{r}=\frac{u}{2}\left(A\frac{u}{c^2}\lambda_t+\lambda_r\right),\end{gathered} \tag{2}$$

where

$$A=e^{(\nu-\lambda)/2},\quad \vartheta^2=1-u^2/c^2$$

and u is the ordinary three-dimensional radial velocity. In deriving these equations for the transformation of the derivatives of λ and ν, we used one of the transformed field equations:

$$A(1+u^2/c^2)\lambda_t+u(\nu+\lambda)_r=0. \tag{3}$$

Let us write the second field equation in the form

$$A\lambda_t+u\lambda_r+u\left(\frac{e^\lambda-1}{r}+\varkappa pre^\lambda\right)=0. \tag{4}$$

If we write the equations of conservation of entropy for the given particle, $d\sigma/ds=0$, then we arrive at the equation

$$A\sigma_t+u\sigma_z=0, \tag{5}$$

$$\sigma=\sigma(p,\varepsilon). \tag{6}$$

The six equations (2)–(6) constitute a complete system for the determination of the six sought-for functions p, ϵ, σ, u, λ, and ν. The solution of these equations in the independent variables t and r is, however, apparently impossible. Earlier [1], we wrote down systems of self-similar equations in which as the independent variable we chose the quantity $\bar{z}=r/t$ or $z=\lambda=z(\bar{z})$. These self-similar systems are, however, also not convenient for further use. To obtain more compact and visible expressions, it is convenient to proceed in the following manner.

Let us use the equation $T_{i,k}^k=0$, which can be reduced to the form

$$\left[\frac{r^2(\varepsilon+pu^2/c^2)}{\vartheta^2}\right]_t+\left[\frac{ur^2(p+\varepsilon)}{A\vartheta^2}\right]_r=0. \tag{7}$$

Equation (7) is a consequence of the Eqs. (2). Let

$$\frac{r^2(\varepsilon+pu^2/c^2)}{\vartheta^2}=m_r,\quad -\frac{ur^2(p+\varepsilon)}{\vartheta^2}=Am_t. \tag{8}$$

The field relations yield

$$\frac{r^2}{\vartheta^2}\left(\varepsilon+p\frac{u^2}{c^2}\right)=\frac{1}{\varkappa}[1-(re^{-\lambda})_r]=1-e^{-\lambda}(1-r\lambda_r). \tag{9}$$

From (8) and (9) we have

$$m=\frac{r}{\varkappa}(1-e^{-\lambda}),\quad e^{-\lambda}=1-\frac{\varkappa m}{r}. \tag{10}$$

Let us now use m and r as the independent variables; since

$$\frac{\partial(t,r)}{\partial(m,r)}=\left(\frac{\partial t}{\partial m}\right)_r\neq0,$$

it is easy to express the Eqs. (8) in terms of these variables:

$$ut_r=\frac{A}{p+\varepsilon}\left(\varepsilon+p\frac{u^2}{c^2}\right),\quad ut_m=-\frac{A\vartheta^2}{r^2(p+\varepsilon)}, \tag{11}$$

which yields

$$\begin{gathered}A=u(t_r-pr^2t_m),\\t_r+t_m\frac{r^2}{\vartheta^2}\left(\varepsilon+p\frac{u^2}{c^2}\right)=0.\end{gathered} \tag{12}$$

Further, it is easy to express the Eqs. (2) and (5) in terms of these variables: eliminating t_r and t_m from the transformed equations, we obtain the equations[2]

$$\frac{u}{c^2\vartheta^2}(u_r-pr^2u_m)+\frac{1}{p+\varepsilon}(p_r+\varepsilon r^2p_m)+\frac{\varkappa}{2r(1-\varkappa m/r)}\left(\frac{m}{r}+pr^2\right)=0, \tag{13}$$

$$\frac{1}{p+\varepsilon}(\varepsilon_r-pr^2\varepsilon_m)+\frac{1}{\vartheta^2u}(u_r+\varepsilon r^2u_m)+\frac{2}{r}+\frac{\varkappa}{2r(1-\varkappa m/r)}\left(\frac{m}{r}-\varepsilon r^2\right)=0, \tag{14}$$

$$\sigma_r-pr^2\sigma_m=0, \tag{15}$$

The system of equations (10–(15) determines p, ϵ, u, λ, ν, and t as functions of m and r. In the basic system, all the variables are mixed up, and it is necessary to solve the six equations simultaneously. A similar situation obtains for the self-similar motions. In the present case, if $\sigma\neq$ const, then it is necessary to solve the three equations (13), (14), and (15) simultaneously, while if $\sigma=$ const, then only two equations (13) and (14) need be solved simultaneously. In either case the functions $t=t(m, r)$ and $\nu=\nu(m, r)$ have to be determined after the solution of the equations; $e^{-\lambda}=1-\kappa m/r$ is already known.

We should, for what follows, make certain thermodynamic transformations. With the aid of the relations

$$d\sigma=c_v\frac{dT}{T}+\left(\frac{\partial p}{\partial T}\right)_v dv,$$

$$dv=\left[c_v\,dp-\left(\frac{\partial p}{\partial v}\right)_T v\,d\varepsilon\right]\left\{c_v\left(\frac{\partial p}{\partial v}\right)_T+\left(\frac{\partial p}{\partial T}\right)_v\left[(p+\varepsilon)-T\left(\frac{\partial p}{\partial T}\right)_v\right]\right\}^{-1},$$

$$dT=\left\{v\,d\varepsilon\left(\frac{\partial p}{\partial v}\right)_T+dp\left[(p+\varepsilon)-T\left(\frac{\partial p}{\partial T}\right)_v\right]\right\}\cdot$$

$$\times\left\{c_v\left(\frac{\partial p}{\partial v}\right)_T+T\left(\frac{\partial p}{\partial T}\right)_v\left[(p+\varepsilon)-T\left(\frac{\partial p}{\partial T}\right)_v\right]\right\}^{-1}$$

we express $d\sigma$ in terms of $d\epsilon$ and dp; Eq. (15) then assumes the form

$$(p_r-pr^2p_m)-\left(\frac{Tp_T^2}{c_v}-p_v\right)\frac{v}{p+\varepsilon}(\varepsilon_r-pr^2\varepsilon_m)=0. \tag{16}$$

The system of equations (13), (14), and (16) is complete, and determines p and ϵ as functions of m and r.

For the ideal gas $p=RT/v$, $p_T=p/T$, $p_V=-p/v$, where R is the gas constant. After simple thermodynamic transformations Eq. (16) assumes in this case the form

$$(p_r-pr^2p_m)-\frac{kp}{p+\varepsilon}(\varepsilon_r-pr^2\varepsilon_m)=0, \tag{17}$$

where $k=c_p/c_V$ is the specific-heat ratio.

Let us now find the relation between m and the Lagrange coordinate R, which is introduced in such a way as to allow the fulfilment of the relation $A(\partial r/\partial t)_R=u$, in which case $A(\partial\sigma/\partial t)_R=0$ and $\sigma=\sigma(R)$. Going over to the independent variables r and m, we have

$$AR_m=ut_rR_m-ut_mR_r;$$

substituting the expressions for ut_r and ut_m from (11), we arrive at the equation

$$pr^2R_m-R_r=0, \tag{18}$$

which determines $R=R(m, r)$; from (18) we can derive the interesting relation $(\partial m/\partial r)_R+pr^2=0$ for $p=0$ and $m=m(R)$.

Let us now proceed to find the self-similar equations. Let

$$u/c=a=a(z),\quad pr^2=\xi(z),\quad \varepsilon r^2=\eta(z)\quad (z=m/r); \tag{19}$$

in this case $e^{-\lambda}=1-\kappa z$. Upon making these substitutions, we find that the indicated system of three equations allows self-similar motions; as can easily be verified, generalizations with other powers of r and m do not yield self-similar motions.

Let us transform (14) and (16) with the aid of (19) to the equations

$$\begin{gathered}\left[\frac{aa_z}{1-a^2}-\frac{\varkappa}{2(1-\varkappa z)}\right](\xi+z)+\frac{2\xi-\xi_z(\eta-z)}{\xi+\eta}=0,\\ \left[\frac{aa_z}{(1-a^2)a^2}-\frac{\varkappa}{2(1-\varkappa z)}\right](\eta-z)+\frac{2\xi-\eta_z(\xi+z)}{\xi+\eta}=0,\\ [2\xi+\xi_z(\xi+z)]-\left(\frac{Tp_T^2}{c_v}-p_v\right)\frac{v}{p+\varepsilon}[2\eta+\eta_z(\xi+z)]=0.\end{gathered} \tag{20}$$

The last equation in (20) will yield a class of self-similar motions only under the condition that

$$\left(\frac{Tp_T^2}{c_v}-p_v\right)\frac{v}{p+\varepsilon}=\frac{p}{p+\varepsilon}\left[\left(\frac{\partial\ln p}{\partial\ln T}\right)_v^2\frac{pv}{c_vT}-\left(\frac{\partial\ln p}{\partial\ln v}\right)_T\right]=f(z),$$

where $f(z)$ is an arbitrary function of z. For the ideal gas

$$f(z)=\left(\frac{R}{c_v}+1\right)\frac{p}{p+\varepsilon}=\frac{kp}{p+\varepsilon}.$$

This expression also defines a selection of classes of equations of state for which the motion is self-similar. In this case

$$\frac{d\sigma}{c_v}=\frac{dp-f(z)\,d\varepsilon}{p}\left[\left(\frac{\partial\ln p}{\partial\ln T}\right)_v-\frac{c_vT}{pv}f(z)\right]^{-1}.$$

The last equation in (20) accordingly assumes the form

$$[2\xi+\xi_z(\xi+z)]-\frac{k\xi}{\xi+\eta}[2\eta+\eta_z(\xi+z)]=0. \tag{21}$$

The self-similar analog of the Eqs. (12) will, if we set $t=r^\alpha T(z)$, assume the form

$$r^{1-\alpha}A=ca[\alpha T-T'(\xi+z)]=r^{1-\alpha}e^{(\lambda-\nu)/2},$$

$$\alpha T=T'\left(z-\frac{\eta+\xi a^2}{1-a^2}\right).$$

Hence, since $e^{-\lambda}=1-\kappa z$ and $\nu=\nu(z)$, it follows from the first equation that $\alpha=1$, and the equation determining ν assumes the form

$$e^{-\nu/2}=ca(1-\varkappa z)^{1/2}[T-T'(\xi+\eta)].$$

Further, we find that

$$\frac{T'}{T}=\frac{1-a^2}{z(1-a^2)-(\eta+\xi a^2)} \tag{22}$$

$$e^{-\nu/2}=caT(1-\varkappa z)^{1/2}\left[\frac{\xi+\eta}{(\eta+\xi a^2)-z(1-a^2)}\right]. \tag{23}$$

Knowing ξ, η, and a as functions of z, we find from (22) that $T=T(z)=t/r$ and obtain from (23) the function $\nu=\nu(z)$, which completely solves the formulated problem. Introducing $R=r^\gamma\omega(z)$, we write Eq. (18) in the form

$$\omega'(\xi+z)-\gamma\omega=0. \tag{24}$$

Let us consider the case when $p=(k-1)\epsilon=\alpha\epsilon$, or

$$\xi=(k-1)\eta=\alpha\eta.$$

Then Eq. (21) is, as can easily be verified, satisfied identically. Equations (20) assume the form

$$\begin{gathered}\left(\frac{a_z^2}{1-a^2}-\frac{\varkappa}{1-\varkappa z}\right)(\alpha\eta+z)+\frac{2\alpha}{\alpha+1}\left[2-\frac{\eta_z}{\eta}(\eta-z)\right]=0,\\ \left(\frac{a_z^2}{(1-a^2)a^2}-\frac{\varkappa}{1-\varkappa z}\right)(\eta-z)+\frac{2\alpha}{\alpha+1}\left[2-\frac{\eta_z}{\eta}\left(\eta+\frac{z}{\alpha}\right)\right]=0.\end{gathered} \tag{25}$$

Let us transform this system into one second-order equation. We first determine

$$a^2=\left\{\frac{\varkappa}{1-\varkappa z}-\frac{2\alpha}{\alpha+1}\frac{1}{\alpha\eta+z}\left[2-\frac{\eta_z(\eta-z)}{\eta}\right]\right\}$$

$$\times\left\{\frac{\varkappa}{1-\varkappa z}-\frac{2\alpha}{\alpha+1}\frac{1}{\eta-z}\left[2-\frac{\eta_z}{\eta}\left(\eta+\frac{z}{\alpha}\right)\right]\right\}^{-1}=\frac{F}{\Phi}.$$

Further, we find

$$\frac{d}{dz}\ln\frac{F}{\Phi}=\Phi-F$$

and finally obtain

$$\frac{d}{dz}\ln\left\{\left[\frac{\varkappa}{1-\varkappa z}-\frac{2\alpha}{\alpha+1}\frac{1}{\alpha\eta+z}\left(2-\frac{\eta_z(\eta-z)}{\eta}\right)\right]\right.$$

$$\left.\times\left[\frac{\varkappa}{1-\varkappa z}-\frac{2\alpha}{\alpha+1}\frac{1}{\eta-z}\left(2-\frac{\eta_z}{\eta}\left(\eta+\frac{z}{\alpha}\right)\right)\right]^{-1}\right\}$$

$$+\frac{2\alpha}{\alpha+1}\left[\frac{1}{\eta-z}\left(2-\frac{\eta_z}{\eta}\left(\eta+\frac{z}{\alpha}\right)\right)-\frac{1}{\alpha}\left(\eta+\frac{z}{\alpha}\right)^{-1}\left(2-\frac{\eta_z}{\eta}(\eta-z)\right)\right]=0.$$

The investigation and numerical solution of this equation can be carried out by ordinary methods. It is easy to indicate some particular solutions to Eqs. (25).

Let us set $\eta=z$. Then the second equation in (25) is identically satisfied when $k=2$ and $\alpha=1$. In this case

$$\xi=\eta=z,\quad p=\varepsilon=z/r^2=m/r^3.$$

The first equation in (25) yields

$$\frac{da^2}{1-a^2}=\frac{dz}{z}\frac{2\varkappa z-1}{1-\varkappa z}$$

and integrating it, we obtain

$$a^2=1-A_0^2 z(1-\varkappa z),$$

where $A_0^2=\text{const}$; the constant is determined from the condition $a=1$ for $\kappa z=1$. Equation (22) assumes the form

$$\frac{dT}{T}=-\frac{1}{2}\frac{A_0^2(1-\varkappa z)\,dz}{1-A_0^2 z(1-\varkappa z)},$$

which, on integration, yields

$$\frac{t}{r}=T=B[1-A_0^2 z(1-\varkappa z)]^{(1-\beta)/4\beta}\left[\frac{2}{1+\beta}-A_0^2 z\right]^{\beta/2}$$

$$\beta=(1-4\varkappa/A_0^2)^{1/2},\quad B=\text{const}.$$

Further, Eq. (23) determines

$$\begin{aligned} e^{-\nu/2}&=(1-\varkappa z)^{1/2}\frac{cT}{a}\\ &=(1-\varkappa z)^{1/2}\frac{cB[1-A_0^2 z(1-\varkappa z)]^{(1-\beta)/4\beta}}{[1-A_0^2 z(1-\varkappa z)]^{1/2}}\left[\frac{2}{1+\beta}-A_0^2 z\right]^{\beta/2}. \end{aligned} \tag{26}$$

For $\kappa\to 0$ and $\lambda\to 0$, we have $\nu\to 0$ and $cB=1$. Thus,

$$\frac{ct}{r}=cT=[1-A_0^2 z(1-\varkappa z)]^{(1-\beta)/4\beta}\left[\frac{2}{1+\beta}-A_0^2 z\right]^{\beta/2},$$

$$e^{-\nu/2}=\left[\frac{1-\varkappa z}{1-A_0^2(1-\varkappa z)z}\right]^{1/2}[1-A_0^2 z[1-\varkappa z)]^{(1-\beta)/4\beta}\left[\frac{2}{1+\beta}-A_0^2 z\right]^{\beta/2}.$$

Since

$$z=\frac{1}{2\varkappa}\left\{1\mp\left[1-\frac{4\varkappa}{A_0^2}(1-a^2)\right]^{1/2}\right\}, \tag{27}$$

we can express ct/r in terms of a^2:

$$\frac{ct}{r}=\left(\frac{A_0^2}{2\varkappa}\right)^{\beta/2}a^{(1-\beta)/2\beta}\left[\pm\left(1-\frac{4\varkappa}{A_0^2}(1-a^2)\right)^{1/2}-\left(1-\frac{4\varkappa}{A_0^2}\right)^{1/2}\right]^{\beta/2}. \tag{28}$$

The relations (26)–(28) completely solve the problem of the self-similar motion of a medium for $\epsilon=p$. The equation $\epsilon=p$ is called the most rigid equation of state [3]. It makes sense in the presence of an extremely strong electromagnetic field [4], or in the case when the particle system interacts with a vector field [3]. We can also find the solution to the system (26)–(28) when $\alpha\eta+z=0$; in this case $k=-\frac{2}{3}$ and $\epsilon+3p=0$. However, this equation of state is not realistic in general relativity, and we shall not investigate it here.

Let us give still another particular solution, which was suggested by I. A. Fedoseev. Let $a^2=\kappa z$; then from Eqs. (20) we have

$$2\xi=\xi_z(\eta-z),\quad \frac{\eta-z}{2z}=\frac{\eta_z(\xi+z)-2\xi}{\xi+\eta}.$$

These equations are identically satisfied if $\xi=\gamma z$, where γ is any quantity. Noting that $\eta=3z$, we can write the equation of state in the form

$$p=\tfrac{1}{3}\gamma\varepsilon,\quad k=1+\tfrac{1}{3}\gamma.$$

Further, we find from (22) that

$$\frac{dT}{T}=-\frac{dz}{z}\frac{1-\varkappa z}{2+\varkappa(\gamma+1)z},$$

which yields

$$\frac{t}{r}=T=\frac{B}{z^{1/2}}[2+\varkappa(\gamma+1)z]^{(\gamma+3)/2(\gamma+1)},$$

where $B=\text{const}$.

Then from (23) we obtain

$$e^{-\nu/2}=Bc\varkappa^{1/2}(\gamma+3)[2+\varkappa(\gamma+1)z]^{(1-\gamma)/2(\gamma+1)}.$$

Since $\nu\to 0$ as $\kappa\to 0$, $e^{-\nu/2}=1$. Hence

$$B=\frac{2}{\gamma+3}\frac{1}{c\sqrt{\varkappa}\,2^{(\gamma+3)/2(\gamma+1)}}$$

which finally yields

$$\begin{aligned} \frac{t}{r}=T&=\frac{2}{c\sqrt{\varkappa z}(\gamma+3)}\left[1+\frac{\varkappa(\gamma+1)}{2}z\right]^{(\gamma+3)/2(\gamma+1)}\\ &=\frac{2}{u(\gamma+3)}\left[1+\frac{\gamma+1}{2}\frac{u^2}{c^2}\right]^{(\gamma+3)/2(\gamma+1)},\\ e^{-\nu/2}&=\left[1+\frac{\varkappa(\gamma+1)z}{2}\right]^{(1-\gamma)/2(\gamma+1)}(1-\varkappa z)^{1/2}. \end{aligned}$$

as $\kappa\to 0$, $a\to 0$, and $t\to\infty$, and we arrive for this particular solution, in the limit when there is no gravitational field, at the equilibrium conditions for a stationary medium, when $p=\text{const}$ and $\epsilon=\text{const}$.

It is likewise easy to find the Tolman solution, when $p=0$ ($\alpha=0$). In this case we first have from (20) that

$$\frac{a_z^2}{1-a^2}=\frac{\varkappa}{1-\varkappa z},$$

which determines $a^2=1-A_0^2(1-\kappa z)$, and then we find that

$$\frac{a_z^2}{a^2}(\eta-z)=\frac{\eta_z z}{\eta},$$

whence we determine η. This problem has already been solved by us [5], and we shall not reproduce the solution here.

It is also easy to find both the self-similar and the general solutions for $\epsilon+p=0$.

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