The gravitational radiation emitted by an ultrarelativistic charged particle in an external electromagnetic field

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It is shown how the intensity of gravitational radiation by an ultrarelativistic charge in an external electromagnetic field can be determined. The method is applicable to a wide class of problems. A closed expression is obtained for the intensity of the radiation emitted in circular motion in a Coulomb field.

The problem of gravitational radiation emitted by an ultrarelativistic charged particle in an external electromagnetic field has been discussed before in[1-4]. However the discussion of this problem in these papers seems to us not quite satisfactory for the reasons which are discussed in detail below.

In the linear approximation (in the gravitational field) the Einstein equation can be written in the form

$$\Box\psi_{\mu\nu}=\varkappa(T_{\mu\nu}{}^p+T_{\mu\nu}{}^f), \quad (1)$$

$$\partial_\mu\psi_{\mu\nu}=0. \quad (2)$$

Here we set $c = 1$ and $\kappa^2 = 16\pi k$, where k is the Newton gravitational constant,

$$\psi_{\mu\nu}=h_{\mu\nu}-{}^1/{}_2\delta_{\mu\nu}h_{\lambda\lambda}$$

and $\kappa h_{\mu\nu}$ is the deviation of the metric from the flat space metric; $T^p_{\mu\nu}$ is the energy-momentum tensor of the particle, and $T^f_{\mu\nu}$ is the energy-momentum tensor of the electromagnetic field. The latter is quadratic in the field, i.e., $T^f \sim FF + 2Ff + ff$. The square FF of the external field F bears no relation to the motion of the particle and will therefore not be discussed in the sequel. There is also no need to discuss the square ff of the field f of the particle, since almost everywhere $f \ll F$, and at small distances from the particle, where f is large, its contribution is taken into account by a mass renormalization, i.e., is already contained in $T^p_{\mu\nu}$. Thus, only the term 2Ff is important here, therefore in the sequel we shall interpret $T^f_{\mu\nu}$ to mean this quantity. This term is necessary for the conservation of the energy-momentum tensor, since for a particle in an external field the tensor $T^p_{\mu\nu}$ by itself is not conserved. It is incorrect to consider only $T^p_{\mu\nu}$ in Eq. (1) (the way the problem was solved in[1,4]).

However, the contributions of $T^p_{\mu\nu}$ and $T^f_{\mu\nu}$ to the field of the gravitational wave are different. Whereas the first is a usual divergent spherical wave, the second contribution does not fall off with the distance when the external field does not decrease at infinity and

$$T^f\sim r^{-1}\exp\{i(kr-\omega t)\}.$$

The reason for this is that the electromagnetic field of the particle goes over in a resonant manner into gravitational radiation. This effect was pointed out for the first time by Gertsenshteĭn[5] (cf. also[2]).

The problem of gravitational radiation in a uniform magnetic field has been considered by Pustovoĭt and Gertsenshteĭn[2]. They have attributed independent meaning to the part $\psi^{(1)}_{\mu\nu}$ of the gravitational field which decreases with the distance. However, it is easy to see that $\partial_\mu\psi^{(1)}_{\mu\nu} \neq 0$, the nonvanishing divergence of $\psi^{(1)}_{\mu\nu}$ being necessary for the cancellation of terms $\sim 1/r$ in the divergence of the resonant part of the field. In itself the separation of a four-dimensionally transverse part from $\psi^{(1)}_{\mu\nu}$ is not an unambiguous operation. Thus, the determination of the nonresonant part of the gravitational radiation turns out somewhat difficult in the case of an infinite homogeneous external field. A model with an arbitrarily cutoff homogeneous field is not admissible, since such a cutoff violates the Maxwell equations, and thus leads to a nonconservation of the energy-momentum tensor.

It is therefore natural to consider an inhomogeneous-field problem of the simplest possible variety: the motion along a circular trajectory in a Coulomb field. The wavelength of the electromagnetic radiation of the ultrarelativistic particle is much smaller than the radius r_0 of the orbit, so that the external field varies little over a wavelength. Consequently, one may speak of a resonant transition of the electromagnetic radiation into gravitational radiation. Let us calculate this effect.

Let Q be the charge at the center; e, m are the charge and mass of the particle. Then $eQ \approx \gamma m r_0 = \epsilon r_0$, where $\gamma = (1 - v^2)^{-1/2}$. In the ultrarelativistic case ($\gamma \gg 1$) the synchrotron radiation is concentrated in a narrow cone along the tangent to the trajectory. Inside the cone one may assume that the external field depends only on the coordinate x along the ray. Therefore we look for a solution of Eq. (1) in the form

$$\psi_{\mu\nu}{}^f=\frac{1}{x}\sum_\omega a_{\mu\nu}(\omega,x)\exp\{i\omega(x-t)\},$$

where $a_{\mu\nu}(\omega, x)$ is a slowly varying function of x. Substituting into the right-hand side of Eq. (1) the expression of $T^f_{\mu\nu}$ in the wave zone of the synchrotron radiation:

$$T_{\mu\nu}{}^f=\frac{1}{x}\sum_\omega \tau_{\mu\nu}(\omega,x)\exp\{i\omega(x-t)\},$$

we arrive at the equation (for $\omega x \gg 1$)

$$2i\omega\frac{da_{\mu\nu}}{dx}+\frac{d^2a_{\mu\nu}}{dx^2}=\varkappa\tau_{\mu\nu}. \quad (3)$$

We neglect the second derivative of the slowly varying function $a_{\mu\nu}(\omega, x)$. Then the solution at infinity has the form

$$a_{\mu\nu}(\omega)=\frac{\varkappa}{2i\omega}\int_0^\infty dx\tau_{\mu\nu}(\omega,x). \quad (4)$$

Only the following components contribute to the gravitational radiation: a_{yy}, a_{yz}, a_{zz}. Taking into account the explicit form to $\tau_{\mu\nu}$ it is easy to see that the resonance transformation comes about on account of the components of the external field which are orthogonal to the direction of the wave[4]. Simple calculations lead us to the following expression for the intensity of the resonant gravitational radiation:

$$I_{res}=\frac{kQ^2}{r_0^2}I_{em}=\frac{2}{3}\frac{k\varepsilon^2}{r_0^2}\gamma^4=\frac{2km^2}{3r_0^2}\gamma^6, \qquad (5)$$

where

$$I_{em}=\frac{2}{3}\frac{e^2}{r_0^2}\gamma^4 \qquad (6)$$

is the intensity of the synchrotron radiation. It is obvious that the angular and spectral distributions of I_{res} and I_{em} coincide.

Let us estimate the intensity of the nonresonance gravitational radiation, I_{gr}. For this purpose we compare it with I_{em}. These quantities differ, first, in their coupling constant: $k\epsilon^2$ in I_{gr}, versus e^2 in I_{em}; second, the component of the vector potential which contributes to the electromagnetic radiation is three-dimensionally transverse, i.e., $A_\perp \sim \theta \sim \gamma^{-1}$, whereas the potential of the gravitational field is doubly transverse $\psi_{\perp\perp} \sim \theta^2 \sim \gamma^{-2}$. (Here we have taken into account the purely kinematic circumstance that any radiation emitted by an ultrarelativistic particle is concentrated in a cone with opening angle $\theta \sim \gamma^{-1}$.) Taking these two differences into account yields, together with Eq. (6)

$$I_{gr}\sim\frac{ke^2}{r_0^2}\gamma^2\sim\frac{km^2}{r_0^2}\gamma^4. \qquad (7)$$

This result was obtained by Doroshkevich, Novikov, and Polnarev[6], who have considered the problem of the ultrarelativistic rotator.

It is now clear that the expression (5) is the correct ultrarelativistic approximation to the total intensity of the gravitational radiation by a charge moving along a circular orbit in a Coulomb field. This result is also confirmed by a direct calculation (cf. the Appendix).

The problem under consideration was solved by Peters[7] by numerical methods. Qualitatively, his result agrees with ours, but the numerical coefficient obtained by Peters is about four times larger than ours. (see note added in proof-transl.) We cannot indicate the concrete cause for this discrepancy, in view of the numerical character of the calculation in[3].

It is quite clear that the intensity I_{res} can be computed for motions in any slowly varying field, whenever I_{em} is known. Let us find in the general case the conditions for which I_{res} will make up the major fraction of the gravitational radiation. Let us first consider the case when the deviation angle α of the particle in the field is much larger than the opening angle of the cone in which the radiation is concentrated, $\theta \sim \gamma^{-1}$. Then the estimates (6) and (7) hold for I_{em} and I_{gr}, with r_0 replaced by a characteristic impact parameter. In this case

$$I_{res}\sim kF^2a^2I_{em}\sim kF^2a^2\frac{e^2}{r_0^2}\gamma^4, \qquad (8)$$

where F is the characteristic intensity of the external field, a is the path traversed by light in this field. From Eqs. (7), (8) it follows that

$$I_{res}/I_{gr}\sim a^2\gamma^2/R^2, \qquad (9)$$

where $R = \epsilon/eF$ is the radius of curvature of the trajectory of the particle. If the paths traversed by the particle and light in the external field are of the same order of magnitude, then $a/R \sim \alpha \gg \gamma^{-1}$ and, consequently, resonant radiation dominates. This is the general solution of the problem of gravitational radiation emitted in fields which are sufficiently strong and vary slowly.

Similar estimates for $\alpha \ll \gamma^{-1}$ (here both I_{em} and I_{gr} turn out to be γ^2 times smaller than for $\alpha \gg \gamma^{-1}$) show that in this case the contribution of the resonant radiation is small.

Let us now prove that the resonant and nonresonant parts of the gravitational radiation do not interfere. (This circumstance was noted in[2] for the case of a uniform magnetic field.) From the absence of interference it is clear than I_{res} is in any case a lower bound for the total intensity of gravitational radiation.

For the proof we expand the solution of Eq. (1) in a Fourier series in time:

$$\psi_{\mu\nu}(t)=\sum_n\psi_{\mu\nu}{}^n e^{-in\omega_0t}. \qquad (10)$$

Here ω_0 is the circular frequency of revolution of the particle. The solution of the stationary wave equation

$$(\Delta+n^2\omega_0^2)\psi_{\mu\nu}{}^n(\mathbf{r})=\varkappa T_{\mu\nu}{}^n(\mathbf{r}) \qquad (11)$$

in the wave zone has the usual form

$$\psi_{\mu\nu}{}^n(\mathbf{R})=-\frac{\varkappa}{4\pi}\frac{e^{ikR}}{R}T_{\mu\nu}{}^n(\mathbf{k}), \qquad (12)$$

where $T^n_{\mu\nu}(\mathbf{k})$ is the space Fourier transform of $T^n_{\mu\nu}(\mathbf{r})$, $\mathbf{k}$ being the propagation vector of the radiation detected at the point $\mathbf{R}$. The contribution of the electromagnetic field to $T^n_{\mu\nu}(\mathbf{k})$ can be represented as a convolution of the Fourier transforms of the external field and of the field of the particle:

$$T'^n(\mathbf{k})=\frac{1}{(2\pi)^3}\int d\mathbf{q}\,\mathcal{F}(\mathbf{k}-\mathbf{q})f(\mathbf{q}). \qquad (13)$$

The Fourier transform $f(\mathbf{q})$ of the field of the particle is related in the following manner to the Fourier transform $j(\mathbf{q})$ of the current:

$$f(\mathbf{q})\sim\frac{j(\mathbf{q})}{k^2-q^2+i\varepsilon}=j(\mathbf{q})\left[\mathrm{P}\frac{1}{k^2-q^2}-i\pi\delta(k^2-q^2)\right]. \qquad (14)$$

The second term in the square brackets is a solution of the wave equation and therefore its contribution corresponds to resonance radiation. The phase of this term is shifted by $\pi/2$ with respect to the nonresonant part $\psi^f_{\mu\nu}$, which in turn must have the same phase as $\psi^p_{\mu\nu}$, in order to guarantee the transversality of their sum. This proves the foregoing assertion.

Unfortunately, the problem under consideration presents a purely methodological interest, since in all known real situations the intensity of the radiation turns out to be minute.

In conclusion, we wish to thank sincerely A. I. Vaĭnshteĭn, V. V. Flambaum and E. V. Shuryak for valuable discussions.

APPENDIX

We now carry out a direct computation of the gravitational radiation emitted by an ultrarelativistic particle moving in a Coulomb field along a circular orbit. Regarding the part $\psi^p_{\mu\nu}$ of the gravitational field which has $T^p_{\mu\nu}$ as source, it is calculated in the usual manner (cf. e.g.,[2]). We do not carry out these calculations here, the more so, since $\psi^p_{\mu\nu}$ is a quantity of higher order in γ^{-1} than $\psi^f_{\mu\nu}$.

The calculation of all the components of the tensor $\psi^f_{\mu\nu}$ is useful in the sense that it allows for an independent verification of the calculation which makes use of the condition (2), taking, of course, into account $\psi^{p}_{\mu\nu}$. However here we calculate only those components of $\psi^{f}_{\mu\nu}$ which contribute to the intensity, leaving out systematically from ψ^{f}_{mn} components proportional to δ_{mn} and $k_m b_n$ (m, n = 1, 2, 3), where k is the propagation vector of the radiation and b is an arbitrary vector.

The tensor T^{f}_{mn} has the form

$$T_{mn}{}^{f}\to-\frac{1}{4\pi}(\mathscr{F}_mE_n+\mathscr{F}_nE_m), \tag{A.1}$$

where $\mathscr{F}_m$ is the external field and E_m is the field of the particle. The arrow denotes here and in the sequel that terms proportional to δ_{mn} and $k_m b_n$ have been omitted.

Making use of Eqs. (12), (13), (A.1), as well as of the explicit form of the Fourier transforms $\mathscr{F}_m(k - p)$ and $E_m(q)$, we find the following expression for ψ^{fn}_{ij} in the wave zone:

$$\psi_{ij}{}^{fn}=\frac{\varkappa}{(2\pi)^3}\frac{e^{ikR}}{R}\int_0^T\frac{dt}{T}e^{ikt}\int d\mathbf{r}\,\delta(\mathbf{r}-\mathbf{r}_0(t))$$
$$\times\int\frac{d\mathbf{q}\,e^{-i\mathbf{q}\mathbf{r}}}{(\mathbf{k}-\mathbf{q})^2(k^2-q^2+i\varepsilon)}\{(\mathbf{k}-\mathbf{q})_i(k\mathbf{v}-\mathbf{q})_j+(\mathbf{k}-\mathbf{q})_j(k\mathbf{v}-\mathbf{q})_i\}. \tag{A.2}$$

Here $r_0(t)$ and $v(t)$ are the coordinate and velocity of the particle. The integral

$$\int d\mathbf{q}\,e^{-i\mathbf{q}\mathbf{r}}\frac{(\mathbf{k}-\mathbf{q})_i(k\mathbf{v}-\mathbf{q})_j}{(\mathbf{k}-\mathbf{q})^2(k^2-q^2+i\varepsilon)}$$

can be reduced with the help of the Feynman parametrization

$$a^{-1}b^{-1}=\int_0^1 dx[ax+b(1-x)]^{-2}$$

to the form

$$-k^2\int_0^1 dx\exp\{ixkr-i(1-x)\mathbf{k}\mathbf{r}\}\frac{r_i}{r}\left[ikv_j+\frac{r_j}{r}\left(ikx-\frac{1}{r}\right)\right]. \tag{A.3}$$

After substituting (A.3) into (A.2) the integration with respect to r can be carried out trivially.

We now define the coordinate system. Assume the trajectory to be in the (x, y) plane and the observation point to be in the (y, z) plane, so that R = R(0, sin θ, cos θ). The integration with respect to t then leads to Bessel functions and we obtain the following equations for the components of ψ^{n}_{ij} in terms of which one can express the intensity of the radiation:

$$\psi_{xx}{}^{fn}-\psi_{yy}{}^{fn}\cos^2\theta=\frac{\varkappa}{4\pi}\frac{ke^{ikR}}{R}\int_0^1 dx\,e^{inv(1-x)}$$
$$\times\left\{J_n'(\xi)\frac{1+\cos^2\theta}{x\sin\theta}\left[1-\frac{1}{nv}\left(i(1-x)-\frac{1}{nv}\right)\right]\right.$$
$$\left.+J_n(\xi)\left[\left(\frac{1+\cos^2\theta}{x^2v^2\sin^2\theta}-\cos^2\theta\right)\left(i(1-x)-\frac{1}{nv}\right)-\frac{1+\cos^2\theta}{nvx^2\sin^2\theta}\right]\right\}, \tag{A.4}$$

$$\psi_{xy}{}^{fn}\cos\theta=\frac{\varkappa}{4\pi}\frac{ke^{ikR}}{R}\int_0^1 dx\,e^{inv(1-x)}\left\{J_n'(\xi)\frac{i\cos\theta}{xv\sin\theta}\right.$$
$$\cdot\left[-\frac{v}{n}+i(1-x)-\frac{1}{nv}\right]-i\cos\theta J_n(\xi)\left[\frac{v}{2}-\frac{1}{x^2v\sin\theta}\right.$$
$$\left.\left.+\frac{1}{nx^2v^2\sin^2\theta}\left(i(1-x)-\frac{1}{nv}\right)\right]\right\}. \tag{A.5}$$

Here ξ = nvx sin θ, Integrals of the type

$$\int_0^1 dx\,J_n(\xi)\exp\{inv(1-x)\}$$

can be calculated for large n (and for ultrarelativistic particles n ≫ 1 are essential) by expanding J_n(nvx sin θ) in the neighborhood of x = 1 in powers of x − 1. Then the integration yields a series in powers of $n^{-1/3}$. Since the main contribution comes from n ~ γ^3 we obtain the gravitational field in the wave zone as an expansion in powers of γ^{-1}. The intensity of radiation after all substitutions reduces to the form (only the leading terms in γ are written out)

$$\frac{dI}{d\Omega}=\frac{k\varepsilon^2}{2\pi r_0^2}\sum_{n=1}^{\infty}n^2[J_n'^2(nv\sin\theta)+\mathrm{ctg}^2\,\theta J_n^2(nv\sin\theta)], \tag{A.6}$$

or

$$dI=\frac{kQ^2}{r_0^2}dI_{em}, \tag{A.7}$$

in complete agreement with the result contained in the main text.

Note added in proof (20 November 1973). As Professor Peters kindly informed us, after receiving our preprint he has discovered a programming error, the correction of which has led to agreement between his results and ours.

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