Nonlinear interaction of space-charge waves with trapped electrons

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The partial numerical simulation method is used to examine the interaction of an intense flux of electrons with a relatively small number of electrons trapped in a potential well. It is found that if the well is not very deep, $w_2/w_1 \lesssim 10^{-2}$ (where w_2 and w_1 are the energies of the trapped electrons and of the electrons in the flux, respectively), a "monochromatic" space-charge wave is unstable with respect to excitation of satellites with shorter wavelength. When the potential well depth is increased ($w_2/w_1 > 10^{-2}$), the dominant role in the space-charge wave packet is assumed by the wave with the maximal increment. The nonlinear stage of the interaction is characterized by heating of the electrons trapped in the potential well.

We consider in this paper the nonlinear stage of the interaction between the space-charge waves of an intense electron beam and a relatively small number of electrons trapped in a potential well. Such a problem arises in the study of the behavior of an electron beam produced by an adiabatic gun in a number of microwave devices. A similar situation also obtains in experiments with a toroidal discharge. Skarsgard et al.[1], for example, investigated the free acceleration of electrons in an argon plasma. They observed a rapid departure of the electrons from the free-acceleration regime with simultaneous heating of the plasma electrons. The authors of that paper have demonstrated experimentally that the cause of the departure of the electrons from the free-acceleration regime is the appearance in the plasma plasma of an relatively small number of electrons trapped in potential wells produced by the corrugated magnetic field that confines the plasma.

The linear theory of space-charge oscillations in an electron beam passing through a region with trapped electrons was considered by one of us[2] in the geometrical-optics approximation. It was shown that the presence of trapped particles leads to instability in the beam. With increasing oscillation amplitude, the interaction of the beam with the trapped electrons becomes essentially nonlinear and cannot be described analytically. To solve the problem we therefore used the method of partial numerical simulation[3,7-9] in the present study.

At a density n_2 of the trapped electrons that is low in comparison with the density N_0 of the electrons in the beam ($n_2 \ll N_0$), the amplitude of the excited wave is sufficiently low and it can be assumed that the motion of the beam electrons is described by the linear approximation. This makes it possible to combine an analytic description of the particle behavior in the beam, regarded as a continuous medium, with simulation of the trapped electrons by individual particles. At the instant $t = 0$ the trapped electrons are uniformly distributed over the length of the space-charge wave, an equation of motion is written for each particle, and the trajectory of each individual particle is determined.

1. We consider the following simplified one-dimensional model. Trapped electrons are contained in a given potential well. A homogeneous monoenergetic electron beam passes through them. It is assumed that the energy of the beam electrons is high enough so that we can neglect the influence of the well potential on their motion. The potential energy w_2 of the trapped electrons is chosen for simplicity in the form

$$w_2 = \frac{1}{4} m V_2^2 \left(1 - \cos\frac{2\pi x}{L}\right),$$

where V_2 is the maximum velocity of the trapped electrons and L is the dimension of the well. We assume also the presence of an immobile background that compensates for the space charge.

We obtain equations describing the interaction of the monoenergetic beam of electrons with the trapped particles. We use for this purpose the Poisson equation

$$\partial E/\partial x = -4\pi e(n_1 + n_2), \tag{1}$$

where n_1 is the perturbation of the beam density and n_2 is the density of the trapped electrons, and also the equations of motion and of charge conservation

$$\left(\frac{\partial}{\partial t} + V_1\frac{\partial}{\partial x}\right) v_1 = -\frac{e}{m}E, \quad \left(\frac{\partial}{\partial t} + V_1\frac{\partial}{\partial x}\right) n_1 = -N_0\frac{\partial v_1}{\partial x},$$

where V_1 and N_0 are the unperturbed velocity and density of the electron beam, e and m are the charge and mass of the electron, and v_1 is the velocity perturbation of the beam electrons.

From the equations of motion and charge conservation for the particles we have

$$\left(\frac{\partial}{\partial t} + V_1\frac{\partial}{\partial x}\right)^2 n_1 = \frac{e}{m}N_0\frac{\partial E}{\partial x}. \tag{2}$$

Applying to the Poisson equation (1) the operator $(\partial/\partial t + V_1\partial/\partial x)^2$ and substituting (2) in this equation, we obtain

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial t} + V_1\frac{\partial}{\partial x}\right)^2 E + \omega_{pl}^2 E\right] = -4\pi e\left(\frac{\partial}{\partial t} + V_1\frac{\partial}{\partial x}\right)^2 n_2. \tag{3}$$

At $n_2 = 0$, Eq. (3) describes the propagation of a fast and a slow space-charge wave in the electron beam, the phase velocity of the waves being determined by the relation $v_{ph} = V_1 \pm \omega_{pl}/k$, the plus and minus signs referring to the fast and slow waves, respectively. It is clear that the trapped electrons will interact most effectively with the slow space-charge wave, since the synchronism condition $v_{ph} \approx v_2 \ll V_1$ is satisfied for them in this case. The maximum instability increment is reached in this case at $kV_1 \approx \omega_{pl}$.

We seek the solution of (3) at $n_2 \neq 0$ by the Bogolyubov method in the form

$$E(x, t) = E(t)\sin(kx + \alpha(t)), \tag{4}$$

which determines the slow space-charge wave. Here α is the oscillation phase shift, and the wave vector k is determined by the condition $k = \omega_{pl}/V_1$, where $\omega_{pl} = 4\pi e^2 N_0/m$ is the plasma frequency of the beam electrons. We regard the quantities E and α as slow functions of the time, the characteristic time of variation of these functions being $T \gg 1/\omega_{p1}$.

The density of the trapped electrons is represented in the form

$$n_2(x,t)=\frac{1}{S}\sum_{j=1}^{M}\delta(x-x_j(t)), \qquad (5)$$

where $x_j(t)$ are the coordinates of the trapped particles, M is the number of particles in the well, and S is the beam cross section area.

Assuming $\epsilon \sim n_2/N_0 \sim (T\omega_{p1})^{-1} \ll 1$, substituting (4) and (5) in Eq. (3), discarding all terms of order higher than ϵ, and using the orthogonality of the trigonometric functions, we obtain a system of equations describing the slow variation of the amplitude and phase of the wave:

$$\frac{d\mathscr{E}}{d\tau}=-\frac{1}{M}\sum_{j=1}^{M}\sin(\varkappa\xi_j+\alpha),$$
$$\frac{d\alpha}{d\tau}=-\frac{1}{\mathscr{E}M}\sum_{j=1}^{M}\cos(\varkappa\xi_j+\alpha). \qquad (6)$$

We have introduced here the dimensionless variables

$$\tau=(n_2/N_0)^{1/3}\omega_{p1}t, \quad \xi=2\pi x/L,$$
$$\mathscr{E}=E[4\pi n_2 m V_1^2(n_2/N_0)^{1/3}]^{-1/2}.$$

The quantity $\kappa = l\omega_{p1}/2\pi V_1$ is an integer that shows how many space-charge wavelengths are spanned by a segment of length l.

The trajectories of the trapped electrons are determined by integrating the equations of motion (in dimensionless variables)

$$d\xi_j/d\tau=\nu_j,$$
$$\frac{d\nu_j}{d\tau}=-\frac{\beta\nu_0^2}{4}\sin(\beta\xi_j)-\frac{1}{\varkappa}\mathscr{E}\sin(\varkappa\xi_j+\alpha), \qquad (7)$$

where

$$\nu=\frac{v_2}{\varkappa V_1(n_2/N_0)^{1/3}}, \qquad \nu_0=\frac{V_2}{\varkappa V_1}\left(\frac{N_0}{n_2}\right)^{1/3};$$

here v_2 is the velocity of the trapped electrons and $\beta = l/L$. The quantity l was introduced for convenience. The case when $\beta = 1$ and $\kappa = 10$, for example, describes the interaction of a space-charge wave of length $\lambda = L/10$ with trapped electrons; the case $\beta = 10 = \kappa$ describes the interaction between a wave with $\lambda = L$ and trapped electrons.

The system (6) and (7) was computer-integrated by the Runge-Kutta method[5].

The electrons trapped in a potential well were simulated by individual particles distributed in the interval $-\pi \leq \xi \leq \pi$. The number of particles was varied between 200 and 300. The results of the calculation did not depend in this case on the number of particles. The particles were distributed over the segment at each point, with the points equally spaced $4\pi/m$ apart. The initial particle velocity was specified by the formula $\nu = \pm\nu_0\cos(\beta\xi/2)$ (i.e., the trapped electrons initially had equal total energies). One of the particles was given a plus sign, and the other a minus sign. A constant integration interval was used. In accordance with the requirement that the problem be correct, the interval chosen was such as to ensure stability of the solution.

Several variants of the calculation were performed. The ratio of the density of the trapped electrons to the density of the beam electrons was always chosen to be the same at 10^{-3}. The ratio κ of the well length to the wavelength was assumed to be either 1 or 10. The calculations were carried out with intervals ranging from 2×10^{-2} to 5×10^{-2} and with initial wave amplitude $\mathscr{E} = 10^{-3}-10^{-2}$. The initial amplitude was chosen here not only to satisfy the condition that it be small compared with the maximum amplitude (the wave amplitude must not exceed the noise level), but also from the condition that the perturbation of the trapped electrons by the wave field be small in comparison with the action of the well potential on the electrons.

Figure 1 shows plots of the wave electric-field amplitude (in dimensionless variables) in terms of the time τ. Figure 1a describes the behavior of the field amplitude when a wave of length $\lambda = L/10 \ll L$ is excited in the electron beam. Figures 1b and 1c describe the behavior of the field amplitude when a wave with length equal to the dimension of the potential well is excited in the beam, at different depths of the well.

At $\mathscr{E} \ll 1$ it is easy to find the linear instability increment. At $\lambda \ll L$ the instability increment coincides with the increment obtained in[2] in the geometrical-optics approximation, and is equal to $0.684\omega_{p1}(n_2/N_0)^{1/3}$. For $\lambda = L$, the instability increment during the linear stage is equal to $2.5\omega_{p1}(n_2/N_0)^{1/3}$ (Fig. 1b) and $2.8\,\omega_{p1}(n_2/N_0)^{1/3}$ (Fig. 1c). The subsequent time dependence of the electric field (Fig. 1) shows the presence of amplitude oscillations, which (as in[9]) have an irregular character. Figures 2 and 3 show plots of the velocity ν of the trapped electrons against the coordinate ξ at different instants of time τ for $\lambda \ll L$ and $\lambda = L$. As follows from Figs. 2b and 3b, which correspond to the maximum of the wave amplitude, the trapped electrons have no time to become completely bunched and have a noticeable spread both in velocity and in the coordinates.

The oscillations have a different character for each different case, and are connected with the presence of two characteristic times in the system comprising the beam and trapped electrons, the period τ_1 of the oscillations of the trapped electrons in the well and the period τ_2 of the oscillation of the bunch of trapped electrons

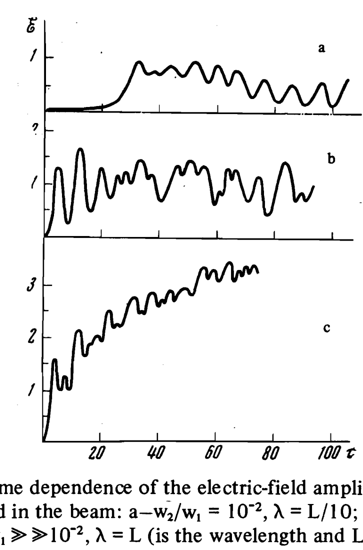

FIG. 1. Time dependence of the electric-field amplitude when one wave is excited in the beam: a—$w_2/w_1 = 10^{-2}$, $\lambda = L/10$; b—$w_2/w_1 = 10^{-2}$, $\lambda = L$; c—$w_2/w_1 \gg\gg 10^{-2}$, $\lambda = L$ (is the wavelength and L is the dimension of the potential well).

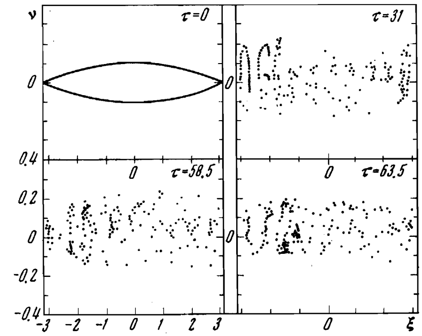

FIG. 2. Dynamics of the phase plane of the trapped electrons, $w_2/w_1 = 10^{-2}$, $\lambda = L/10$.

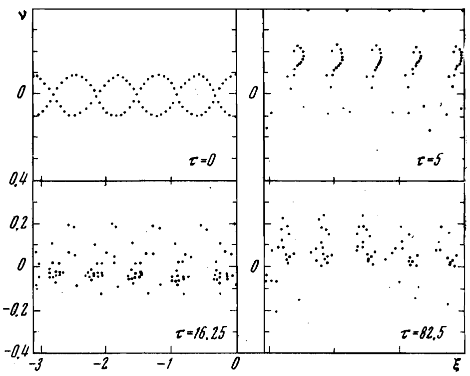

FIG. 3. Dynamics of the phase plane of the trapped electrons, $w_2/w_1 = 10^{-2}$, $\lambda = L$.

captured by the wave field near the maximum of the field. The times τ_1 and τ_2 can easily be determined by linearizing the right-hand sides of (7) ($\xi \ll 1$, $\alpha = 0$)[6]:

$$\tau_1 = 4\pi/\beta\nu_0, \quad \tau_2 = 2\pi/\sqrt{\mathscr{E}_{max}}.$$

The quantities ν_0 and β were chosen as follows: $\nu_0 = 0.1$ and $\beta = 1$ for Fig. 1a, $\nu_0 = 0.1$ and $\beta = 10$ for Fig. 1b, and $\nu_0 = 0.25$ and $\beta = 10$ for Fig. 1c. This yields for τ_1 and τ_2 the values

for Fig. 1a $\quad \tau_1 = 120, \quad \tau_2 \approx 6;$

for Fig. 1b $\quad \tau_1 = 13, \quad \tau_2 \approx 6;$

for Fig. 1c $\quad \tau_1 \approx 4, \quad \tau_2 \approx 4.$

Figure 1b, for example, clearly shows the beats of the wave amplitude, with characteristic times coinciding with the times τ_1 and τ_2. Amplitude beats can also be traced in Fig. 1a. For the third case (Fig. 1c), there should be no beats, since the times τ_1 and τ_2 are approximately equal.

The incomplete capture of the trapped electrons by the wave field does not lead to a damping of the oscillations, since the violation of the condition for the buildup of oscillations as a result of the decreased velocity of the wave-captured trapped particles is compensated for by the change in the phase shift α of the wave. The trapped electrons "attune themselves" as it were, and the wave amplitude, dropping to a certain level, begins to increase again.

However, a numerical experiment on the excitation of only one oscillation mode cannot provide an accurate picture of the considered instability. When an electron beam interacts with trapped electrons, the increment of the resultant instability is not a δ function (see[2]). The trapped electrons excite primarily oscillations whose growth increment is close to maximal. As a result there is excited in the passing beam a narrow packet of waves with wave vectors k close to ω_{pl}/V_1. The excitation of the wave packet can significantly distort the character of the interaction of the electron beam with the trapped electrons and lead to cessation of the instability.

2. The system of equations describing the behavior of the system consisting of the passing beam and the trapped trapped electrons when a wave packet is excited in the electron beam is obtained in analogy with the preceding system, and takes the following form (in dimensionless variables):

$$\frac{d\mathscr{E}_i}{d\tau} = -\frac{1}{M}\sum_{j=1}^{M}\sin(\varkappa_i\xi_j+\alpha_i),$$

$$\frac{d\alpha_i}{d\tau} = \frac{1}{2}\left(\frac{N_0}{n_2}\right)^{1/3}\frac{1-(\varkappa_i/\varkappa)^2}{\varkappa_i/\varkappa} - \frac{1}{\mathscr{E}_i M}\sum_{j=1}^{M}\cos(\varkappa_i\xi_j+\alpha_i), \tag{8}$$

$$\frac{d\xi_j}{d\tau} = \nu_j,$$

$$\frac{d\nu_j}{d\tau} = -\frac{\beta\nu_0^2}{4}\sin(\beta\xi_j) - \frac{1}{\varkappa}\sum_{i=1}^{N}\mathscr{E}_i\sin(\varkappa_i\xi_j+\alpha_i).$$

Here $j = 1, \ldots, M$; $i = 1, \ldots, N$; M is the number of particles, N is the number of waves in the packet, $\kappa_i = l/\lambda_i$, λ_i is the length of the i-th wave, and $\kappa = l\omega_{pl}/2\pi V_1$ is an integer showing the number of waves with maximum increment spanned by a segment of length l. The numbers κ_i are integers for all the waves of the packet. We investigated two cases, $l = L$ and $l = \kappa L$.

The system (8) was integrated by the Runge-Kutta method for different numbers of particles M. The results of the calculations were in this case independent of M. The number of particles ranged from 200 to 300. The particle velocities and coordinates were specified in the same manner as when one mode was excited. We considered the time variation of the amplitude and phase of waves whose length was smaller than the segment l by factors of 8, 9, 10, 11, and 12 respectively (i.e., we investigated the 8th, 9th, 10th, 11th, and 12th harmonics). The growth increment was largest for the 10th harmonic.

A plot of $\mathscr{E}_i(\tau)$ for five harmonics is shown in Fig. 4. Figure 4a corresponds to the case $l = L$, and Figs. 4b and 4c to the case $l = 10L$. The potential well has different depths in Figs. 4b and 4c. In Fig. 4b we have $w_2/w_1 = 10^{-2}$, and in Fig. 4c we have $w_2/w_1 = 6 \times 10^{-2}$ (here w_2 and w_1 are the energies of the trapped and beam electrons, respectively).

At a small depth of the potential well (Figs. 4a and 4b), the 11th and 12th harmonics increase in the packet in addition to the wave with the maximum increment (10th harmonic), and in the case $l = L$ (Fig. 4a) the wave with the maximum increment is rapidly suppressed. In the case $l = 10L$ (Figs. 4b and 4c), the wave with the maximum increment is excited during the initial stage of instability development (cf. Figs. 1 and 4). However, whereas in Fig. 4b the wave with the maximal increment is suppressed, and the 11th and 12th harmonics begin to grow rapidly in the packet, in Fig. 4c the wave with the maximum increment drops first to a value $0.3\,E_{max}$ and then starts increasing again. The energy of the 10th harmonic (Fig. 4c) exceeds the energy of the 9th by one order of magnitude in this case, and the energies of the other waves in the packet by two orders of magnitude.

The behavior of the electrons on the phase plane fol-

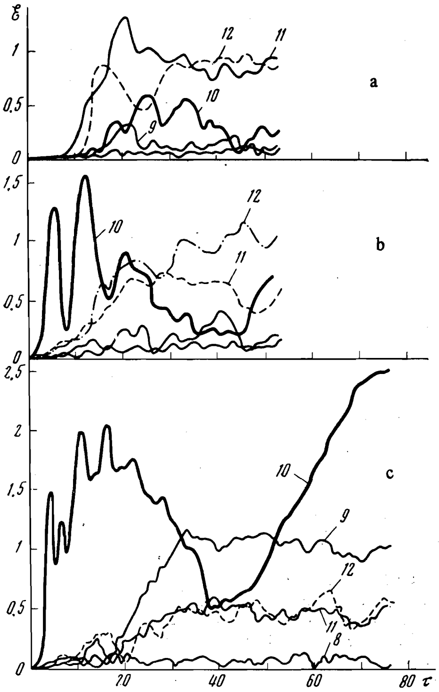

FIG. 4. Dynamics of wave packet: a—λ_{10} = L/10, $w_2/w_1 = 10^{-2}$; b—λ_{10} = L, $w_2/w_1 = 10^{-2}$; c—λ_{10} = L, $w_2/w_1 \gg 10^{-2}$. The numbers are those of the harmonics.

lowing excitation of a wave packet in the beam has an irregular character. For the case l = L ($\lambda \ll$ L), rapid randomization of the electrons sets in. For the case l = 10L ($\lambda \sim$ L), the electrons behave during the initial stage in the same manner as when only one oscillation mode is excited. At succeeding times, appreciable scatter of the electron velocity occurs in all cases. The excitation of the wave packet in the beam is accompanied by heating of the trapped electrons, whose energy increases by one order of magnitude. An exception is the case of a large well depth, when the heating is negligible.

The results can be understood from the following physical considerations. The trapped electrons draw from the electric field of the excited wave an additional energy (in dimensionless variables) $\delta w_2 \sim \mathscr{E}/\kappa^2$. At a small potential-well depth $w_2/w_1 \approx 10^{-2}$, the energy acquired by the trapped electrons from the field is much larger than their initial energy $\mathscr{E}/\kappa^2 \gg \nu_0^2/2$. The behavior of the trapped electrons is determined in this case mainly by their interaction with the field of the excited wave. In a coordinate system connected with the beam electrons, the trapped electrons constitute a low-density beam passing through a plasma. The phase velocity v_{ph} of the excited waves differs and amounts to V_1 for the wave with $\lambda = l/10$. Shorter waves will have a velocity $v_{ph} < V_1$, and longer ones will have $v_{ph} > V_1$. The growth of the waves in the packet is accompanied by slowing down of the beam and by violation of the resonance condition $v_{ph} = V_1$ for the maximum-increment wave $\lambda = l/10$. Its growth increment $\gamma \sim kV_1(n_2/N_0)^{1/3}$ decreases as a result. With decreasing beam velocity, the shorter waves with $v_{ph} < V_1$ enter into resonance. Their growth increment increases and becomes larger than the increment of the wave with $\lambda = l/10$. This leads to a transformation of the packet in the direction of the shorter wavelengths (Figs. 4a and 4b).

With increasing well depth, the potential gradient increases and at $\nu_0^2/2 > \mathscr{E}/\kappa^2$ (Fig. 4c) the decisive role in the behavior of the trapped particle is played by the well potential. The fraction of the resonant particles decreases noticeably in this case, since the only resonant particles are those located in the narrow region at the edges of the well (the width of the region is determined by the quantity $\nu_0^2/2 - \mathscr{E}/\kappa^2$). The decrease in the number of resonant particles leads only to a slight heating of the trapped electrons (in comparison with the initial energy). Although the trapped electrons slow down somewhat in this case, the change in their velocity is negligible and consequently the wave with the maximum increment, which is at resonance with the beam, is the dominant one in the packet.

We note that the initial distribution of the trapped electrons was chosen to be homogeneous (nonequilibrium). However, the results also remain valid for an inhomogeneous initial distribution. Indeed, the equilibrium distribution of the trapped electrons is characterized by the fact that the bulk of the electrons are concentrated at the edges of the well. In the coordinate system connected with the passing beam, the trapped electrons constitute a beam of bunched electrons. For waves of length $\lambda \ll$ L this causes the grouping of the trapped electrons into bunches captured by the excited-wave field to start first with the electrons located at the edges of the well. Further development of the instability proceeds in the same manner as when the trapped electrons are uniformly distributed inside the well. For waves of length $\lambda \sim$ L, allowance for the equilibrium initial distribution leads only to a somewhat more rapid growth of the wave with the maximum increment at the initial stage of the excitation. Subsequently, the behavior of the waves in the packet is the same.

In an electron beam, as already noted above, there exist, besides slow space-charge waves, also fast waves of frequency $\omega = kV_1 + \omega_{p1} > \omega_{p1}$. Therefore, for a packet of fast waves it follows from (3), after averaging over the rapid oscillations, that $dE_i/d\tau = d\alpha_i/d\tau = 0$, i.e., the amplitudes of the fast space-charge waves remain at the level of the thermal noise in the beam.

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