Kinetics of a quantum particle in a one-dimensional random potential

V. L. Berezinskiĭ

(Submitted April 17, 1973)
Zh. Eksp. Teor. Fiz. **65**, 1251–1266 (September 1973)

The motion of a quantum particle in a one-dimensional random potential of the white-noise type is considered. The asymptotic density-density and current-current correlation functions are found for high particle energies ($E\tau_B \gg \hbar$) and large scales $k^{-1} \sim l_B$, $\omega^{-1} \sim \tau_B$, where l_B and τ_B are the Boltzmann mean free path and time. The following results have been obtained: (a) Anderson localization occurs in the system, i. e., particles emitted at a point x_0 do not spread out along a straight line as $t \to \infty$ but are distributed near a point x_0 with a density $p_\infty(x-x_0)$ that decreases for $|x-x_0| \gg l_B$; (b) the conductivity for $\omega \to 0$ is of the form $\sigma \simeq -i\omega\alpha$, which is typical of a dielectric ($\alpha \sim l_B^2$ is the "polarizability"), and the initial term of the expansion of the dissipative part is $\mathrm{Re}\,\sigma \sim (\omega\tau_B)^2 \ln^2|\omega\tau_B|$. These results are in complete agreement with Mott's concepts. It is shown that the localization case can be regarded as a spontaneous violation of translational invariance and in particular, owing to property (a), translationally noninvariant averages can be introduced which are similar to the Bogolyubov quasiaverages.

Mott and, subsequently, other authors (see the review[1]) have put forward the idea, according to which the one-electron states in disordered systems can be of two types: localized and delocalized. The spectral regions corresponding to the localized states are then, to a certain extent, analogous to the forbidden bands in crystals: if the Fermi level E_F falls in this region, then the system behaves as a dielectric as $T \to 0$. The acceptance of Mott's idea raises a number of questions about the physical properties and the mathematical description of the system in the region of the localized states. In particular, it is not quite clear how we can reconcile the existence of localized states with the spatial (macroscopic) homogeneity of the system.

From the mathematical standpoint the elucidation of these problems requires the analysis of the Schrödinger equation for a random potential, which is a difficult problem. Only in the one-dimensional case do we have any general approach to the problem, and is therefore the case which has been most thoroughly investigated thus far. It has been shown (see Mott's[1] and Halperin's[2] reviews) that the eigenfunctions for the one-dimensional random potential are localized at all energies. These results are however incomplete, since they do not allow final conclusions to be drawn about the behavior of the conductivity and the space-time correlation functions.

At the same time, if we accept Mott's idea, then it follows from these results that the conductivity in the one-dimensional random potential should vanish at all energies. This conclusion appears to be paradoxical, since at high energies the scattering probability and the wavelength tend to zero ($\beta^2 \to 0$, $\lambda_E \to 0$), so that the following conditions obtain:

$$l_B = (\beta^2 c)^{-1} \gg r_0 = c^{-1} \gg \lambda_E \qquad (I)$$

(l_B is the Boltzmann mean free path and $r_0 = c^{-1}$ is the mean distance between the scattering centers). In the three-dimensional case, the kinetic equation is applicable under the conditions (I), and it seems strange that this is not the case in the one-dimensional case (the one-dimensional kinetic equation clearly leads to a finite conductivity).

It is clear from the foregoing that apart from the consideration of the individual ψ-functions, which has been carried out in the above-cited papers (see[1,2]), the computation of the correlation functions and the conductivity, especially for high energies where the conditions (I) are fulfilled, is also of interest. This is done in the present paper (for the model with a white-noise type of potential). The results obtained corroborate fully Mott's idea and, as seems to us, they clarify a number of questions connected with the idea.

Notice that the problem of the computation of the conductivity (for the same model) has been considered by Halperin[3], who expressed the conductivity in terms of the solutions of a system of partial differential equations. These equations were solved for $\lambda_E \ll l_B$ in a recent paper by Bychkov[4] [1)]. He found the dominant term of the expansion of the conductivity for $\omega \to 0$ to have the form $\mathrm{Re}\,\sigma \sim A/|\omega|$, but it can be shown that in this term (which coincides exactly with the real part of our expression (49b)), the integral giving the constant A vanishes, so that it is necessary to seek the next term of the expansion, which leads to our formula (63). (In the equations of[4], certain terms of order $l^{-2} = (\lambda_E/l_B)^{-2}$ are retained, owing to which one can obtain $A \neq 0$, but this is connected with the fact that not all the terms of order l^{-2} are taken into account there.)

1. THE CHOICE OF THE IMPORTANT DIAGRAMS AND THE DERIVATION OF THE BASIC EQUATIONS

1. Determination of the principal quantities

The motion of a quantum particle of mass m and charge e in a one-dimensional random potential V(x) is considered. We shall be interested in the correlation functions for the density and current operators defined as

$$j^0(x) = \delta(\hat{x} - x), \quad j^1(x) = \frac{1}{2m}\{\hat{p}j^0(x) + j^0(x)\hat{p}\}. \qquad (1)$$

Here $\hat{x}$ and $\hat{p}$ are the coordinate and momentum operators; the unified notation $j^a(x)$ (a = 0, 1) has been introduced for convenience only (the problem is nonrelativistic). Another definition of the operators (1) in terms of their matrix elements ($\hbar = 1$ and $\varphi_n(x)$ are the wave functions of the states $|n\rangle$) is

$$\langle 1|j^0(x)|2\rangle = \varphi_1^*(x)\varphi_2(x), \quad \langle 1|j^1(x)|2\rangle = \frac{1}{2m}\left\{\varphi_1(x)\frac{\partial}{\partial x}\varphi_2^*(x) - \varphi_1^*(x)\frac{\partial}{\partial x}\varphi_2(x)\right\}. \qquad (1')$$

The potential V(x) is assumed to be time independ-

ent, which implies the neglect of transitions induced by the interaction with the thermostat. Energy is then conserved and the distribution function f(E) can be arbitrary (not necessarily a Fermi distribution). The physical averages are given by the formula

$$\langle \hat{A} \rangle = n_0 \int f(E) M_V \{ \mathrm{Sp}(\delta_E \hat{A}) \} dE, \tag{2}$$

where the symbol $M_V\{\dots\}$ denotes averaging over the realizations of the random potential, δ_E is the density matrix of the microcanonical distribution:

$$\delta_E = \sum_{(n)} |n\rangle \delta(E - E_n) \langle n| = \frac{1}{2\pi i} \left\{ \frac{1}{E - H - i0} - \frac{1}{E - H + i0} \right\} \tag{3}$$

(E_n and $|n\rangle$ are the eigenvalues and eigenfunctions of the Hamiltonian $H = \hat{p}^2/2m + V(x)$ for the given realization of $V(x)$), and n_0 is the particle density (for normalization to one particle $n_0 = L^{-1}$, where L is the length of the system), f(E) being normalized by the condition

$$\int f(E) N(E) dE = 1, \text{ where } N(E) = M_V \{ \mathrm{Sp}(j^0(x) \delta_E) \}$$

is the density of states.

We shall consider two forms of correlation functions, the Fourier transforms of which we shall denote by $\widetilde{jj}^a_E(\omega, k)$ and $\overline{jj}^a_E(\omega, k)$ (a = 0, 1) and define by the equalities

$$\int_0^\infty e^{i\omega t} M_V \{ \mathrm{Sp}(\delta_E j^a(x,t) j^a(x)) \} dt = \int_{-\infty}^{\infty} \widetilde{jj}_E{}^a(\omega, k) e^{ik(x-x')} dk/2\pi, \tag{4}$$

$$M_V \{ \mathrm{Sp}(\delta_E j^a(x) \delta_{E+\omega'} j^a(x')) \} = \int_{-\infty}^{\infty} \overline{jj}_E{}^a(\omega', k) e^{ik(x-x')} dk/2\pi. \tag{5}$$

In the definitions (4) and (5), the $j^a(x, t)$ are the Heisenberg operators (1) for the moment of time t, ω is a complex frequency (Im $\omega > 0$), and ω' is a real frequency. The functions (4) and (5) are connected by the relations

$$\widetilde{jj}_E{}^a(\omega, k) = i \int_{-\infty}^{\infty} \frac{\overline{jj}_E{}^a(\omega', k)}{\omega - \omega'} d\omega' \qquad (\mathrm{Im}\,\omega > 0), \tag{6}$$

$$\overline{jj}_E{}^a(\omega', k) = \pi^{-1} \mathrm{Re}\, \widetilde{jj}_E{}^a(\omega' + i0, k). \tag{7}$$

The functions (6) are, according to (4), the Fourier transforms of the retarded correlation functions, while the spectral densities (7) are connected with the response functions by the dynamic susceptibility $\chi(\omega, k)$ (the density-external potential response function) and by the conductivity $\sigma(\omega, k)$ (the current-external field response function). The expression for the latter has the form

$$\sigma(\omega, k) = ie^2 n_0 \iint \frac{f(E) - f(E+\omega')}{\omega - \omega'} \frac{1}{\omega'} \overline{jj}_E{}^1(\omega', k) dE\, d\omega', \tag{8}$$

$$\mathrm{Re}\, \sigma(\omega', k) = \pi e^2 n_0 \int \frac{f(E) - f(E+\omega')}{\omega'} \overline{jj}^1(\omega', k) dE. \tag{8'}$$

In particular, for a Fermi distribution, we have as $T \to 0$

$$\begin{aligned} \sigma(\omega, k) &= e^2 n_0 \widetilde{jj}_E{}^1(\omega, k), \\ \mathrm{Re}\, \sigma(\omega', k) &= \pi e^2 n_0 \overline{jj}_E{}^1(\omega, k), \quad E = E_F. \end{aligned} \tag{9}$$

The functions introduced can, according to the second equality in (3), be expressed in terms of the averages of the product of two Green's functions. Let us denote by $G^\pm$ and G' the Green functions for the energies E and $E + \omega$:

$$G^\pm = (E - H \pm i0)^{-1}, \quad G' = (E + \omega - H)^{-1}. \tag{10}$$

We obtain in accordance with (3) and the definitions (4) and (5)

$$\widetilde{jj}_E{}^a(\omega, k) = \int e^{ik(x-x')} \{ \widetilde{jj}_-{}^a(\omega, x - x') - \widetilde{jj}_+{}^a(\omega, x - x') \} dx', \tag{11}$$

where

$$2\pi \widetilde{jj}_\pm{}^0(x - x') = M_V \{ \langle x' | G^\pm | x \rangle \langle x | G' | x' \rangle \}, \tag{11a}$$

$$2\pi \widetilde{jj}_\pm{}^1(x - x') = M_V \left\{ \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) \left(\frac{\partial}{\partial x_1'} - \frac{\partial}{\partial x_2'} \right) \cdot \right.$$
$$\left. \times \langle x_1' | G^\pm | x_1 \rangle \langle x_2 | G' | x_2' \rangle \right\} \Big|_{\substack{x_1 = x_1' = x \\ x_2 = x_2' = x'}} \tag{11b}$$

($\langle x | G | x' \rangle$ is the coordinate representation for (10)). These expressions will be used below.

2. Formulation of the problem

As can be seen from the above-cited formulas, the sought functions can be considered separately for each energy shell, whose energy E will then be a parameter of the problem (in the Fermi case the role of E is played by E_F). Let us introduce the system of units

$$\hbar = E = 2m = 1, \tag{12}$$

in which the units of length, energy, and time will respectively be the de Broglie wavelength $\lambda_E = \hbar(2mE)^{-1/2}$, the energy E, and the time $\hbar/E$, while all the quantities having the dimensions of length, energy, and time will be dimensionless ratios in these units. Starting from this point, the system of units (12) will, unless otherwise stipulated, be used everywhere.

To concretize the problem, we must specify the statistical distribution of the potential V(x). We shall consider a model in which V(x) is a Gaussian random function with the correlators

$$M_V\{V(x)\} = 0, \quad M_V\{V(x)V(x')\} = l^{-1}\delta(x - x'). \tag{13}$$

The model (13) for a value of the parameter l equal to

$$l = 1/cv_0^2, \quad v_0 = \int V(x - x_i)\, dx, \tag{14}$$

describes a system of randomly distributed scattering centers (c is the concentration of the centers and $V(x - x_i)$ is the potential of a single center) when the range of action of each center is much less 1 and $v_0 \ll 1$ (the condition of applicability of the Born approximation). For other cases where the model (13) is applicable, see Halperin's paper[3].

Under the indicated assumptions, the quantity l can be arbitrary (depending on the concentration c), but we shall restrict ourselves to the case

$$l \gg 1, \tag{15}$$

which corresponds exactly to the conditions (I). In accordance with what we said in the introductory part of the article, we shall be interested in the correlation functions for $\omega \sim k \sim l^{-1}$, i.e., strictly speaking, in the asymptotic limit when $\omega \to 0$, $k \to 0$, and $l \to \infty$, but the quantities $\nu = 4\omega l$ and $\kappa = 4kl$ tend to finite limits (the factor 4 has been introduced for convenience):

$$\omega \to 0, \; k \to 0, \; l \to \infty, \; 4\omega l \to \nu, \; 4kl \to \varkappa. \tag{16}$$

Of particular interest here is the case $\nu \ll 1$; in the opposite case, when $\nu \gg 1$, the results should correspond to those obtained from the one-dimensional kinetic equation.

3. Description of the diagrammatic technique and the choice of the important diagrams

The expressions (11) allow a diagrammatic representation on the basis of the well-known perturbation-theory series expansion of the Green functions. The

unperturbed Green functions (henceforth denoted by GF) have the form

$$\langle x_1|G_0^{\pm}|x_2\rangle = \mp i\cdot{}^1/_2 \exp\{\pm i|x_2-x_1|\},$$
$$\langle x_1|G_0'|x_2\rangle \approx -i\cdot{}^1/_2 \exp\{i(1+\omega/2)|x_2-x_1|\},\ \mathrm{Im}\,\omega>0,\ |\omega|\ll 1. \quad (17)$$

The unaveraged diagram consists of two electron lines going from the point x' to the point x. Each line consists of segments $(x', x_1), \ldots, (x_i, x_{i+1}), \ldots, (x_n, x)$; the segments represent the GF (17), while the points x_i represent the factors $V(x_i)$. As a result of the averaging, these factors group together in pairs, each pair being associated with a correlator from (13). In the diagrams, correlators will be represented by wavy lines, while the GF (17) will be represented by ordinary or double continuous lines (the retarded GF, G' and G^+, by ordinary, and the advanced GF, G^-, by double lines). An example of such a diagram for the average $M_V\{\langle x'|G^-|x\rangle\langle x|G'|x'\rangle\}$ is shown in Fig. 1. Notice that the unfolding of the drawing along the vertical axis was done only for the sake of clarity and does not carry any semantic weight.

The most important feature is the use of "ordered" diagrams, in which the scattering points x_i (over which the integration is performed) are ordered along a straight line in a definite manner with respect to each other and the fixed points x' and x. In the general case an ordered diagram corresponds to an integral over the variables x_i varying in a region of the form

$$-\infty < x_l \leqslant \ldots \leqslant x_i \leqslant x' \leqslant x_{i+1} \leqslant \ldots \leqslant x_c \leqslant x \leqslant x_{c+1} \leqslant \ldots \leqslant x_r < \infty. \quad (18)$$

In such diagrams the coordinate x plays a role similar to that of time. If we move along a diagram, say from right to left, then we can associate with each interval between neighboring vertices an intermediate state defined by the number pair (g, g'), where g and g' are the numbers of the G^-- and G'-lines in this interval. For example, to the diagram in Fig. 1 corresponds the sequence (2.2), (4.4), (4.4), (2.2), (1.1), (3.3), (3.3), (2.2), (2,2). On going through each vertex (the points x' and x are included in the number of vertices), the numbers g and g' undergo definite changes Δg and $\Delta g'$ which are uniquely determined by the type of vertex (e.g., for the diagram e in Fig. 2, $\Delta g = \Delta g' = +2$).

The ordered diagrams are convenient in that each has a definite order for the passage to the limit (16). To see this, we use the fact that the differences $x_i - x_j$ do not change sign in the integration over the region (18), so that the GF (17) are factorable. For example, if $x_i < x_2$, then $\langle x_1|G_0^-|x_2\rangle = ({}^1/_2 i)^{1/2} e^{-ix_1}({}^1/_2 i)^{1/2} e^{ix_2}$. Let us now transfer the first factor to the vertex x_1 and the second to the vertex x_2, and let us perform a similar operation with respect to all the G_0^-- and G_0'-lines of the diagram, thereby transferring the dependence on the x_i's from the lines to the vertices (this operation is similar to the transition to the interaction representation in time-dependent perturbation theory).

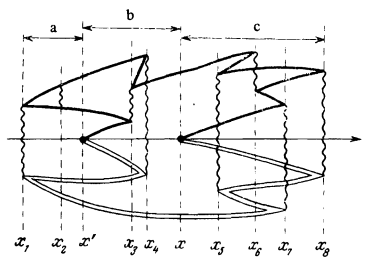

FIG. 1. An example of the diagram for the average $\langle x'|G^-|x\rangle$ $\langle x|G'|x'\rangle$. The diagram pertains to important diagrams of the type Γ_{+-} and has a left-hand (a), a central (b), and a right-hand (c) part of the form R_1, Z_{10} and R_2 (see the text), respectively.

Figures 2 and 3 show the vertices having the property that $\Delta g = \Delta g'$, together with the corresponding factors. Notice that in the limit $\omega \to 0$, these factors are equal to the constants $\pm(4l)^{-1}$, whereas to vertices with $\Delta g \neq \Delta g'$ (not shown in the figures) there correspond in this case additional factors of the form $\exp\{i(\Delta g - \Delta g')x\}$. This difference turns out to be important for order-of-magnitude estimates for the diagrams when passing to the limit (16).

For such estimates, we set $\omega = k = 0$ in the integral and simultaneously replace the limits $\pm\infty$ in (18) by $\pm L$. Then the estimate for such an integral yields for $L \to \infty$ the estimate for the original diagram, since the integrals are cut off at $L \sim \omega^{-1}, k^{-1}$. To the vertices shown in Figs. 2 and 3 correspond the factors $\pm(4l)^{-1}$, and for the diagrams containing only these vertices we have the estimate $l^{-r}L^r L$ (the extraneous factor L is connected with the integration with respect to dx when computing the Fourier transform (11)). Such diagrams give contributions of the form

$$\tilde{jj}_E{}^a(\omega, k) \sim lf(4\omega l, 4kl) \to lf(\nu, \varkappa). \quad (19)$$

To the vertices for which $\Delta g \neq \Delta g'$ correspond at $\omega = 0$ the factors $\sim\exp(\pm 2ix_j)$, the integrals of which converge at the upper limit, and for diagrams containing such vertices the power of L for $L \to \infty$ will be less than, or equal to, the power of $(4l)^{-1}$; they will, upon passage to the limit (16), give contributions of the form $l^{-n}f(\nu, \kappa)$ with $n = 0, 1, \ldots$, which are small compared to (19).

It is easy to see that the same contributions are made by all the diagrams for the second term in (11), i.e., for the average of the product of the retarded GF, G' and G^+. In fact, in such diagrams only the interaction vertices a, b, and c in Figs. 2 have the limit $\pm(4l)^{-1}$ for $\omega \to 0$. But in diagrams containing only such vertices, to the points x' and x correspond the factors $e^{\pm 2ix}$, so that the maximum order of magnitude of the diagrams will be $l^0 f(\nu, \kappa)$.

Thus, in the approximation (19), only the diagrams containing no other vertices except those shown in Figs. 2 and 3 are important. Such diagrams may be divided into four classes, the sums of the contributions of which

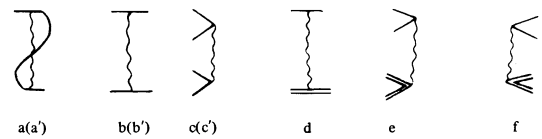

FIG. 2. The types of interaction vertices that form the important diagrams. The vertices a', b', and c' not shown in the figure differ from the vertices a, b, and c by having double lines in place of the ordinary lines. To the vertices correspond the following factors: $-1/4l$(a, a', b, b', c, c'), $+1/4l$ (d), $+\exp(i\omega x_i)/4l$ (e), and $+\exp(-i\omega x_i)/4l$ (f).

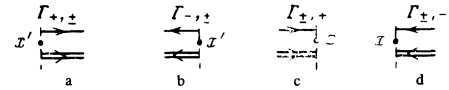

FIG. 3. The types of incoming and outgoing vertices in the important diagrams. At the top are indicated the types of $\Gamma_{\pm\pm}$ for the diagrams containing the given vertex. To the vertices correspond the following factors: $\frac{1}{2}\exp\{-i\omega x'/2\}$(a), $\frac{1}{2}\exp\{i\omega x'/2\}$(b), $\frac{1}{2}\exp\{i\omega x/2\}$(c), and $\frac{1}{2}\exp\{-i\omega x/2\}$(d).

(without the factors of $1/2$ pertaining to the points x' and x) we shall denote by $\Gamma_{\pm\pm}$. The left subscript $\pm$ indicates the type of x' vertex in Fig. 3, a) and b) (or according as $\Delta g = \Delta g' = \mp 1$), while the right subscript indicates the type of x vertex in Fig. 3, c) and d), or according as $\Delta g = \Delta g' = \pm 1$. For the quantities (11a) we have

$$2\pi\tilde{j}j_E^0(\omega; x-x') = 1/4\{\Gamma_{++}(x-x') + \Gamma_{--}(x-x') + \Gamma_{+-}(x-x') + \Gamma_{-+}(x-x')\}. \quad (20a)$$

In computing (11b), we must imagine that each of the points x' and x has split up into two close points, differentiate the contributions according to (11b), and then let the points merge. In our approximation, each differentiation yields a factor of $\pm i$, and we obtain

$$2\pi\tilde{j}j_E^1(\omega, x-x') = \Gamma_{++}(x-x') + \Gamma_{--}(x-x') - \Gamma_{+-}(x-x') - \Gamma_{-+}(x-x'). \quad (20b)$$

4. Derivation of the basic equations

Let us distinguish the left-hand, right-hand, and central parts of the selected diagrams as the parts of the diagrams lying to the left of x', to the right of x, and between x' and x, respectively (for $x' < x$). Since for the vertices of Fig. 2, $\Delta g = \Delta g' = 0, \pm 2$, in the intermediate states of the left-hand and right-hand parts $g = g' = 2n$, while in the central part $g = g' = 2n + 1$ ($n = 0, 1, 2, \ldots$).

The contribution of each diagram is an integral over the region (18), an integral which breaks up into a product of three integrals over $(x_1, \ldots, x_l)$, $(x_{l+1}, \ldots, x_c)$, and $(x_{c+1}, \ldots, x_r)$, which we shall respectively call the contributions of the left-hand, central, and right-hand parts. Let us denote by $\tilde{R}_m(x)$ the sum of the contributions of all the right-hand parts that have at the boundary with the central part (i.e. immediately to the right of the point x) the state $g = g' = 2m$, by $\tilde{R}_{m'}(x')$ the analogous sum of the contributions of the left-hand parts, and by $Z_{m'm}(x', x)$ the sum of the contributions made by the central parts with left and right boundary states $g = g' = 2m' + 1$ and $g = g' = 2m + 1$.

The right-hand parts of the diagrams can be constructed by successively joining all the possible vertices in Fig. 2 to the right of already constructed right-hand parts. In doing this, in order not to obtain diagrams with electron loops, diagrams that bear no relation to the original diagrams, we must take account of only those possibilities for which the segments of the G_0'- and G_0^--lines are joined up into two continuous electron lines going from x' to x. For this purpose we can number the lines on the boundary of the right-hand part by assigning No. 1 to the segment corresponding to the first entry of the electron line into the right-hand part, No. 2 to the first emergence from it, No. 3 to the second entry, and so forth from 1 to 2m. In constructing the diagrams, we should take account of the fact that the angles at the vertices in Fig. 2 can be formed by only the segments of the G_0-lines with consecutive numbers, e.g., $2l - 1$ and $2l$.

The process of constructing the diagrams of the right-hand part is schematically illustrated in Fig. 4 (we illustrate the addition of only the vertices d, e, and f in Fig. 2 and not of all the vertices in the figure). To this process corresponds also the equation for $\tilde{R}_m(x)$

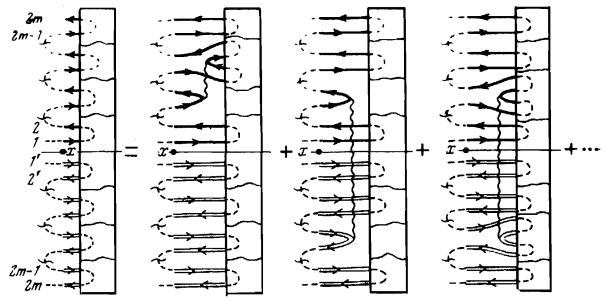

FIG. 4. Schematic representation of the process of constructing the right-hand parts and the equation for $\tilde{R}_m(x)$. The right-hand parts are represented by the rectangles; the dashed curves indicate the shape of the electron lines inside the right-hand and central parts. For each of the the two electron lines G' and G^- are shown four segments with consecutive entry and exit of the electron line into and from the right-hand part: the initial pair 1 and 2, the two intermediate segments $2l - 1$, $2l$ and $2n - 1$, $2n$, and the last lines $2m - 1$, $2m$. Summation over all the possible ways of selecting l, n, etc., and over all the possible ways of joining up the vertices a, a', b, b', c', and d in Fig. 2, not shown here, is implied.

(the integral and the corresponding differential equation) having the form

$$-\frac{d\tilde{R}_m}{dx} = \frac{1}{4e}\{m^2\tilde{R}_{m-1}e^{i\omega x} + m^2\tilde{R}_{m+1}e^{-i\omega x} - 2m^2\tilde{R}_m\}. \quad (21)$$

A similar procedure for constructing the central parts yields the equation

$$\frac{d}{dx}Z_{\cdot,m} = \frac{1}{2}i\omega Z_{\cdot,m} + \frac{1}{4l}\{m^2e^{i\omega x}Z_{\cdot,m-1} + (m+1)^2e^{-i\omega x}Z_{\cdot,m+1} - (m^2+(m+1)^2)Z_{\cdot,m}\} \quad (22)$$

(the dot stands for the index m', which plays the role of a parameter in (22)).

The terms with $\Delta m = \pm 1$ on the right-hand sides of (21) and (22) correspond to the joining of the diagrams e and f in Fig. 2, while the terms with $\Delta m = 0$ correspond to the sum of the terms contributed by the diagrams a--d in Fig. 2 (in the order (a, a') + (b, b') + (c, c') + d):

$$-2m^2\tilde{R}_m = \left\{-2\cdot 2m - 2\cdot\frac{2m(2m-1)}{2} - 2\cdot m(m+1) + (2m)^2\right\}\tilde{R}_m,$$

$$-(m^2+(m+1)^2)Z_{\cdot,m} = \left\{-2\cdot(2m+1) - 2\cdot\frac{(2m+1)2m}{2} - 2\cdot m^2 - (2m+1)^2\right\}Z_{\cdot,m}.$$

It is easy to see that for the diagrams with the central part $Z_{m'm}$, the left-hand and right-hand parts will have the forms $(\tilde{R}_{m'}, \tilde{R}_m)$, $(\tilde{R}_{m'}, \tilde{R}_{m+1})$, $(\tilde{R}_{m'+1}, \tilde{R}_m)$, and $(\tilde{R}_{m'+1}, \tilde{R}_{m+1})$ for the types Γ_{++}, Γ_{+-}, Γ_{-+}, and Γ_{--}, respectively, Therefore

$$\begin{pmatrix}\Gamma_{++} & \Gamma_{+-}\\ \Gamma_{-+} & \Gamma_{--}\end{pmatrix} = \sum_{m'=0}^{\infty}\sum_{m=0}^{\infty}\begin{pmatrix}\tilde{R}_{m'}Z_{m'm}\tilde{R}_m; & \tilde{R}_{m'}Z_{m'm}\tilde{R}_{m+1}\\ \tilde{R}_{m'+1}Z_{m'm}\tilde{R}_m; & \tilde{R}_{m'+1}Z_{m'm}\tilde{R}_{m+1}\end{pmatrix} \quad (23)$$

(the arguments x' and x have been dropped).

In an infinite system the quantities $\tilde{R}_m(x)$ have the form $\tilde{R}_m = e^{i\omega mx}R_m$, where we have for R_m from (21) the equations

$$i\nu R_m + m\{R_{m+1} + R_{m-1} - 2R_m\} = 0 \quad (m = 1, 2, \ldots); \quad R_0 = 1. \quad (24)$$

On account of (23), the quantities (20a) and (20b) depend bilinearly on the quantities

$$P_m^0 = 1/2(R_m + R_{m+1}), \quad P_m^1 = R_m - R_{m+1}. \quad (25)$$

Let us perform in the bilinear expression the summation over m' and introduce the notation

$$Q_m^a(\omega,k)=\frac{1}{4l}\sum_{m'=0}^{\infty}\int_{x'}^{\infty}e^{ik(x-x')}e^{+i\omega mx}Z_{m'm}(x',x)e^{-i\omega m'x'}P_{m'}^a. \quad (26)$$

From (22) and the condition $Z_{m'm}(x, x+0) = \delta_{m'm}$, we obtain for (26) the equations

$$i\nu(m+1/2)Q_m^a+(m+1)^2\{Q_{m+1}-Q_m^a\}-m^2\{Q_m^a-Q_{m-1}^a\}-i\varkappa Q_m^a+P_m^a=0. \quad (27)$$

The sought-for quantities can be expressed in terms of (25) and (26) in the form

$$2\pi\tilde{jj}^a(\omega,k)=4l\sum_{m=0}^{\infty}P_m^a(\omega)\{Q_m^a(\omega,k)+Q_m^a(\omega,-k)\} \quad (28)$$

(the term with $-k$ takes account of the contribution from the region $x < x'$). We recall that

$$\nu=4\omega l,\ \varkappa=4kl.$$

Thus, the summation of the most important diagrams has been reduced to the solution of the Eqs. (24) and (27) and the substitution of the solutions into (28).

2. ASYMPTOTIC SOLUTION OF THE EQUATIONS

1. Interpretation of the equations

Before proceeding to the solution of the equations, let us consider their interpretation. Let us denote by ψ_E the solutions of the Schrödinger equation $\psi'' + (E - V(x))\psi = 0$, by $z_E = \psi_E'/\psi_E$ their logarithmic derivatives, and by $A_E^{\pm}$ the running-wave amplitudes, which are determined from the equalities

$$\psi_E=A^+ + A^-;\ \psi_E'=ik_E(A^+ - A^-),$$

where $k_E = E^{1/2}$.

Let us define the coefficient r_E of reflection to the right by the formula

$$r_E=\frac{A_E^+}{A_E^-}=\frac{ik_E-z_E}{ik_E+z_E}. \quad (29)$$

By comparing the diagrams for r_E with the above-considered diagrams for the right-hand parts, we can see that the R_m's have the meaning of averages:

$$R_m=\bar{R}_me^{-i\omega mx}=M_V\{\rho^m\},\ \rho=r_{E+\omega}r_E^*. \quad (30)$$

The Eqs. (21) and (24) can be written in the form of the relations

$$\frac{\partial}{\partial x}M_V\{f(\rho_x)\}+M_V\left\{i\nu\rho_x\frac{\partial}{\partial\rho_x}f(\rho_x)+(1-\rho_x)^2\frac{\partial}{\partial\rho_x}(\rho_x f(\rho_x))\right\}=0, \quad (31)$$

$$i\nu M_V\{f(\rho)\}+M_V\left\{(1-\rho^2)\frac{\partial f}{\partial\rho}\right\}=0 \quad (31')$$

for the averages of an arbitrary analytic function of the quantities (30). In fact, substituting $f(\rho) = \rho^m$ into (31) and (31'), we obtain the Eqs. (21) and (24). Equation (31) is an approximate Fokker-Planck equation, which, for $l \gg 1$, is valid for intervals of the type

$$1 \ll \Delta x \ll l. \quad (32)$$

There are in the present problem exact Fokker-Planck type of equations that depend on two variables (thus, the first Halperin equation[3] is an equation for the distribution function of the quantities z_E and $z_{E+\omega}$). The approximate equations for the intervals (32) can also be derived (besides the diagrammatic method used by us) from the exact Fokker-Planck equations by expanding the latter in powers of the small parameter l^{-1}.

2. Solution of (24)

We shall, taking (31') into account, seek the solution of (24) in the form of a contour integral, where the weight function $w(\rho)$ should satisfy the equation conjugate to (31'), i.e.,

$$R_m=\int w(\rho)\rho^m d\rho,\ i\nu w-((1-\rho)^2w)'=0.$$

Solving the equation for $w(\rho)$ and selecting the contour and its parametrization, we can represent the solution in the following form:

$$R_m=-i\nu\int_0^{\infty}ds\,e^{i\nu s}\left(\frac{s}{s+1}\right)^m=-i\nu\int_0^{\infty}ds\,e^{i\nu s}(1+s^{-1})^{-m} \quad (33)$$

(other representations are possible). It is not difficult to verify (through integration by parts) that (33) is indeed a solution to (24). Notice that $R_m = 1$ when $\nu = 0$, i.e., $M_V\{|r_E|^{2m}\} = 1$. So we have

$$r_E=e^{i\theta_E};\ z_E=k_E\,\mathrm{tg}\,(1/2\theta_E), \quad (34)$$

where θ_E is the random angle (which is real when $\mathrm{Im}\,E = 0$). Therefore

$$\rho_{E,\omega}=\exp[i(\theta_{E+\omega}-\theta_E)]=\exp(i\delta),\quad R_m=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(im\delta)p(\delta)d\delta, \quad (35)$$

where $p(\delta) = \theta_{E+\omega} - \theta_E$. This function can be found from (33). Let us denote by

$$\mathcal{R}(\zeta)=\sum_{m=0}^{\infty}R_m\zeta^m \quad (36)$$

the generating function of the quantities R_m; then the function $p(\delta)$ is equal to

$$p(\delta)=\mathrm{Re}\,\mathcal{R}(e^{-i\delta})-1. \quad (37)$$

From (33), we can obtain for the function (36) the explicit representation

$$\mathcal{R}(\zeta)=\frac{1}{1-\zeta}\left(1-\zeta F\left(\frac{i\nu}{1-\zeta}\right)\right),\ F(z)=ze^{-z}\,\mathrm{Ei}(z) \quad (38)$$

(Ei(z) is an exponential integral function), and it can be verified that (37) satisfies all the necessary requirements.

The expression (33) is irregular with respect to the passage to the limits $m \to \infty$ and $\nu \to 0$. It is easy to see that for $\nu \ll 1$ and $m \gg 1$, (33) depends asymptotically on $m\nu$, i.e., it has a definite limit when

$$m\to\infty,\ \nu\to 0,\ p=-im\nu\to \text{ to a finite limit.} \quad (39)$$

In fact, for $\nu \ll 1$ and $m \gg 1$, large m are important in the interval (33) and, replacing $(1 + s^{-1})^{-m}$ by $\exp\{-m/s\}$, we obtain

$$R_m(\nu)\to R(p)=p\int_0^{\infty}e^{-ps}e^{-1/s}ds=2p^{1/2}K_1(2p^{1/2}) \quad (40)$$

($K_1(x)$ is the modified Bessel function).

The asymptotic form of (40) can be found directly from Eq. (24) by taking the limit as prescribed by (39). Replacing the differences by derivatives, we obtain

$$i\nu R_m+m\frac{d^2R}{dm^2}=0,\ \text{i.e.,}\ R(p)-p\frac{d^2R}{dp^2}=0. \quad (41)$$

The function (40) is a solution to Eq. (41) provided $R(0) = 1$ and $R(\infty) = 0$ (which follow from the limits of R_m for $\nu \to 0$ and $m \to \infty$). Indeed,

$$R(p)\underset{p\ll 1}{\sim}1+p(\ln p+2C-1),\quad R(p)=O(p^{1/4}\exp(-p^{1/2})),\quad p\to\infty, \quad (42)$$

where $C = 0.5772\ldots$ is the Euler constant.

3. Asymptotic solution of (27)

It is not possible to solve (27) in closed form. Thus, for the generating functions $Q(\zeta)$ (connected with the

Q_m's in the same way as (36) is connected with the R_m's) we have the equation

$$i\nu\left(\zeta\frac{dQ^a}{d\zeta}+\frac{1}{2}Q^a\right)+(1-\zeta)\frac{d}{d\zeta}\left\{\zeta\frac{d}{d\zeta}[(1-\zeta)Q^a]\right\}-i\varkappa Q^a+\mathcal{P}^a=0 \tag{43}$$

($\mathcal{P}^a(\zeta)$ are the generating functions for (25) and can be found from (38)). This equation is of a more complex type than the hypergeometric equation (four singular points: $\zeta = 0$, $\zeta = \infty$, and the two-fold degenerate point $\zeta = 1$), and its solutions cannot be obtained even in the form of contour integrals.

At the same time, the asymptotic forms of the Q_m's for $m \gg 1$ and $\nu \ll 1$ can be obtained in the form $Q^a_m = Q^a(-im\nu)$, where $Q^a(p)$ are solutions to the equations

$$-pQ^a(p)+\frac{d}{dp}\left(p^2\frac{dQ^a}{dp}\right)-i\varkappa Q^a(p)+P^a(p)=0, \tag{44}$$

the inhomogeneous terms being equal, according to (25) and (40), to

$$P^0(p)=R(p),\ P^1(p)=-dR_m/dm=i\nu R'(p). \tag{45}$$

Equations (44) are obtained by passage to the limit (39) from (27). To them must be added the boundary conditions $Q(\infty) = 0$ and the condition for $p \to 0$. To find the latter condition, we must solve Eqs. (27) for $Q_m - Q_{m-1}$, neglecting in (27) the terms $\sim\nu$; these solutions are applicable when $m\nu \ll 1$, and their asymptotic forms for $m \gg 1$, expressed in terms of $p = -i\nu m$, are matched with the asymptotic forms of $dQ(p)/dp$, which yields the conditions:

$$\frac{dQ^0}{dp}\sim p^{-1},\quad \frac{d}{dp}Q^1(p)\sim p^{-1}\ln p \text{ as } p\to 0.$$

The solutions of (44) for $\kappa = 0$ have the form

$$Q^0(p,\varkappa=0)=-R'(p)\sim-\ln p \text{ as } p\to 0, \tag{46a}$$

$$Q^1(p,\varkappa=0)=\frac{-i\nu}{2p}[\ln pR(p)+2CR(p)-R'(p)]\sim\frac{-i\nu}{2}(\ln p)^2 \tag{46b}$$

as $p \to 0$.

In fact, using the fact that the function $R(p)$ satisfies Eq. (41), we can easily verify that the expressions (46) are solutions to (44) (at $\kappa = 0$). The addition to (46) of solutions of the form $Cp^{-1}R(p)$ to the corresponding homogeneous equation violates the condition for $p \to 0$, so that the solutions (46) are unique.

The explicit solution of the Eqs. (44) for $\kappa \neq 0$ is, if at all possible, complicated. We shall consider only the asymptotic form of $Q^0(p, \kappa)$ for $\kappa \ll 1$. The function $Q^0(p, \kappa)$ is regular with respect to κ for $\kappa \to 0$ everywhere except in the vicinity of the point $p = 0$; in the latter region, however, the effect of small κ amounts to the replacement of $\ln p$ in the asymptotic form (46a) by $(i\kappa)^{-1}(p^{i\kappa} - 1)$. Choosing p_0 such that $|p_0| \ll 1$, but $|\kappa \ln p_0| \ll 1$, we can write with sufficient accuracy for what follows:

$$Q^0(p,\varkappa\ll 1)\approx\begin{cases}-R'(p), & p\geqslant p_0\\ (i\varkappa)^{-1}(1-p^{i\varkappa}), & p\leqslant p_0,\ p_0\ll 1,\ |\varkappa||\ln p_0|\ll 1.\end{cases} \tag{47}$$

4. Computation of the dominant terms of the expansion as $\nu \to 0$

Let us now consider the behavior of the quantities (28) for $\nu \to 0$. The dominant terms of the corresponding expansions are determined by the behavior of the P_m's and Q_m's for $m \gg 1$ and can be found with the aid of the above-obtained asymptotic forms (45)–(47).

In fact, let us consider the sum of the terms (28) for $m \geq M$. Let us choose $M \gg 1$ (but $M\nu \ll 1$), then we can replace the sum by an integral and use the asymptotic forms (45)–(47). We arrive then at the integrals

$$4l\int_M^\infty P^a(-i\nu m)Q^a(-i\nu m)\,dm. \tag{48}$$

The integrals (48) may be regarded as integrals along the straight lines $p = -i\nu m$ ($M \leq m \leq \infty$) in the plane of the complex variable p. Since $|\nu M| \ll 1$, the lower limit can be replaced by zero (the integrals then converge, which can be seen from (42) and (45)–(47)). Further, the integrand is analytic in the right-hand half-plane $\mathrm{Re}\ p > 0$, so that the integration contour can be shifted to the real axis. On doing this, we obtain for the quantities (28) the expressions

$$2\pi\widetilde{jj}^0(\omega,k)=\frac{4l}{-i\nu}\int_0^\infty R(p)\{Q^0(p,\varkappa)+Q^0(p,-\varkappa)\}\,dp, \tag{49a}$$

$$2\pi\widetilde{jj}^1(\omega,0)=4l(+i\nu)\int_0^\infty R'(p)[\ln pR(p)+2CR(p)-R'(p)]\frac{dp}{p}. \tag{49b}$$

(We are considering (49b) for only $k = 0$.)

Solving (27) in the region $m|\nu| \ll 1$ (with allowance for the matching with (46)), we can show that as $\nu \to 0$ and for a fixed m, the quantities P_m and Q_m have the following orders of magnitude: $P^0_m = O(1)$, $P^{(1)}_m = O(\nu \ln \nu)$, $Q^0_m = O(1)$, and $Q^1_m = O(\nu \ln^2 \nu)$, so that the sum of any finite number of terms from (28) will have a higher order of smallness as $\nu \to 0$ than (49). Thus, the expressions (49) give the dominant terms of the expansions as $\nu \to 0$ for the respective quantities.

Let us first consider (49a). We write the integral in (49a) in the form

$$\int_{p_0}^\infty -2RR'\,dp+2\,\mathrm{Re}\int_0^{p_0}\frac{1-p^{i\varkappa}}{i\varkappa}dp=1+2\mathrm{Re}\int_0^{p_0}e^{i\varkappa\ln p}e^{\ln p}\,d(-\ln p) \tag{50}$$

(the expression (47) has been used and the second integral to the right has been integrated by parts, the double substitution cancelling out exactly the dependence on p_0 of the first integral (on account of the choice of p_0)). Further, since $\nu = 4\omega l$, the factor in front of the integral is equal to $(-i\omega)^{-1}$. If the Fourier transform of a retarded function has the form $-A/i\omega$, then the limit of this function as $t \to \infty$ is equal to A, so that in view of (4) we obtain

$$\lim_{t\to\infty}M_V\{\mathrm{Sp}(\delta\varrho j^0(x,t)j^0(x_0))\}=p_\infty(x-x_0), \tag{51}$$

where, according to (50), the function $p_\infty(x)$ is such that

$$\int p_\infty(x)\,dx=1,\quad p_\infty(x)\underset{x\gg l}{\sim}(4l)^{-1}\exp\left\{-\frac{|x|}{4l}\right\}. \tag{52}$$

Let us now consider (49b). The integral can also be evaluated:

$$-8\int_0^\infty K_0(x)\left\{xK_1(x)\left[\ln\frac{x}{2}+C\right]+K_0(x)\right\}\frac{dx}{x}=-2(\pi^2-C^2). \tag{53}$$

Therefore, (49b) assumes the form

$$\widetilde{jj}^1(\omega)=-i\omega\alpha,\ \alpha=32(\pi^2-C^2)l^2\approx 316l^2. \tag{54}$$

5. Computation of $\mathrm{Re}\sigma$ as $\omega \to 0$

According to (9) and (10), Eq. (54) gives the reactive part of the conductivity as $\omega \to 0$. To compute the dissipative part, we must find the next terms of the expansion of $\widetilde{jj}^1(\omega)$. To do this in the discrete representation of (28) is not very convenient (although it is possible). It is better to go over to integrals, using the ap-

paratus of generating functions. For this purpose, let us substitute into (28) the integral representation for $P^1_m = R^1_m - R^1_{m+1}$ that follows from (33) and sum over m under the integral sign. We obtain (the second integral is obtained from the first by integration by parts)

$$2\pi\tilde{jj}^1(\omega) = 8l(-i\nu)\int_0^\infty e^{i\nu s}B(s)\,ds = 8l\int_0^\infty (e^{i\nu s}-1)\frac{dB}{ds}\,ds, \qquad (55)$$

where we have introduced for the function B(s) the notation:

$$B(s) = \frac{1}{s+1}\mathcal{Q}^1\left(\frac{s}{s+1}\right) = \frac{1}{s+1}\sum_{m=0}^{\infty} Q_m{}^1\left(\frac{s}{s+1}\right)^m. \qquad (56)$$

From (43), we obtain for this function the equation:

$$\frac{d}{ds}\left(s(s+1)\frac{dB}{ds}\right) + i\nu\left[s\frac{d}{ds}((s+1)B) + \frac{1}{2}B\right] + i\nu e^{-i\nu(s+1)}\,\mathrm{Ei}(i\nu(s+1)) = 0 \qquad (57)$$

(the expression for the generating function $\mathcal{P}^1(\zeta)$ following from (38) has been used). Equation (57) should be solved under the conditions $B(s) = O(1/s)$ as $s \to \infty$ and the condition of regularity of B(s) as $s \to 0$.

To the asymptotic forms of Q^1_m as $m \to \infty$ correspond the asymptotic forms of (56) as $s \to \infty$. Let us therefore introduce the variable $u = -i\nu s$ and rewrite (57) in the form

$$\frac{d}{du}\left(u^2\frac{dB}{du}\right) - u\frac{d}{du}(uB) + i\nu\left\{-\frac{d}{du}\left(u\frac{dB}{du}\right) + u\frac{dB}{du} + \frac{1}{2}B\right\} + i\nu e^{-i\nu}e^{u}\,\mathrm{Ei}(-u+i\nu) = 0. \qquad (58)$$

In order to determine the asymptotic forms with the aid of (58), we must find the boundary condition as $u \to 0$. For this purpose, let us use the already mentioned procedure for matching the asymptotic forms. To wit, we solve (57) in the region $|\nu s| \ll 1$, where the terms enclosed in the square brackets can be neglected, and use the expansion of $\mathrm{Ei}(z) \approx \ln(-z) + C$ for $|z| \ll 1$. We obtain

$$\frac{dB}{ds} \approx -i\nu\left\{\frac{C-1}{s+1} + \frac{\ln[(-i\nu)(s+1)]}{s+1} + \frac{\ln(s+1)}{s(s+1)}\right\} \quad \text{for } |\nu s| \ll 1. \qquad (59)$$

(Notice that there is no point in integrating (59): the unknown quantity B(0) will come in.)

Expanding (59) with respect to $s \gg 1$ and expressing the result in terms of the variable $u = -i\nu s$, we obtain

$$\frac{dB}{du} \sim -i\nu\left\{\frac{\ln u + C - 1}{u}\right\} + (i\nu)^2\left\{\frac{d}{du}\left[\frac{\ln u + C - 1}{u}\right] + \frac{1}{u^2}\ln\left(\frac{u}{-i\nu}\right)\right\} + \ldots \qquad (60)$$

This is the expression for the expansion of the boundary condition as $u \to 0$ for Eq. (58). It can be seen from (60) that the expansion of B(u) should have the form

$$B(u) = i\nu B_1(u) + (i\nu)^2\ln(-i\nu)\hat{B}(u) + (i\nu)^2 B_2(u) + \ldots, \qquad (61a)$$

yielding as $u \to 0$ the expression

$$\frac{dB_1}{du} \to -\left(\frac{\ln u + C - 1}{u}\right), \quad \frac{d\hat{B}}{du} \to -\frac{1}{u^2}, \quad \frac{dB_2}{du} \to \frac{2-C}{u^2} \ldots \qquad (61b)$$

Substituting the expansion (60) into (58) and expanding the nonhomogeneous term, we can find $B_1(u)$, $\hat{B}(u)$, and $B_2(u)$. In particular, $\hat{B}(u) = u^{-1}$.

Let us now split the second integral in (55) into two: over the region $0 \le s \le s_0$ and over $s_0 \le s < \infty$, where $s_0 \gg 1$, but $|\nu| s_0 \ll 1$. Let us make a change of variables $u = -i\nu s$ in the region $s \ge s_0$ and use (60); in the region $s < s_0$, let us set $e^{i\nu s} - 1 \approx i\nu s$ and use (59).

Evaluating the integral over the region $s < s_0$ explicitly and expressing its asymptotic form for $s_0 \gg 1$ in terms of $u_0 = -i\nu s_0$, we obtain

$$\frac{2\pi}{8l}\tilde{jj}_1(\nu) = -i\nu\int_{u_0}^\infty \frac{dB_1}{du}(e^{-u}-1)\,du - (i\nu)^2\ln(-i\nu)\int_{u_0}^\infty (e^{-u}-1)\frac{du}{u^2} + (i\nu)^2\int_{u_0}^\infty \frac{dB_2}{du}(e^{-u}-1)\,du + \ldots + i\nu u_0(\ln u_0 + C - 2) - (i\nu)^2\ln(-i\nu)(C - 2 - \ln u_0) + (i\nu)^2(C-2)\ln u_0 + \nu^2\ln^2(-i\nu). \qquad (62)$$

Here we can set $u_0 = 0$ (as can be seen from (61), the divergences of the integrals are exactly cancelled out by the terms with $\ln u_0$).

The first integral in (62) converges and yields precisely the dominant term of (49b) found earlier; the second integral is equal to $\ln u_0 + C - 1$, and we obtain

$$\tilde{jj}_E{}^1(\omega) = -i\omega\alpha + \frac{8l}{2\pi}\nu^2\{\ln^2(-i\nu) + (2C-3)\ln(-i\nu) + \mathrm{const}\} + o(\nu^2). \qquad (63)$$

The expansion of Re σ can be obtained from the second term if we take account of the fact that

$$\ln(-i\nu) = \ln|\nu| + i(\mathrm{Arg}\,\nu - \pi/2).$$

3. DISCUSSION OF THE RESULTS

Let us now consider the meaning of the results obtained in the light of the Mott concept. Let us begin with the expression (51).

The function standing under the "limit" sign on the left-hand side of (51) has the meaning of a transition probability. In fact, we can write

$$\langle F_1(\hat{x}_t)F_2(\hat{x})\rangle = n_0\int dE\,f(E)\iint F_1(x)F_2(x_0)M_\nu\{\mathrm{Sp}(\delta_E j^0(x,t)j^0(x_0))\}\,dx\,dx_0. \qquad (64)$$

($\hat{x}_t$ is a Heisenberg coordinate operator for the moment of time t, and $F_1(x)$ and $F_2(x)$ are arbitrary functions), whereas the classical transition probability $W_E(x_0; x; t)$ is given by the equality

$$\langle F_1(x_t)F_2(x)\rangle = n_0\int dE\,f(E)N(E)\iint F_1(x)F_2(x_0)W_E(x_0,x;t)\,dx\,dx_0 \qquad (65)$$

(true, it is possible to ascribe to the function in (64) the meaning of transition probability at energy E only for scales much larger than λ_E, but our functions are in fact asymptotic forms for scales of order $l \gg 1$).

Thus, the relation (51) implies that the particles emitted at t = 0 from the point x_0 do not, as $t \to \infty$, spread out along a straight line, but remain localized in the vicinity of x_0. The function $p_\infty(x - x_0)$ gives the density distribution in this limiting state. This conception of localization can be traced to Anderson[5] (this paper was entitled: "The Absence of Diffusion in Certain Disordered Lattices"). From the mathematical standpoint, the property (51) implies the absence of ergodicity in the energy shell. If we introduce the x_0-dependent averages defined by the formula

$$\langle\hat{A}\|x_0,E\rangle = \frac{1}{N(E)}\lim_{t\to\infty} M_\nu\{\mathrm{Sp}(\delta_E j^0(x,t)\hat{A})\}, \qquad (66)$$

then, on account of the identity $\int j^0(x)dx = 1$, we obtain for the averages (2) the expression

$$\langle A\rangle = n_0\int dE f(E)N(E)\int\langle\hat{A}\|x_0,E\rangle\,dx_0, \qquad (67)$$

which gives the expansion of the averages (2) in terms of the ergodic components.

It is natural to assume that the property (51) with $p_\infty(x) \neq 0$ is also typical in the general case for energies pertaining to the region of localized states,

whereas to the delocalized bands should correspond the case $p_\infty(x) = 0$. In three-dimensional systems, transitions between bands of both types can occur at certain energies. Since (51) corresponds to the spontaneous violation of translational invariance, these transitions are similar to second-order phase transitions, which are also connected with spontaneous symmetry breaking. In particular, the averages (66) are similar to the Bogolyubov quasi-averages.

It can be shown that (54) follows from (51), with

$$\alpha \sim \int x^2 p_\infty(x)\, dx.$$

The function $\sigma \cong -i\omega\alpha$ is typical for a dielectric and α may be regarded as the "polarizability" of the localized state (to the extent that we can at all speak of one-dimensional electrodynamics). Notice in connection with (63) that Mott[1] adduced arguments to show that $\mathrm{Re}\,\sigma \sim \omega^2$. These arguments have only logarithmic accuracy, and the function $\mathrm{Re}\,\sigma \sim \omega^2 \ln^2|\omega|$ is not at variance with them.

From the form of the summed diagrams, it is easy to understand why the kinetic equation is inapplicable in the one-dimensional case. This inapplicability is connected with the fact that in the present case the wave properties of the particles are important at $\lambda_E \to 0$ too. Let us consider, for example, the scattering of a wave propagating from left to right by three scattering centers 1, 2, and 3, such that $x_1 < x_2 < x_3$. Let the particle-wave, after reaching the center 2, split up into two particle-waves propagating along the paths 2-1-3-"left" and 2-3-1-2-"left" (to each change in direction corresponds a scattering event). Then it is easy to see that the two waves reflected to the left will have a nonzero phase difference for any distances between the centers, i.e., the effects of the scattering by such sets of three (and more complicated configurations) scattering centers will be integrated.

[1)] I am grateful to Yu. A. Bychkov for making his work available to me before publication.

[1] N. F. Mott, Adv. Phys. 16, 49 (1967) (Russ. Transl. in book form, Mir, 1969).
[2] B. I. Halperin, Adv. Chem. Phys. 13, 123 (1968).
[3] B. I. Halperin, Phys. Rev. A139, 104 (1968).
[4] Yu. A. Bychkov, Zh. Eksp. Teor. Fiz. 65, 427 (1973) [Sov. Phys.-JETP 38, 209 (1974)].
[5] P. W. Anderson, Phys. Rev. 109, 1942 (1958).

Translated by A. K. Agyei
126