Diffusion of magnetization in a Heisenberg ferromagnet above the Curie point

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The dynamics of long-wave magnetization fluctuations above the Curie point is usually described by the Van Hove diffusion equation. The accuracy of this description is discussed. It is shown that in reality the diffusion coefficient is a function of the energy ω and momentum $\mathbf{k}$. The major corrections to the static diffusion coefficient D_0 are calculated for small values of $\mathbf{k}$ and ω. These corrections are due to the presence, in the perturbation-theory expansion for the diffusion coefficient, of intermediate states which contain two diffusion poles, i.e., of two "quasiparticles" or "diffusions." The possibility of applying the results to other systems and in particular to a liquid-gas system near the critical point or to helium above the λ point is discussed. The effect of the corrections computed on diffusion is considered. It is shown that the diffusion law is of the form $\langle r^2 \rangle = 6 (D_0 t + A + B t^{-1/2})$ and the temperature dependences of A and B are determined.

1. INTRODUCTION

In this paper we investigate the dynamics of long-wavelength fluctuations in a Heisenberg ferromagnet in the critical region, above the Curie temperature. We elucidate the question of the consistency of the pattern of diffusion of the magnetization and the question of the role of corrections to the diffusion. We confine ourselves to the case of zero external magnetic field, neglect the magnetic dipole forces[1)], and consider for simplicity the case of cubic symmetry. Under these conditions, the dynamics of the critical phenomena are described by the Halperin-Hohenberg scaling law[1], according to which the dynamic susceptibility has the form

$$\chi(\mathbf{k}, \omega) = \chi(k) F\left(\frac{k}{\varkappa}, \frac{\omega}{T_c \tau^{\nu z}}\right),$$
$$\chi(k) = \frac{1}{k^{2-\eta}} f\left(\frac{k}{\varkappa}\right). \qquad (1)$$

Here $\chi(\mathbf{k})$ is the static susceptibility, $\tau = (T - T_c)T_c^{-1}$, κ is the inverse static correlation length: $R_c = \kappa^{-1} = a\tau^{-\nu}$, a is of the order of the lattice constant, $\nu \approx 2/3$ and z is the dynamic critical index. For ferromagnets, this index is not independent but is expressed in terms of the Fisher parameter η : $z = (5 - \eta)/2$. This relation was obtained in[1] and, as shown in[2], arises from the law of conservation of the total spin.

In the region of large momenta ($k \gtrsim \kappa$), the theory makes no prediction about the form of the function F; its only assertion is essentially that the characteristic energy has the form $\tilde{\omega}(\mathbf{k}) = T_c \tau^{\nu z} \psi(\mathbf{k}/\kappa)$ and $\tilde{\omega}(\mathbf{k}) \propto k^z$ for $k \gg \kappa$. On the other hand, in the limit of very long wavelengths, the dynamics of the magnetization fluctuations are described by the van Hove macroscopic diffusion equation[3]

$$\partial \mathbf{M} / \partial t = D_0 \nabla^2 \mathbf{M}. \qquad (2)$$

by virtue of which, for $k \to 0$,

$$\chi(\mathbf{k}, \omega) = \chi(0) D_0 k^2 / (-i\omega + D_0 k^2). \qquad (3)$$

Comparison of this formula with (1) leads to

$$D_0 = T_c a^2 \tau^{\nu(1-\eta)/2} d \approx T_c a^2 \tau^{1/2} d, \qquad (3a)$$

where $d \sim 1$. Below we shall write the second argument of the function F in (1) in the form $\omega/D_0\kappa^2$.

An experimental verification of dynamic scaling by means of neutron scattering was performed for iron and nickel by Minkiewicz, Collins, et al.[4,5], and led to somewhat unexpected results. In the region of large k ($k \gg \kappa$) good agreement with the scaling law was observed, viz., the index z was found to be equal to 2.46 ± 0.25 for nickel and 2.7 ± 0.3 for iron (we recall that in three-dimensional systems $\eta \ll 1$, and so $z \approx 2.5$). At the same time, in the hydrodynamic region (small k) the temperature dependence of the diffusion coefficient was found to differ sharply from the $\tau^{1/3}$ law: for nickel $D_0 \propto \tau^{0.51 \pm 0.05}$, and for iron $D_0 \propto \tau^{0.14 \pm 0.04}$. In addition, the measurements were performed in practice at fairly large momentum transfers ($k \lesssim \kappa/2$), and therefore the above discrepancy may be due to the approximate character of the formula (3), which was used to treat the experimental data. In this connection, the question arises of calculating corrections to the expression (3). It is to this question that the present paper is devoted.

Lying at the basis of the corresponding calculations is an idea widely used in high-energy physics (see, e.g., the works of Gribov[6,7]): if the one-particle Green function has a pole lying near the real axis, then allowance for this pole in the diagrams of the perturbation-theory series leads to closely positioned branch points, all such adjacent singular points having a "pole" origin.

In our case the "bare" pole is the diffusion pole in the expression (3). As we shall see, the singularities generated by this in the "hydrodynamic" regime (small k and ω) make a comparatively small contribution to $\chi(\mathbf{k}, \omega)$, and this can be taken into account easily. Outside the hydrodynamic regime, the contribution of these singularities is large and the simple diffusion picture ceases to be valid. Actually, for small k and ω we are concerned with an analog of the cutoff technique, and the actual small parameter is not the weakness of the interaction but the smallness of the corresponding phase-space volumes of the intermediate states. It should be noted here that the "cutoffs" under consideration lie at purely imaginary energies, this being due to the purely imaginary position of the pole in (3) or, in other words, to the purely imaginary energies of the quasi-particles in the intermediate states (for convenience, these quasi-particles may be called "diffusons").

The theory under consideration is a variant of the mode-mode coupling theory, differing principally in the form of the bare vertices from that proposed by Kawasaki[8]. We shall use the vertex parts from static scaling theory, which were introduced in papers by Migdal[9] and Polyakov[10], whereas Kawasaki used the coefficients of the Mori expansion[11].

The use of the static vertices can be justified by means of analytic continuation of the temperature dia-

grams of the perturbation-theory series. It should be noted that the continued temperature diagrams have already been used previously by Polyakov[12] to prove dynamic scaling. However, the character of the principal singularity of the Green function and the consequences stemming from this character were not analyzed in[12].

To conclude this Section, we note that the results we have obtained depend only to a slight degree on the specific properties of the Heisenberg ferromagnet and are therefore applicable, in particular, to the diffusion of density in a liquid-gas system near the critical point and in helium above the λ-point, if the role of the distant sound poles is neglected.

2. THE DYNAMIC DIFFUSION COEFFICIENT

Thus, we are interested in the dynamic susceptibility of a cubic Heisenberg ferromagnet in zero magnetic field above the Curie temperature. This susceptibility is related to the retarded spin Green function by the equality

$$\chi(\mathbf{k}, \omega) = (g\mu_0)^2 v_0^{-1} G(\mathbf{k}, \omega),$$

where V_0 is the volume of the unit cell and

$$G(\mathbf{k}, \omega) = i\int_0^\infty dt\, e^{i\omega t} \langle [S_{\mathbf{k}}^z(t), S_{-\mathbf{k}}(0)] \rangle,$$

$$S_{\mathbf{k}} = \frac{1}{\sqrt{N}} \sum_i e^{i\mathbf{k}\mathbf{R}_i} S_i. \tag{4}$$

As is well known (cf., e.g., the papers of Vaks, Larkin and Pikin[13,14]), this Green function is the analytic continuation of the temperature spin Green function from the upper half of the imaginary axis on to the real axis. To realize the program formulated in the Introduction, it seems necessary, at first sight, to write an expression for $G(\mathbf{k}, \omega)$ in terms of its irreducible part:

$$G(\mathbf{k}, \omega) = \Sigma(\mathbf{k}, \omega) / [1 - V_{\mathbf{k}}\Sigma(\mathbf{k}, \omega)] \tag{5}$$

($V_{\mathbf{k}}$ is the Fourier transform of the exchange integral), and then investigate $\Sigma(\mathbf{k}, \omega)$ by means of the diagram technique developed in[13,14] with allowance for the fact that the internal renormalized interaction $V(\mathbf{k}, \omega) = V_{\mathbf{k}} + V_{\mathbf{k}} G(\mathbf{k}, \omega) V_{\mathbf{k}}$ (cf.[13]) has the diffusion pole of (3). However, up to now it has not been possible, starting from the temperature diagrams, to extract from $\Sigma(\mathbf{k}, \omega)$ the part responsible for the diffusion and the correction terms. In addition, as is well known (cf., e.g., the paper by Kadanoff and Martin[15]), there exists a simple expression for the diffusion coefficient:

$$D_0 = \lim_{\substack{\mathbf{k}\to 0 \\ \omega\to 0}} k^{-2} \Phi_{L_{\mathbf{k}}^z L_{-\mathbf{k}}^z}(\omega) G^{-1}(\mathbf{k}, 0), \tag{6}$$

where $L_{\mathbf{k}} = dS_{\mathbf{k}}/dt$, and $\Phi_{L_{\mathbf{k}}L_{-\mathbf{k}}}(\omega)$ is the Kubo function of the operators $L_{\mathbf{k}}$ and $L_{-\mathbf{k}}$. We recall that the Kubo function of operators A and B is related to the corresponding generalized susceptibilities by the equality

$$\Phi_{AB} = \frac{1}{i\omega}[\chi_{AB}(\omega) - \chi_{AB}(0)]. \tag{7}$$

The question arises as to whether it is possible in formula (6) to remove the constraint associated with taking the limit $\mathbf{k}, \omega \to 0$, and thereby introduce a dynamical diffusion coefficient $D(\mathbf{k}, \omega)$. Recently, Schwabl and Michel[16] have proposed a general method for deriving hydrodynamic equations. Using their procedure (see the Appendix), one can obtain an expression for the dynamic susceptibility in the form

$$G(\mathbf{k}, \omega) = \frac{G(\mathbf{k}, 0) D(\mathbf{k}, \omega) k^2}{-i\omega + D(\mathbf{k}, \omega) k^2},$$

$$D(\mathbf{k}, \omega) k^2 = \Phi_{L_{\mathbf{k}}^z L_{-\mathbf{k}}^z} G^{-1}(\mathbf{k}, 0) \{1 + \Phi_{S_{\mathbf{k}}^z L_{-\mathbf{k}}^z}(\omega) G^{-1}(\mathbf{k}, 0)\}^{-1}. \tag{8}$$

These formulas are general in character. It must be emphasized that the expression for $D(\mathbf{k}, \omega)$ differs from that obtained from (6) by removing the passage to the limit. Formula (6) is a particular case of the expression for the damping coefficient of a macroscopic quantity in terms of the kinetic coefficients, based on the assumption of linearity of the relaxational forces (cf. the book by Landau and Lifshitz[17]). Formula (8) is exact and takes into account the nonlinear character of these forces. In the case of purely exchange interaction, $L_{\mathbf{k}}^z \propto k$; the Kubo function in the denominator vanishes at $k = 0$ and we arrive at the expression (6). However, in studying the dispersion of the diffusion coefficient, i.e., its dependence on $\mathbf{k}$ and ω, we cannot, generally speaking, neglect this function in the denominator.

The function $\chi_{L_{\mathbf{k}}L_{-\mathbf{k}}}(\omega)$ is completely analogous in its properties to an ordinary susceptibility: its real part is an even function of ω and its imaginary part is an odd function of ω, and $\chi_{L_{\mathbf{k}}L_{-\mathbf{k}}}(0) \neq 0$. Furthermore, if we represent $\chi_{L_{\mathbf{k}}L_{-\mathbf{k}}}$ in the form of a sum over intermediate states (cf., e.g.,[18]), then $\mathrm{Im}\chi_{LL}$ coincides with the so-called absorptive part of this function, i.e., the part which contains an energy δ-function in the summand. The properties of the function $\chi_{SL}(\omega)$ are different, this being connected with the different behavior of the operators S and L under time reversal: the absorptive part of χ_{SL} is real and is an even function of ω, and the dispersive part is odd and imaginary; in addition, $\chi_{SL}(0) = 0$ and $\chi_{SL}(\omega) = -\chi_{LS}(\omega)$. All these properties are easily obtained from the expansions over the intermediate states.

Using the usual expression for the exchange interaction energy, it is not difficult to obtain the formulas

$$\chi_{L_{\mathbf{k}}^z L_{-\mathbf{k}}^z} = \frac{(a^2 T_c \gamma)^2}{6} \int_0^\infty dt\, e^{i\omega t} \frac{1}{N} \sum_{\mathbf{k}_1\mathbf{k}_2} (\mathbf{k}\mathbf{k}_1)(\mathbf{k}\mathbf{k}_2) \times i\langle [(S^o_{\mathbf{k}_1+\mathbf{k}/2} S^\mu_{-\mathbf{k}_1+\mathbf{k}/2})_t, (S^o_{-\mathbf{k}_2-\mathbf{k}/2} S^\mu_{\mathbf{k}_2-\mathbf{k}/2})_0] \rangle, \tag{9a}$$

$$\chi_{S_{\mathbf{k}}^z L_{-\mathbf{k}}^z} = \frac{a^2 T_c \gamma}{6} \int_0^\infty dt\, e^{i\omega t} \frac{1}{\sqrt{N}} \sum_{\mathbf{k}_1} (\mathbf{k}\mathbf{k}_1)\, \varepsilon_{\alpha\mu\lambda} \times i\langle [S^\alpha_{\mathbf{k}}(t), (S^o_{\mathbf{k}_1-\mathbf{k}/2} S^\lambda_{-\mathbf{k}_1-\mathbf{k}/2})_0] \rangle. \tag{9b}$$

In writing these expressions, we have confined ourselves to the lowest terms of the expansion in powers of ka and have put $\nabla_{\mathbf{k}_1} V_{\mathbf{k}_1} \approx -\mathbf{k}_1 T_c a^2 \gamma$, where $\gamma \sim 1$. In the following, we shall be interested in the correction terms of order $(kR_c)^2$ and $(kR_c)^3$; we shall consistently neglect the $(ka)^2$ and $(ka)^3$ corrections, since $R_c \gg a$. The functions (9a) and (9b) are the analytic continuations on to the real axis of the corresponding Matsubara functions, for which we can use a diagram technique[13,14]. The appropriate diagrams are depicted in Fig. 1, where the solid lines are the renormalized interaction $V(\mathbf{k}, \omega)$, the empty circles are the bare vertices, and the shaded circles are the exact vertices. The properties of these vertices are discussed below.

Near T_c, the effective interaction $V(\mathbf{k}, \omega)$ can be replaced by $V_0^2 G(\mathbf{k}, \omega)$, since in this region

$$G(\mathbf{k}, 0) \approx \frac{A}{T_c \tau^{\nu(2-\eta)} (k^2/\varkappa^2 + 1)}, \tag{10}$$

where $A \sim 1$ and $\tau \ll 1$, and therefore $G(\mathbf{k}, 0) \gg T_c^{-1} \propto V_0^{-1}$. In the following, we shall include the factors V_0 in the vertices and associate $G(\mathbf{k}, \omega)$ with the solid lines.

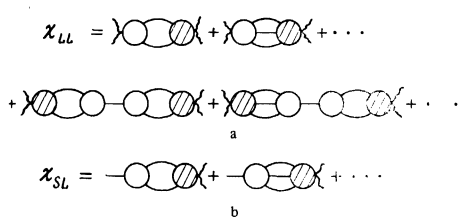

FIG. 1

The diagrams of Fig. 1b for χ_{SL} start from an external solid line, and therefore $\chi_{SL}(\mathbf{k}, \omega) = G(\mathbf{k}, \omega)R(\mathbf{k}, \omega)$, where R is the set of all diagrams that cannot be separated into two parts by cutting one line. Further, among the diagrams for χ_{LL} are some which contain one solid line in an intermediate state; we shall call them single-particle diagrams. Clearly, the set of all diagrams standing to the right of a single-particle state is equal to $R(\mathbf{k}, \omega)$, while those to the left, correspondingly, are equal to $\chi_{LS}G^{-1}$. We have already noted that $\chi_{LS} = -\chi_{SL}$, and $\chi_{SL}(\mathbf{k}, 0) = 0$. Therefore, taking (7) into account, we obtain for $D(\mathbf{k}, \omega)$ the equation

$$D(\mathbf{k},\omega)k^2 = \left\{G^{-1}(\mathbf{k},0)K(\mathbf{k},\omega) + \frac{R^2(\mathbf{k},\omega)D(\mathbf{k},\omega)k^2}{i\omega[-i\omega + D(\mathbf{k},\omega)k^2]}\right\} \times \left\{1 - \frac{R(\mathbf{k},\omega)D(\mathbf{k},\omega)k^2}{i\omega[-i\omega + D(\mathbf{k},\omega)k^2]}\right\}^{-1}, \quad (11)$$

where $K(\mathbf{k}, \omega) = (i\omega)^{-1}[N(\mathbf{k}, \omega) - N(\mathbf{k}, 0)]$ and N is the set of all diagrams for χ_{LL} without poles. This equation is valid both for $k \gtrsim \kappa$ and for $k \ll \kappa$. In the limit $k \to 0$, the terms with R in (11) vanish, since $R \propto k^2$. For $k \neq 0$, R needs to be taken into account only if this quantity has an order of magnitude corresponding to the scaling law:

$$R(\mathbf{k},\omega) = i\omega\left(\frac{k}{\varkappa}\right)^2 r\left(\frac{k}{\varkappa}, \frac{\omega}{D_0\varkappa^2}\right),$$

where the factor $i\omega$ allows for the facts that $R(\mathbf{k}, 0) = 0$ and that the dispersive part of R is imaginary. We cannot ascertain accurately the order of magnitude of R. A calculation of R in the self-consistent field approximation[13,14] for non-zero discrete imaginary frequencies gives the estimate

$$R \sim \left(\frac{k}{\varkappa_0}\right) ka \ll \left(\frac{k}{\varkappa_0}\right)^2,$$

where κ_0 is the quantity κ in this theory. We therefore believe that $R \sim (k/\kappa)^2\kappa a$ in the critical region also, and we shall neglect this quantity in the following treatment. Thus, we assume that

$$D(\mathbf{k},\omega) = k^{-2}G^{-1}(\mathbf{k},0)K(\mathbf{k},\omega). \quad (12)$$

3. CORRECTIONS TO THE DIFFUSION COEFFICIENT

Thus, we must investigate the non-single-particle diagrams of Fig. 1a and, in accordance with the program outlined in the Introduction, determine the behavior of $D(\mathbf{k}, \omega)$ in the region of small k and ω, i.e., when $k \ll \kappa$ and $\omega \ll D_0\kappa^2$. As we shall see, in this region the main role is played by two-particle intermediate states, and we shall begin with these. From the diagrams without poles in Fig. 1a, we choose any diagram containing at least one two-particle intermediate state. The corresponding contribution to χ_{LL} can be written in the form

$$\delta\chi_{LL}^{(2)}(\mathbf{k}, i\omega_n) = (aT_c\gamma)^2(ak)^2\frac{T\upsilon_0}{2(2\pi)^3}\sum_{\omega_1\omega_2}\int d\mathbf{q}\frac{1}{T}\delta_{\omega_1+\omega_2,\omega}\Lambda^{(1)}(\mathbf{k},\mathbf{q},i\omega,i\omega_1,i\omega_2) \times G\left(\mathbf{q}+\frac{\mathbf{k}}{2}, i\omega_1\right)G\left(-\mathbf{q}+\frac{\mathbf{k}}{2}, i\omega_2\right)\Lambda^{(2)}(\mathbf{k},\mathbf{q},i\omega_1,i\omega_2,i\omega), \quad (13)$$

where the factor 1/2 takes into account the indistinguishability of the particles in the intermediate state, and $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are the parts standing to the right and to the left of the selected state in the vector vertices describing the creation and annihilation of two particles. It should be emphasized that $\Lambda^{(1,2)}$ can themselves have two-particle intermediate states. We have included all factors of order unity, apart from those written out explicitly, in the definition of the vertices.

From the point of view of the energy dependence, the vertices $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are three-point functions, and, as such, are analytic functions of all three frequencies ω, ω_1 and ω_2, in each of which they have a cut along the real axis; they have no other singularities in the physical region of the variables $\omega_{1,2}$ and ω. These properties of the three-point functions, and also of vertices with a larger number of points, are discussed in a paper by the author[19]. Using these properties, it is not difficult to perform the summation over ω_1 and ω_2 and the analytic continuation in ω. As a result, we obtain

$$\delta\chi_{LL}^{(2)}(\mathbf{k},\omega+i\delta) = (aT_c\gamma)^2(ka)^2\frac{\upsilon_0}{2(2\pi)^3}\int d\mathbf{q}\frac{1}{\pi^2}\int dx_1\,dx_2 \times \operatorname{sh}\left(\frac{x_1+x_2}{2T}\right)\left[\operatorname{sh}\left(\frac{x_1}{2T}\right)\operatorname{sh}\left(\frac{x_2}{2T}\right)(x_1+x_2-\omega-i\delta)\right]^{-1} \times \Delta_{x_1}\Delta_{x_2}\left[G\left(-\frac{\mathbf{k}}{2}+\mathbf{q}, x_1\right)G\left(\frac{\mathbf{k}}{2}-\mathbf{q}, x_2\right)\Lambda^{(1)}(\mathbf{k},\mathbf{q},\omega,x_1,x_2)\Lambda^{(2)}(\mathbf{k},\mathbf{q},x_1,x_2,\omega)\right]. \quad (14)$$

Here Δ_X denotes the discontinuity at the cut:

$$\Delta_x f(x) = [f(x+i\delta) - f(x-i\delta)]/2i,$$

and, in particular, $\Delta_X G(\mathbf{k}, x) = \operatorname{Im} G(\mathbf{k}, x)$. The combination of hyperbolic sines is the statistical weight of the intermediate state, written symmetrically (there is a formula of the type (14) in the Appendix of the paper[19]). Near T_c, all the interesting dynamic phenomena develop in the region of energies that are small compared with the temperature, and in the following we shall therefore replace the hyperbolic sines by their arguments. We can separate out from the vertices $\Lambda^{(1,2)}$ a static part $\Lambda^{(1,2)}(\mathbf{k}, \mathbf{q}) = \Lambda^{(1,2)}(\mathbf{k}, \mathbf{q}, 0, 0, 0)$ and a dynamic correction which vanishes at zero energy. First we shall consider the contribution to $\delta\chi_{LL}^{(2)}$ due to the static vertices. Choosing the two-particle partitions in all possible ways, separating out the static vertices each time, and combining all the expressions thus obtained, we arrive at a formula differing from (14) by the replacement of $\Lambda^{(1)}$ and $\Lambda^{(2)}$ by the exact static vertices. As a result, the corresponding contribution to K is found to be equal to

$$\delta K_2(\mathbf{k},\omega+i\delta) = (aT_c\gamma)^2(ka)^2\frac{T_c\upsilon_0}{(2\pi)^3 i}\int d\mathbf{q}\,\Lambda_2(\mathbf{kq})\Lambda_2(\mathbf{q},\mathbf{k}) \times \frac{1}{\pi^2}\int\frac{dx_1\,dx_2}{x_1x_2(x_1+x_2-\omega-i\delta)}\operatorname{Im}G\left(\frac{\mathbf{k}}{2}+\mathbf{q}, x_1\right)\operatorname{Im}G\left(\frac{\mathbf{k}}{2}-\mathbf{q}, x_2\right). \quad (15)$$

Clearly, the quantities Λ_2 can be represented in the form of the aggregate of the diagrams of the static theory[9,10], the sum of which should give the vector vertex of static scaling theory. On the other hand, the static Green functions $G(\mathbf{k}, 0)$ "do not know" the direction of the momentum; in other words, the lines corresponding to them in the diagrams do not possess direction. Therefore, the aggregate of diagrams for $\Lambda_2(\mathbf{k}, \mathbf{q})$ describes both the creation of two particles, with momenta $\mathbf{k}/2 + \mathbf{q}$ and $\mathbf{k}/2 - \mathbf{q}$, and the scattering of a particle with momentum $\mathbf{k}/2 + \mathbf{q}$ by a static vector field with transfer of momentum $\mathbf{k}$ to this field. But for such a scattering process, in the limit $k \to 0$ the Ward identity holds[2]:

$$\partial G^{-1}/\partial \mathbf{q} = a^2 T_c c \Lambda_2(0, \mathbf{q}). \qquad (16)$$

Here we have separated out the dimensional factor a^2T_c, since in our normalization G^{-1} has the dimensions of energy, and Λ_2 of momentum; the constant $c \sim 1$ is also connected with the normalization and its value is unimportant for what follows.

First we shall estimate the contribution of the part K under consideration to the static diffusion coefficient. Substituting (16) into (15), and taking into account (12) and (1) and the fact that $\mathrm{Im}G(\mathbf{q}, \omega)$ is an odd function of the energy, we obtain

$$\delta D_{02} = CT_c \frac{v_0}{(2\pi)^3} \int d\mathbf{q} \left(\frac{\partial G^{-1}(\mathbf{q}, 0)}{\partial \mathbf{q}}\right)^2 G^2(\mathbf{q}, 0) \times \frac{1}{\pi} \int \frac{dx}{x^2} \left[\mathrm{Im}\, F\left(\frac{q}{\varkappa}, \frac{x}{D_0 x^2}\right)\right]^2 G_0^{-1}(0,0), \qquad (17)$$

where $C \sim 1$. In this expression, the integrals over x and $\mathbf{q}$ converge well. In fact, for small x, $\mathrm{Im}F \sim x$, and for large x, ImF falls off, since $\int dx x^{-1}\,\mathrm{Im}F = \pi$. Therefore, the integral over x converges. Furthermore, for $q \gg \kappa$ the function ImF does not depend on the temperature, i.e., on κ and D_0: for such q we are effectively at the Curie point itself. This means that for large q the integral over x is proportional to $q^{-5(1-\eta)/2}$. This decrease of this integral with increasing q ensures the good convergence of the integral over $\mathbf{q}$, since for large q we shall have $G(\mathbf{q}, 0) \sim q^{-(2-\eta)}$. As a result, for δD_{02} we obtain the estimate

$$\delta D_{02} \sim \frac{T_c^2}{D_0} (\varkappa a)^{1-\eta} a^4, \qquad (18)$$

which agrees with the expression (3a) for the diffusion coefficient in the dynamic scaling theory. It is interesting to note that, if it were possible to neglect all contributions to K apart from (15) and calculate the latter by means of (16) and (3), in the limit $\mathbf{k}, \omega = 0$ we would obtain an expression for the diffusion coefficient D_0 satisfying the scaling hypothesis (3a).

We now study the expression (15) for finite but small $\mathbf{k}$ and ω. As is well known[9,10], as functions of the momenta $\mathbf{p}_i$ all the quantities of the static theory have singularities at the points $p_i^2 = -n^2\kappa^2$, where n are integers, with $n \geq 1$, and therefore, for $|p_i^2| \ll \kappa^2$, they can be expanded in a series in powers of p_i^2/κ^2. This means that, for $\omega = 0$,

$$\delta D(\mathbf{k}, 0) = \delta D_0 \left(1 + \alpha \frac{k^2}{\varkappa^2} + \ldots\right), \qquad (19)$$

where $\alpha \sim 1$ is a coefficient which cannot be calculated at the present time.

We proceed to analyze the energy dependence of δK_2. In order to understand the essence of the problem, we first substitute expression (3) into (15). Then the integrals over $\mathbf{x}_1$ and $\mathbf{x}_2$ are easily taken and we obtain

$$\delta K_2(\mathbf{k}, \omega) = (aT_c\gamma)^2 (ka) \frac{T_c v_0}{(2\pi)^3} \times \int d\mathbf{q} \frac{\Lambda_2(\mathbf{k}, \mathbf{q}) \Lambda_2(\mathbf{q}, \mathbf{k}) G(\mathbf{q} + \mathbf{k}/2, 0) G(-\mathbf{q} + \mathbf{k}/2, 0)}{2D_0 q^2 + D_0 k^2/2 - i\omega}. \qquad (20)$$

It is clear from the form of the denominator that δK_2 has a singular point $\omega = -iD_0k^2/2$ in the ω plane; a cut runs downwards from this point to $-i\infty$ along the imaginary axis (cf. Fig. 2). Furthermore, according to (16) and (1), for small q we shall have $\Lambda_2(0, \mathbf{q}) \propto \kappa^{-\eta}\mathbf{q}$. To conserve the static-scaling properties, it is necessary that

$$\Lambda_2(\mathbf{k}, \mathbf{q}) = (\varkappa a)^{-\eta} (C_1 \mathbf{q} + C_2 \mathbf{k}). \qquad (21)$$

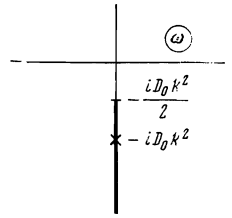

FIG. 2

for small $\mathbf{k}$ and $\mathbf{q}$. This form of Λ_2 leads to the following expression for δK_2 in the region of small $\mathbf{k}$ and ω:

$$\delta K_2(\mathbf{k}, \omega) = \delta K_{02} \left\{1 + \alpha' \frac{k^2}{\varkappa^2} + \beta \frac{i\omega}{D_0\varkappa^2} + \lambda \left[-C_2^2 \left(\frac{k}{\varkappa}\right)^2 \times \left[\frac{k^2}{2\varkappa^2} - \frac{i\omega}{D_0\varkappa^2}\right]^{1/2} + \frac{1}{2} C_1^2 \left(\frac{k^2}{2\varkappa^2} - \frac{i\omega}{D_0\varkappa^2}\right)^{3/2}\right]\right\}. \qquad (22)$$

Here α', β and λ are constants of order unity, and $\lambda > 0$. In this expression there are terms of two types. First of all there are the regular terms proportional to k^2 and ω. Their appearance is associated with the integration in (20) over the region of large momenta $q \sim \kappa$, and therefore the coefficients α' and β should change if, instead of (3), we substitute the exact formula (1) into (15). Next there are singular terms which are of order k^3 for $k^2 \sim \omega$. Their appearance is associated with the integration over the region of small momenta, and $\delta K_{02}\lambda$ is expressed in terms of the static diffusion coefficient D_0, the static susceptibility $G(0, 0)$ and κ:

$$\delta K_{02}\lambda = \frac{T_c a^2 \gamma^2}{8\pi\sqrt{2} D_0} (ka)^2 (T_c G(0,0))^2 v_0 \varkappa^3 (\varkappa a)^{-2\eta}. \qquad (23)$$

We now show that formulas (22) and (23) determine the behavior of the function $K(\mathbf{k}, \omega)$ at small k and ω. We first analyze the role of the many-particle intermediate states, beginning with states containing an even number of intermediate particles. Proceeding as above, it is not difficult to write for the n-particle intemediate state a formula containing the right and left exact static vector vertices:

$$\delta K_n \propto \frac{(ka)^2}{i} \int d\mathbf{q}_1 \ldots d\mathbf{q}_n \Lambda_n(\mathbf{k}\mathbf{q}_i) \Lambda_n(\mathbf{q}_i\mathbf{k}) \delta(\mathbf{q}_1 + \mathbf{q}_2 + \ldots + \mathbf{q}_n - \mathbf{k}) \times \frac{1}{\pi^n} \int \frac{dx_1 \ldots dx_n\, \mathrm{Im}\, G(\mathbf{q}_1 x_1) \ldots \mathrm{Im}\, G(\mathbf{q}_n x_n)}{x_1 \ldots x_n (x_1 + x_2 + \ldots + x_n - \omega - i\delta)}. \qquad (24)$$

The vertices Λ_n are symmetric functions of the momenta of the intermediate particles. The character of their dependence on $\mathbf{q}_i$ and $\mathbf{k}$ can be determined by starting from the same considerations as for Λ_2. For $\mathbf{k} = 0$, generalized Ward identities analogous to (16) hold for Λ_n; by virtue of these, Λ_n can be expressed in terms of a sum of derivatives of the ordinary n-particle vertices Γ_n of static scaling theory. A "dimensional" estimate exists for the latter[9,10]: $\Gamma_n \propto q_i^{3-n(1+\eta)/2}$ and therefore the vertex Λ_n behaves at small $\mathbf{q}_i$ and $\mathbf{k}$ like a product of $\kappa^{1-n(1+\eta)/2}$ with a linear combination of the momenta $\mathbf{q}_i$ and $\mathbf{k}$, and for large $\mathbf{q}_i$ falls off like $q_m^{2-n(1+\eta)/2}$, where $\mathbf{q}_m$ is the largest of the momenta $\mathbf{q}_i$. This behavior of the vertices ensures the convergence of the expression (24). As a result, for $\omega, \mathbf{k} \to 0$, the estimate obtained for the contribution to D_0 corresponds to the scaling law (3a). Furthermore, an expansion in powers of k^2/κ^2 and $i\omega/D_0\kappa^2$ arises from the region of large $\mathbf{q}_i$ in (24), while the region of small $\mathbf{q}_i$ leads to the appearance of an irregular term having the form

$$P_{3n/2-2}\left(\frac{k^2}{\varkappa^2}, \frac{i\omega}{D_0\varkappa^2}\right) \left(\frac{k^2}{n\varkappa^2} - \frac{i\omega}{D_0\varkappa^2}\right)^{1/2}, \qquad (25)$$

in which the factor in front of the square root is a polynomial of order $3n/2 - 2$, the coefficients of which are completely determined by the behavior of the vertices Λ_n in the region of small momenta and by the quantities $G(0, 0)$, in complete agreement with (22) and (23). Furthermore, in (25) the singularity in ω is situated at the point $\omega_n = -iD_0k^2/n$, from which a cut runs to $-i\infty$. This position of the singularity is analogous to the well-known formula for the so-called Reggeon branch points in the theory of complex angular momenta[6,7], and is established in the same way. Thus, the greater is n, the nearer to the real axis is the singularity. In other words, the singularities cover the pole of (3) which generated them and bunch up toward the coordinate origin.

In the region of k and ω under consideration, all these singularities are small corrections, and therefore the phenomena associated with them (the drift of the pole toward the new cut, and so on[6,7]) are of no special interest. However, in the region $\omega \sim D_0\kappa^2$ and $k \sim \kappa$, the picture becomes very complicated and scarcely amenable to detailed analysis at the present time (cf.[6,7]).

It is necessary now to discuss the consequences of taking into account the energy dependence of the vertices. First of all, we note that the total contribution of the two-particle states can be written in unitary form[12,19]:

$$i\omega\delta K_2(\mathbf{k},\omega) = (aT_c\gamma)^2(ka)^2\frac{T_c v_0}{(2\pi)^3}\int d\mathbf{q}\,\frac{1}{\pi^2}\int dx_1\,dx_2\Lambda_2^+(\mathbf{k},\mathbf{q},x_1,x_2,\omega)\frac{\omega}{x_1x_2}$$
$$\times\frac{\operatorname{Im}G(\mathbf{k}/2+\mathbf{q},x_1)\operatorname{Im}G(\mathbf{k}/2-\mathbf{q},x_2)}{x_1+x_2-\omega-i\delta}\Lambda_2(\mathbf{k},\mathbf{q},x_1,x_2,\omega), \qquad (26)$$

where Λ_2 is the exact vector vertex and Λ_2^+ differs from Λ_2 in the signs of the imaginary parts of the energies. The discontinuity of this expression in ω reduces to the replacement of the denominator by a δ-function and has the form of the usual unitarity condition with a temperature statistical weight of the intermediate state[19]. The expression (22) follows from (26) when two conditions are fulfilled: first, the vertex Λ_2 should depend on $x_{1,2}/D_0\kappa^2$ and $\omega/D_0\kappa^2$ or, in other words, should have a form corresponding to the dynamic scaling law, and, secondly, the integration in (26) should be performed over the "scaling" region, i.e., the main contribution to the integral should be given by $q \leq \kappa$ and $x_{1,2} \lesssim D_0\kappa^2$.

The first of these conditions is natural; it is necessary for the consistency of the entire concept of dynamic scaling[12]. Earlier, in the calculation of δK with static vertices, we saw that the second condition holds. Clearly, for its fulfilment it is now sufficient that the vertex Λ_2 not be an increasing function of x_i in the region $x_{1,2} \sim D_0\kappa^2$. The following consideration convinces one that this is indeed the case: if the integration region $x_{1,2} >> D_0\kappa^2$ were important in (26), then it would be possible to have a dynamic scaling theory with the critical index z expressed in terms of the static index η. In the case of intermediate states with $n > 2$, no such simple unitary expression for δK_n with Λ_n^+ and Λ_n exists (this question is analyzed in detail in[20]). However, all that has been said about two-particle states can be carried over to $n > 2$, if we make use of a somewhat different form of the unitarity condition[12], in which the vertex standing to the left is not exact. We shall not consider this question in more detail.

We turn now to the question of the states with odd n. In the static theory, there are no odd vertices[21] [3]. There should therefore be an energy selection rule, by virtue of which the odd vertices vanish in the limit of zero energies. We shall assume that the odd vertices consist of a series of terms, each of which is a product of $\omega_i/D_0\kappa^2$ with a function of the energies ω_i that is finite at zero. Using the example of the three-point function, it is not difficult to show that such a selection rule is compatible with the general analytic structure of the vertex as analyzed in[19], whereas even powers of ω_i are not permissible. As a result, we obtain the following expression for the irregular contribution to K from the odd states:

$$P_{(3n+1)/2}\left(\frac{k^2}{\varkappa^2},\frac{i\omega}{D_0\varkappa^2}\right)\ln\left(\frac{k^2}{n\varkappa^2}-\frac{i\omega}{D_0\varkappa^2}\right), \qquad (27)$$

where the factor in front of the logarithm is a polynomial of order $(3n + 1)/2$. With regard to the regular contribution, we can repeat word-for-word what was said above about the even states.

The formulas (25) and (27) permit a simple interpretation. The pole of (3) lies on the imaginary axis. If we replace $i\omega$ by ω, the "diffusons" become ordinary quasi-particles. After this, the singular terms can be calculated simply with the aid of the unitarity conditions describing the decay of one particle into n particles. Then the residue Dq^2 of the Green function and the factor $1/x$ arising from the temperature statistical weight cancel, so that these conditions have the form corresponding to zero temperature[20]. As a result, the power dependence in (25) and (27) is determined by the phase-space volume and by the structure of the vertices, and the logarithm in (27) is determined by the requirement of analyticity.

We now give the final expression for the dynamic diffusion coefficient for small k and ω:

$$D(\mathbf{k},\omega) = D_0\left\{1+\alpha\frac{k^2}{\varkappa^2}+\beta\frac{i\omega}{D_0\varkappa^2}\right. \qquad (28)$$
$$\left.+\lambda\left[-C_2\frac{k^2}{\varkappa^2}+\frac{C_1^2}{2}\left(\frac{k^2}{2\varkappa^2}-\frac{i\omega}{D_0\varkappa^2}\right)\right]\left(\frac{k^2}{2\varkappa^2}-\frac{i\omega}{D_0\varkappa^2}\right)^{1/2}\right\}.$$

We also give a formula for the quantity determining the neutron-scattering cross section:

$$\frac{\operatorname{Im}G(\mathbf{k},\omega)}{\omega}\approx\frac{G(\mathbf{k},0)D_0(1+\alpha k^2/\varkappa^2)}{\omega^2(1-2\beta k^2/\varkappa^2)+D_0^2k^4(1+2\alpha k^2/\varkappa^2)}. \qquad (29)$$

Here, we have neglected for simplicity the small corrections arising from the irregular term in (28). We emphasize again that lying at the basis of formular (28) and (29) are the static and dynamic scaling laws and the assumption that all singularities at small k and ω are due to the existence of the diffusion pole in (3). Therefore, these formulas depend only weakly on the specific properties of a Heisenberg ferromagnet and will be valid for any system in which the nearest singularity to the point $\omega = 0$ is a diffusion pole, if for D_0 we substitute the appropriate expression for the diffusion coefficient. Therefore, in particular, (28) and (29) should be valid for the diffusion of density in a liquid-gas system near the critical point and in liquid helium above the λ-point, if we neglect the influence of more distant sound poles; the latter, incidentally, require a more detailed analysis.

It remains to consider one further question—the influence of the calculated corrections on the time correlation of the fluctuations. Defining the latter in the usual way: $\tilde{G}(\mathbf{k}, t) = \langle S^z_{\mathbf{k}}(t)S^z_{-\mathbf{k}}(0)\rangle$, we obtain for $D_0k^2 > 1$:

$$\tilde{G}(\mathbf{k},t) = \frac{T}{\pi}\int\frac{d\omega}{\omega}e^{-i\omega t}\operatorname{Im}G(\mathbf{k},\omega)$$
$$\approx TG(\mathbf{k},0)\left\{\left(1+\beta\frac{k^2}{\varkappa^2}\right)\exp(-D_0k^2)\left[1+(\alpha+\beta)\frac{k^2}{\varkappa^2}\right]t\right.$$

$$+\frac{2\lambda C_2^2}{(D_0\varkappa^2 t)^{3/2}}\exp\left(-\frac{D_0k^2t}{2}\right)+\dots\Big\}. \qquad (30)$$

In this expression the first term takes account of the renormalization, due to the regular terms in (28), of the diffusion pole, and the second describes the contribution of the two-diffusion branch. Allowance for the subsequent branches leads to the appearance of terms proportional to $\exp\{-D_0k^2t/n\}$. This means that for large t the asymptotic form of $\tilde{G}$ is determined by the branches with $n \sim D_0k^2t$. The branches, however, have almost no effect on the diffusion law $\langle \mathbf{r}^2\rangle = 6D_0t$. Indeed, it follows from (30) that

$$\langle r^2\rangle = \int d\mathbf{r}\, r^2 \bar{G}(\mathbf{r},t) \Big/ \int d\mathbf{r}\, \bar{G}(\mathbf{r},t) = 6\left\{D_0 t - \frac{\beta+\beta_1}{\varkappa^2} + \frac{\lambda C_2^2}{\varkappa^2 (D\varkappa^2 t)^{1/2}} + \dots\right\},$$

$$\beta_1 = G^{-1}(\mathbf{k},0)\frac{d}{dk^2}G(\mathbf{k},0)\big|_{k^2=0} \approx -1. \qquad (31)$$

Thus, the principal correction to the diffusion law arises from the regular terms; the correction from the first branch is small, and that from the subsequent branches is still smaller. An important consequence of this formula is the fact that extrapolation of the quantity $\langle \mathbf{r}^2\rangle$ to the point t = 0 gives information on the regular corrections to the diffusion. This result, obviously, is valid for any system. It would be very interesting to discover such corrections experimentally.

APPENDIX

Using the method of the paper by Schwabl and Michel[16], we now give a general derivation of the formulas for the dynamic susceptibilities. First of all, we discuss the quantities for which these formulas hold. These are always operators which are either conserved or almost conserved in the limit $k \to 0$. In our case, the operator is S^z_k, since $L^z_k \sim k$; when the magnetic dipole forces are taken into account, S^z_k is almost conserved. There are other operators with the same momentum k that are not conserved, e.g., $S^z_q + k^{S^z_{-q}}$. Their time derivative at $k = 0$ is non-zero. The method of the paper[16] makes it possible to write general formulas for the dynamic susceptibilities of conserved or almost conserved quantities. Schwabl and Michel were interested only in the limit $k \to 0$, $\omega \to 0$, and made an assumption equivalent to that of linearity of the generalized forces. We now obtain formulas free from these restrictions. Any system is characterized by a complete set of operators, so that any operator can be expanded as a series in this set. For the derivation we use the Mori expansion[11,8], but the final answer is practically independent of the choice of expansion. Following Mori, we define a scalar product of operators:

$$(A_k, B_k^+) = \int_0^{1/T} d\tau \langle A_k(\tau) B_k^+(0)\rangle = \chi_{AB}(k,\omega=0) = \chi_{ABk}, \qquad (A.1)$$

By means of this expression it is easy to construct a complete set of orthogonal operators A_{kj}, such that $\langle A_{kj}\rangle = 0$. The Mori expansion for A_{ki} has the form (the dot denotes time-differentiation):

$$\dot{A}_{ki} = -i\sum_j c_{ij}^k A_{kj},$$

$$c_{ij}^k = i(\dot{A}_{ki}, A_{kj}^+)\chi^{-1}_{jjk} = \langle [\dot{A}_{ki}, A_{kj}^+]\rangle \chi^{-1}_{jjk} = c_{ji}^{k*}\frac{\chi_{iik}}{\chi_{jjk}}. \qquad (A.2)$$

We note that the operators A_{kj} and the coefficients c^k_{ij} depend on the temperature. Integrating the equality

$$i\langle [\dot{A}_{ki}(t),\ A_{kj}^+(0)]\rangle = c_{il}^k \langle [A_{kl}(t),\ A_{kj}^+(0)]\rangle$$

over time from t > 0 to infinity and going over to Fourier components, we obtain

$$(\omega - c^k)_{il}\Phi_{lj}(k,\omega) = i\chi_{iik}\delta_{ij}, \qquad (A.3)$$

where Φ_{lj} is the Kubo function.

We now denote the set of indices referring to conserved quantities by the label 1, and those referring to non-conserved quantities by 2. After this, (A.3) can be rewritten in the form

$$(\omega - c_{11})\Phi_{11} - c_{12}\Phi_{21} = i\chi_{11}, \qquad (A.4a)$$

$$(\omega - c_{11})\Phi_{12} - c_{12}\Phi_{22} = 0, \qquad (A.4b)$$

$$(\omega - c_{22})\Phi_{21} - c_{21}\Phi_{11} = 0, \qquad (A.4c)$$

$$(\omega - c_{22})\Phi_{22} - c_{21}\Phi_{12} = i\chi_{22}. \qquad (A.4d)$$

From (A.4a) and (A.4c), we obtain

$$(\omega - c_{11})\Phi_{11} - c_{12}\frac{1}{\omega - c_{22}}c_{21}\Phi_{11} = i\chi_{11} \qquad (A.5)$$

and from (A.4d) it follows that

$$(\omega - c_{22})^{-1} = \Phi_{22}(c_{21}\Phi_{12} + i\chi_{22})^{-1}. \qquad (A.6)$$

Substituting (A.6) into (A.5), we arrive at the expression

$$[\omega - c_{11}^k + i\Gamma_{11}(k\omega)]\Phi_{11}(k,\omega) = i\chi_{11}(k),$$

$$\Gamma_{11}(k,\omega) = c_{12}^k\Phi_{22}(k,\omega)\chi_{22}^{-1}(k)[1 - ic_{21}^k\Phi_{12}(k,\omega)\chi_{22}^{-1}(k)]^{-1}c_{21}^k. \qquad (A.7)$$

Up to this point, infinite matrices appear in Γ_{11}. But the expansion (A.2) for A_{k1} can be written thus:

$$\dot{A}_{k1} + ic_{11}^k A_{k1} = -ic_{12}^k A_{k2} = -id_k, \qquad (A.8)$$

where d_k is the analog of the divergence of the flux for the conserved quantities. Using this definition and the symmetry property (A.2) of the c_{ij}, we arrive after simple transformations at the formula

$$\Gamma_{11}(k,\omega) = \Phi_{d_k d_{-k}}(k,\omega)\frac{1}{\chi_{11}(k)}\frac{1}{1 - \Phi_{A_{k1}d_{-k}}(k,\omega)\chi_{11}^{-1}(k)}. \qquad (A.9)$$

It should be remembered that, generally speaking, the quantities occurring in this expression are matrices, and the order of the factors is therefore important. Using the formula (7) relating the Kubo function and the susceptibility, we finally obtain

$$\chi_{11}(k,\omega) = [\omega - c_{11}^k + i\Gamma_{11}(k,\omega)]^{-1}[-c_{11}^k + i\Gamma_{11}(k,\omega)]\chi_{11}(k). \qquad (A.10)$$

In the paper[16], the last factor in (A.9) was replaced by unity.

We note that the expansion method used affected only the choice of the matrix c^k_{11}. But this matrix is easily determined, starting from physical considerations, and its choice is practically fixed by the choice of conserved operators in the Mori expansion. In the case treated in the main text, $c^k_{11} = 0$. From (A.9) follows the condition for the existence of hydrodynamic or soft modes, viz., the smallness of $\Gamma_{11}(k,\omega)$ in the limit $k, \omega \to 0$.

[1)] It can be shown that neglect of these forces is legitimate in the region in which $4\pi\chi \ll 1$, i.e., not too close to the Curie temperature.

[2)] It is necessary for the existence of this identity that the contribution from closed loops containing a vector vertex be equal to zero, and this is indeed the case, since such loops change sign on change of sign of the momentum of integration.

[3)] This is connected with the fact that the spin operator changes sign under time reversal.

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