Multipoint energy density correlators for the two-dimensional Ising model

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Energy-density correlation functions of *n* th order are obtained for the plane Ising model with an arbitrary point positions. The somewhat cumbersome general expression can be simplified by introducing a "multivector product." Due to symmetry of the two-dimensional Ising model with respect to the Kramer–Wannier transformation, the even- and odd-order correlators are even and odd functions of $\Delta T/T_c$, respectively. In conclusion, the behavior of energy correlators in the general case is discussed from the view point of scaling laws.

1. INTRODUCTION

A large number of papers devoted to the investigation of correlation functions for the two-dimensional Ising model has been published in recent years. However, the multipoint correlators found thus far[1,2] pertain only to the case when the points are located on a straight line—along a row (column) or diagonal. The multipoint spin correlators then satisfy the scaling-law hypothesis[3,4], whereas the multipoint energy-density correlators, while not contradicting the hypothesis, does not confirm it, Indeed, if at T = T_c the energy-density correlator for an odd number of points vanishes on account of the Kramers-Wannier symmetry, then the even-order correlator is, according to Kadanoff[1], equal to

$$\langle\mathscr{E}_1\ldots\mathscr{E}_{2n}\rangle=\sum\langle\mathscr{E}_1\mathscr{E}_2\rangle\langle\mathscr{E}_2\mathscr{E}_3\rangle\ldots\langle\mathscr{E}_{2n-1}\mathscr{E}_{2n}\rangle, \tag{1.1}$$

where the summation is over all the (2n − 1)!! permutations of the indices. It follows at once from this that the corresponding irreducible correlator $\langle\langle\varepsilon_1\ \ldots\ldots\ \varepsilon_{2n}\rangle\rangle$ for points situated on a straight line also vanishes. Also inapplicable for the two-dimensional Ising model is the expression resulting from the general form of the three-point correlator at T = T_c found by Polyakov[5] on the basis of the theory of conformal symmetry; this expression vanishes here due to the above-indicated Kramers-Wannier symmetry.

However, the high-order energy correlators do not vanish: if n is their order, then their integrals are equal to the n-th derivatives of the free energy:

$$\iint\ldots\int\langle\langle\mathscr{E}_1\ldots\mathscr{E}_n\rangle\rangle\,d\mathbf{r}_1\ldots d\mathbf{r}_n=\frac{\partial^n\ln Z}{\partial K^n}, \tag{1.2}$$

where K = –J/T. Therefore, the investigation of the multipoint energy-density correlators for arbitrary positions of the points, to which the present paper is devoted, is of unquestionable interest.

2. THE GENERAL FORMULAS

To compute the multipoint energy-density correlator, let us consider a two-dimensional M × N Ising lattice for different interaction constants. The statistical sum of such a model is equal to

$$Z=\operatorname{Sp}\left\{\exp\left[\sum_{m=1}^{M}\sum_{n=1}^{N}(K_{mn}\sigma_{mn}\sigma_{m+1,n}+K_{mn}'\sigma_{mn}\sigma_{m,n+1})\right]\right\}. \tag{2.1}$$

Here, as a rule, σ takes on the values ± 1, $K_{mn} = J_{mn}/T$, $K_{mn}' = J_{mn}'/T$, where J_{mn} and J_{mn}' are respectively the interaction constants along the vertical and horizontal lines. Let us introduce the dimensionless vertical and horizontal energy densities:

$$\mathscr{E}_{mn}=\sigma_{mn}\sigma_{m+1,n},\qquad\mathscr{E}_{mn}'=\sigma_{mn}\sigma_{m,n+1}. \tag{2.2}$$

The ordinary energy density is obtained by adding these expressions multiplied beforehand by –J and –J′, respectively.

The energy-density correlation function in any number of arbitrary points of a homogeneous anisotropic lattice is equal to the corresponding n-th derivative of the statistical sum (2.1) with respect to K_{mn} and K_{mn}' for all K_{mn} = K and K_{mn}' = K′:

$$\left\langle\prod_i\mathscr{E}_{m_in_i}^{(\alpha_i)}\right\rangle=Z^{-1}\left(\prod_i\frac{\partial}{\partial K_{m_in_i}^{(\alpha_i)}}\right)Z\Bigg|_{K_{mn}^{(\alpha)}=K^{(\alpha)}}. \tag{2.3}$$

Here the index α denotes the presence or absence of the prime on the corresponding quantities.

Let us now write the statistical sum (2.1) in terms of the trace of the product of the transition matrices:

$$Z=\operatorname{Sp}(\hat P_M'\ldots\hat P_2'\hat P_1'), \tag{2.4}$$

$$\hat P_m'=\hat U_m\hat V_m, \tag{2.5}$$

$$\hat U_m=\prod_n(2\operatorname{sh}2K_{nm})^{1/2}\exp\Big(\sum_n K_{mn}{}^*\hat X_n$$

$$=\prod_n(2\operatorname{sh}2K_{nm})^{1/2}\exp\Big(2i\sum_n K_{nm}{}^*\hat\mu_n{}^+\hat\mu_n{}^-\Big), \tag{2.6}$$

$$\hat V_m=\exp\Big(\sum_n K_{nm}'\hat Z_n\hat Z_{n+1}\Big)=\exp\Big(-2i\sum_n K_{nm}'\hat\mu_n{}^+\hat\mu_{n+1}^-\Big); \tag{2.7}$$

$$\operatorname{th}K^*=e^{-2K}\quad\text{or}\quad\operatorname{sh}2K^*=(\operatorname{sh}2K)^{-1}. \tag{2.8}$$

Here we have written the expression for the transition matrix first in terms of the ordinary Pauli matrices $\hat X$ and $\hat Z$ and then in terms of the Onsager matrices

$$\hat\mu_n{}^+=2^{-1/2}\hat Y_n\hat X_1\ldots\hat X_{n-1},\quad\hat\mu_n{}^-=\hat Z^{-1/2}\hat Z_n\hat X_1\ldots\hat X_{n-1}, \tag{2.9}$$

which satisfy the commutation relations for Fermi operators:

$$[\hat\mu_n{}^\beta,\hat\mu_{n'}{}^{\beta'}]_+=\delta_{nn'}\delta_{\beta\beta'},\quad\beta,\beta'=\pm. \tag{2.10}$$

Since we shall be interested in only bulk effects, we shall not pay any attention here and below to the boundary conditions.

To find the energy-density correlator of interest to us, we shall differentiate, according to (2.3), the statistical sum (2.4). Then, there will appear at the points of the transition-operator product on the right hand side of (2.4) that correspond to the column number the energy-density operators:

$$\langle\mathscr{E}'^{(\alpha_1)}_{m_1n_1}\mathscr{E}^{(\alpha_1)}_{m_2n_2}\ldots\rangle=\operatorname{Sp}(\hat P^{M-m_1}\hat{\mathscr{E}}^{(\alpha_1)}_{n_1}\hat P^{m_1-m_2}\hat{\mathscr{E}}^{(\alpha_2)}_{n_2}\ldots)/\operatorname{Sp}P^M, \tag{2.11}$$

$$\hat{\mathscr{E}}_n = (\operatorname{sh} 2K)^{-1}(\operatorname{ch} 2K - 2i\hat{\mu}_n{}^+\hat{\mu}_n{}^-),$$
$$\hat{\mathscr{E}}_n{}' = -2i\mu_n{}^+\mu_{n+1}^-. \qquad (2.12)$$

To simplify the expression, we carried out a small number of cyclic permutations under the Sp sign, so that now

$$\hat{P} = \hat{V}\hat{U}. \qquad (2.13)$$

The expression (2.11) obtained for the correlator can easily be reduced to a product of factors of the form $A + B\mu_n^\beta\mu_{n'}^{\beta'}$, and then by the Hurst-Green method[6] to the corresponding Pfaffian form, which can be computed with the aid of the hothouse forms. It is easier, however, to use the Kadanoff-Hecht method[7,8], introducing the column-number dependent energy-density operators:

$$\hat{\mathscr{E}}_{mn}^{(\alpha)} = \hat{P}^{-m}\hat{\mathscr{E}}_n\hat{P}^m. \qquad (2.14)$$

Then the sought-for correlator assumes the form

$$\left\langle \prod_i \mathscr{E}_{m_i n_i}^{(\alpha_i)} \right\rangle = \operatorname{Sp}\left(P^M T_m \prod_i \hat{\mathscr{E}}_{m_i n_i}^{(\alpha_i)}\right) \Big/ \operatorname{Sp} P^M. \qquad (2.15)$$

The symbol T_m here stands for the operator that permutes the operators standing after it in the order of decreasing column number m from left to right.

The above expression is clearly analogous to the multipoint temperature Green function[9], the role of the imaginary "time" being played here by the index m: the column-number dependent operators are analogous to the operators in the Heisenberg representation, the operator $\hat{P}$ is analogous to the translation operator $\exp(-\mathscr{H}\tau)$, the operator $\hat{p}M$ is analogous to the operator $\exp(-\beta\mathscr{H})$, and the row number n is analogous to the space variables. The subsequent probram is obvious. The expression (2.12) connecting the energy-density Schrödinger operators and the Schrödinger Fermi-operators is also valid for the Heisenberg operators if we analogously introduce the Heisenberg Fermi-operators:

$$\hat{\mu}_{mn}^{\pm} = \hat{P}^{-m}\hat{\mu}_n{}^{\pm}\hat{P}^m. \qquad (2.16)$$

Since our Hamiltonian is a quadratic function of the Fermi operators (2.6) and (2.7), the Heisenberg representation is in fact the interaction representation and the many-particle Green function, i.e., the required correlator, can, according to Wick's theorem, be expressed in terms of the two-particle Green function:

$$\mathfrak{G}^{\pm\pm}(m-m', n-n') = \operatorname{Sp}(P^M T_m \hat{\mu}_{mn}{}^{\pm}\hat{\mu}_{m'n'}^{\pm})/\operatorname{Sp} P^M. \qquad (2.17)$$

The applicability of Wick's theorem to the case under consideration can be proved by the method given in the book[9].

To do this, we assume that the distances between the points $|m - m'| \ll M$ and $|n - n'| \ll N$, and also that n, $n' \gg 1$, so that the Green function (2.17) depends only on the intervals $|m - m'|$ and $|n - n'|$, on account of the uniformity of the system. The Green function (2.17) is computed in the Appendix.

3. THE ENERGY-DENSITY CORRELATORS

Let us first of all find the pair correlator. Computing according to Wick's theorem the average of the product of four Fermi operators, we find

$$\langle \mathscr{E}_{nm}^{(\alpha)} \mathscr{E}_{n'm'}^{(\alpha')} \rangle = \langle\!\langle \mathscr{E}_{nm}^{(\alpha)} \mathscr{E}_{n'm'}^{(\alpha')} \rangle\!\rangle + \langle \mathscr{E}^{(\alpha)} \rangle \langle \mathscr{E}^{(\alpha')} \rangle. \qquad (3.1)$$

The second term on the right hand side is the result of the bonding of the Fermi operators that enter into one and the same energy-density operator. If we graphically represent each Fermi operator by a point and the pair Green function (2.17) by a line, then the second term in (3.1) graphically has the form shown in Fig. 1b. The first term—the irreducible energy-density—energy-density correlator—is the result of the bonding of the Fermi operators from different energy-density operators: it is shown graphically in Fig. 1a. The interaction of lines in a diagram corresponds, on account of the anticommutation of Fermi operators, to the appearance of the minus sign.

We can now use the expressions (A.8) and (A.9) obtained in the Appendix for the Green functions to find the irreducible pair correlator. The corresponding expressions can be simplified near the transition temperature at small values of

$$\varepsilon = 2 \operatorname{sh} 2K_c'(K' - K^*) \ll 1. \qquad (3.2)$$

Here K_c' is the critical value of K' determined by the equation $K_c' = K_c^*$, i.e., $\sinh 2K_c \sinh 2K_c' = 1$. For an isotropic lattice, $\sinh 2K_c = 1$ and $\epsilon = -4J(T - T_c)/T_c^2$. Using the expressions (A.14) and (A.15) (see the Appendix) for the Green functions, we find for large distances measured in units of the lattice constant that

$$\langle\!\langle \mathscr{E}_{r_1}'\mathscr{E}_{r_2}' \rangle\!\rangle = (\varepsilon/\pi)^2(K_1^2(|\varepsilon|R) - K_0^2(|\varepsilon|R)), \qquad (3.3)$$
$$R^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \operatorname{sh}^2 2K_c \gg 1, \qquad (3.4)$$

where K_0 and K_1 are Macdonald functions.

We have written here the expression for the horizontal (primed) energy-density correlator. The other pair correlators differ only by the appearance of the additional factor $(\sinh 2K_c)^{-1}$ multiplying each unprimed (vertical) energy density:

$$\langle\!\langle \mathscr{E}_1\mathscr{E}_2' \rangle\!\rangle = \langle\!\langle \mathscr{E}_1'\mathscr{E}_2 \rangle\!\rangle = (\operatorname{sh} 2K_c)^{-1}\langle\!\langle \mathscr{E}_1'\mathscr{E}_2' \rangle\!\rangle,$$
$$\langle\!\langle \mathscr{E}_1\mathscr{E}_2 \rangle\!\rangle = (\operatorname{sh} 2K)^{-2}\langle\!\langle \mathscr{E}_1'\mathscr{E}_2' \rangle\!\rangle.$$

At small—in comparison with the correlation radius $R_c \sim |\epsilon|^{-1}$—distances, using (A.17), we obtain

$$\langle\!\langle \mathscr{E}_1'\mathscr{E}_2' \rangle\!\rangle = 1/\pi^2 R^2, \quad 1 \ll R \ll R_c. \qquad (3.5)$$

The pair energy-density correlator was previously obtained by Stephenson in[10] and by Hecht in[8].

Let us now consider the triple energy-density correlator. The application of Wick's theorem to the expression (2.15) in the case of three energy-density operators is demonstrated graphically in Fig. 2. Thus, the triple correlator breaks up into the irreducible correlators:

$$\langle \mathscr{E}_1\mathscr{E}_2\mathscr{E}_3 \rangle = \langle\!\langle \mathscr{E}_1\mathscr{E}_2\mathscr{E}_3 \rangle\!\rangle + \langle \mathscr{E}_1 \rangle \langle\!\langle \mathscr{E}_2\mathscr{E}_3 \rangle\!\rangle + \langle \mathscr{E}_2 \rangle \langle\!\langle \mathscr{E}_1\mathscr{E}_3 \rangle\!\rangle$$
$$+ \langle \mathscr{E}_3 \rangle \langle\!\langle \mathscr{E}_1\mathscr{E}_2 \rangle\!\rangle + \langle \mathscr{E}_1 \rangle \langle \mathscr{E}_2 \rangle \langle \mathscr{E}_3 \rangle. \qquad (3.6)$$

Using (A.14) and (A.15), we can obtain after quite tedious computations the following expression for the irreducible triple energy-density correlator near the transition temperature for large distances:

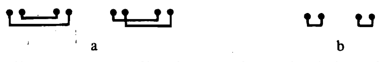

FIG. 1. The diagrams contributing to the reducible pair correlator (see (3.1)); a—contribution to the first term of the expression (3.1)—the irreducible correlator; b—contribution to the second term.

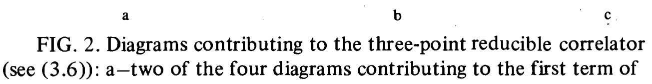

FIG. 2. Diagrams contributing to the three-point reducible correlator (see (3.6)): a—two of the four diagrams contributing to the first term of the expression (3.6)—the irreducible correlator; b—the diagrams contributing to the second term of this expression; c—the contribution to the last term.

$$\langle\!\langle \mathscr{E}_1'\mathscr{E}_2'\mathscr{E}_3'\rangle\!\rangle = 2\left(\frac{\varepsilon}{\pi}\right)^3 \Big[\sum K_0(|\varepsilon|R_{12})K_1(|\varepsilon|R_{23})K_1(|\varepsilon|R_{31})\,(\mathbf{n}_{23}\mathbf{n}_{31}) + K_0(|\varepsilon|R_{12})K_0(|\varepsilon|R_{23})K_0(|\varepsilon|R_{31})\Big], \qquad R_{ij} \gg 1. \tag{3.7}$$

The summation sign denotes summation over three permutations of the indices: 1, 2, and 3; R_{ij} is the distance between the points r_i and r_j in the sense of (3.4); n_{ij} is the unit vector in the direction going from the point $\mathbf{r}_i$ to r_j, also with allowance for the anisotropy:

$$\mathbf{n}_{ij} = \mathbf{R}_{ij}/R_{ij}, \quad \mathbf{R}_{ij} = \{x_i - x_j,\ (y_i - y_j)\operatorname{sh} 2K_c\}. \tag{3.8}$$

As before, the correlator for the unprimed (vertical) energy densities differs from this correlator by the appearance of the factor $(\sinh 2K_c)^{-1}$ multiplying each unprimed energy.

At distances that are small compared to the correlation radius R_c, the dominant contribution is made by the first term of the formula (3.7):

$$\langle\!\langle \mathscr{E}_1'\mathscr{E}_2'\mathscr{E}_3'\rangle\!\rangle = \frac{-2\varepsilon}{\pi^3} \sum \frac{\mathbf{n}_{23}\mathbf{n}_{31}}{R_{23}R_{31}} \ln(|\varepsilon|R_{12}). \tag{3.9}$$

Thus, the triple correlator is an odd function with respect to the transition temperature and vanishes at $T = T_c$. This result is a consequence of the symmetry of the Ising model under the dual Kramers-Wannier transformation. Under such a transformation $(\hat{\varepsilon} - \langle\varepsilon\rangle) \to -(\hat{\varepsilon} - \langle\varepsilon\rangle)$. Therefore, the scaling term in the triple energy correlator predicted by Polyakov[5] vanishes here and the dominant term (3.9) behaves as $\pm R_c^{-1}R^{-2}\ln(R/R_c)$.

Let us consider the fourth-order correlator. It also breaks up into irreducible correlators:

$$\begin{gathered}\langle \mathscr{E}_1\mathscr{E}_2\mathscr{E}_3\mathscr{E}_4\rangle = \langle\!\langle \mathscr{E}_1\mathscr{E}_2\mathscr{E}_3\mathscr{E}_4\rangle\!\rangle + \\ + \langle \mathscr{E}_1\rangle\langle\!\langle \mathscr{E}_2\mathscr{E}_3\mathscr{E}_4\rangle\!\rangle + \ldots + \langle\!\langle \mathscr{E}_1\mathscr{E}_2\rangle\!\rangle\langle\!\langle \mathscr{E}_3\mathscr{E}_4\rangle\!\rangle \\ \ldots + \langle \mathscr{E}_1\rangle\langle \mathscr{E}_2\rangle\langle \mathscr{E}_3\rangle\langle \mathscr{E}_4\rangle.\end{gathered} \tag{3.10}$$

The dots stand for terms obtainable from the explicitly written-out terms by permuting their indices. We leave out the tedious intermediate computations and give the final result for large R_{ij} near T_c, obtained with the aid of the formulas (A.14) and (A.15):

$$\begin{gathered}\langle\!\langle \mathscr{E}_1'\mathscr{E}_2'\mathscr{E}_3'\mathscr{E}_4'\rangle\!\rangle = -2(\varepsilon/\pi)^4 \Big\{ K_1(|\varepsilon|R_{12}) \cdot \\ \cdot K_1(|\varepsilon|R_{23})K_1(|\varepsilon|R_{34})K_1(|\varepsilon|R_{41}) \cdot \\ \times[(\mathbf{n}_{12}\mathbf{n}_{23})(\mathbf{n}_{34}\mathbf{n}_{41}) + (\mathbf{n}_{23}\mathbf{n}_{34})(\mathbf{n}_{41}\mathbf{n}_{12}) - (\mathbf{n}_{12}\mathbf{n}_{34})(\mathbf{n}_{41}\mathbf{n}_{23})] \\ + \sum K_0K_0K_1\mathbf{n}K_1\mathbf{n} + K_0(|\varepsilon|R_{12})K_0(|\varepsilon|R_{23})K_0(|\varepsilon|R_{34})K_0(|\varepsilon|R_{41})\Big\} + \ldots\end{gathered} \tag{3.11}$$

The dots stand for two more analogous terms obtainable from the written-out term by permuting the indices (12) and (34); in other words, to each of the terms correspond the nonequivalent (in the sense that they cannot be transformed into each other by cyclic permutation or inversion) cycles (1234), (1324), and (1243). The summation sign denotes summation over the six permutations of the arguments R_{12}, R_{23}, R_{34}, and R_{41} of the functions under the summation sign.

The expression (3.11) obtained by us for the fourth-order correlator is, of course, quite unwieldy and difficult to examine. To simplify it let us introduce the "multivector product" denoted below by the angular brackets $\langle\ldots\rangle$ and satisfying the following rules:

1) the multivector product is distributive;

2) the scalar quantities that enter into the multivector product commute with each other and with the vectors; to put it simply, they play the role of coefficients and can be taken outside the multivector-product sign;

3) the multivector product is noncommutative with respect to the vectors that enter into it, but cofactors can be cyclically permuted;

4) the transposition of two adjacent vectors satisfies the "anticommutation" condition:

$$\langle \ldots \mathbf{ab} \ldots \rangle + \langle \ldots \mathbf{ba} \ldots \rangle = 2\langle \mathbf{ab}\rangle\langle \ldots \rangle;$$

5) the multivector product of two vectors coincides with the ordinary scalar product: $\langle \mathbf{ab}\rangle = (\mathbf{a}.\,\mathbf{b})$.

It follows from the definitions 3)—5) that the multivector product of an odd number of vectors vanishes, while the product of an even number of vectors can be expanded according to Wick's theorem and expressed in terms of binary scalar products. For example, for four cofactors

$$\langle \mathbf{abcd}\rangle = (\mathbf{ab})(\mathbf{cd}) + (\mathbf{ad})(\mathbf{bc}) + (\mathbf{ac})(\mathbf{bd}). \tag{3.12}$$

With the aid of the multivector product introduced by us we can write the expression (3.11) for the correlator in the quite simple form:

$$\langle\!\langle \mathscr{E}_1'\mathscr{E}_2'\mathscr{E}_3'\mathscr{E}_4'\rangle\!\rangle = -2\left(\frac{\varepsilon}{\pi}\right)^4 \Big\langle \prod_{i=1}^{4} (K_1(|\varepsilon|R_{i,i+1})\mathbf{n}_{i,i+1} + K_0(|\varepsilon|R_{i,i+1})) \Big\rangle + \ldots. \tag{3.13}$$

As before, the dots here stand for the two analogous terms with different nonequivalent arrangements of the indices; $R_{45} \equiv R_{41}$. The unprimed $\mathscr{E}$ lead to additional factors of $(\sinh 2K_c)^{-1}$.

For $1 \ll R_{ij} \sim R \ll R_c$, the dominant contribution is made by the first terms of the cofactors in (3.13), so that the fourth-order correlator behaves, on account of (A.17), in an invariant fashion under scaling:

$$\langle\!\langle \mathscr{E}_1\mathscr{E}_2\mathscr{E}_3\mathscr{E}_4\rangle\!\rangle \sim 1/R^4,$$

with quite a complicated angular dependence, which, as can be verified by direct calculation, leads to the vanishing of this dominant part of the correlator when all the four points are situated on a straight line, in accord with Kadanoff's result[1].

Higher-order energy-density correlators have similar form near T_c. They are each a sum of $(n-1)!/2$ terms, to each of which corresponds a nonequivalent closed cycle composed of the indices entering into the correlator of the points. Each term of the sum can be written in the form of a multivector product:

$$\langle\!\langle \mathscr{E}_1'\mathscr{E}_2'\ldots\mathscr{E}_n'\rangle\!\rangle = -2\left(\frac{-\varepsilon}{\pi}\right)^n \Big\langle \prod_{i=1}^{4} (K_1(|\varepsilon|R_{i,i+1})\mathbf{n}_{i,i+1} + K_0(|\varepsilon|R_{i,i+1})) \Big\rangle + \ldots. \tag{3.14}$$

The unprimed (vertical) energy densities yield additional factors of $(\sinh 2K_c)^{-1}$; R_{ij} should be understood in the sense of (3.8); $R_{n,\,n+1} = R_{n1}$.

For $1 \ll R_{ij} \sim R \ll R_c$, the dominant contribution to the correlator is made by the first terms standing inside the brackets under the product sign in (3.14), so that even-order correlators behave at $T = T_c$ according to scaling theory:

$$\langle\!\langle \mathscr{E}_1\mathscr{E}_2\ldots\mathscr{E}_{2n}\rangle\!\rangle \sim 1/R^{2n}$$

with a complicated angular dependence guaranteeing the vanishing of the correlator when the points are situated on a straight line. An odd-order correlator is an odd function of ϵ and vanishes at $T = T_c$. Since the multivector product is different from zero only for an even number of vectors, the dominant contribution to an odd-order correlator is made by the product in (3.14) that contains the function K_1 $n-1$ times and the function K_0 once:

$$\langle\langle \mathscr{E}_1\mathscr{E}_2\ldots\mathscr{E}_{2n+1}\rangle\rangle \sim \pm R_c^{-1}R^{-2n}\ln(R/R_c). \qquad (3.15)$$

4. CONCLUDING REMARKS

In conclusion, let us answer the naturally arising question: Is the vanishing at $T = T_c$ of odd-order energy correlation functions a general property, or is it connected with the specific properties of the two-dimensional Ising model? According to the scaling-law hypothesis[3,4], the singular part of the free energy is equal to

$$F_{sing}/NT = \begin{cases} A_+\varepsilon^{2-\alpha} & \varepsilon > 0. \\ A_-|\varepsilon|^{2-\alpha}, & \varepsilon < 0. \end{cases} \qquad (4.1)$$

Let us now represent the expression (4.1) as a sum of even and odd functions of ϵ and take into account the connection between the integrals of the irreducible correlators and the derivatives of the free energy with respect to ϵ (see (1.2)), a connection which is valid for any model. Then an even-order correlator is determined at $T = T_c$ by the even part of the expression (4.1):

$$\langle\langle \mathscr{E}_1\ldots\mathscr{E}_{2n}\rangle\rangle \sim (A_+ + A_-)/R^{2n\lambda}, \qquad (4.2)$$

while an odd-order correlator is determined by the odd part of (4.1):

$$\langle\langle \mathscr{E}_1\ldots\mathscr{E}_{2n+1}\rangle\rangle \sim (A_+ - A_-)/R^{(2n+1)\lambda}. \qquad (4.3)$$

The critical-point exponent of the energy density $\lambda = d - 1/\nu$, where d is the dimension of the model and ν is the critical-point exponent of the correlation radius $R_c \sim |\epsilon|^{-\nu}$.

Thus, the vanishing of odd-order energy correlators for the two-dimensional Ising model is connected with the symmetry of the singular part of the specific heat with respect to the transition point, i.e., in the final analysis, with the Kramers-Wannier symmetry. In the general case odd correlators do not vanish, being proportional to $A_+ - A_-$.

In conclusion, the author expresses his thanks to M. A. Mikulinskiĭ for useful discussions.

APPENDIX

A Green function similar to ours (2.17), was found in[7] by Kadanoff; therefore, we shall only outline the method of computing it, citing in detail only the final formulas that are necessary for obtaining the energy-density correlation functions.

Considering the Fermi operators μ^{+}_{mn} as components of a spinor

$$\hat{\mu}_{mn} = \begin{pmatrix} \mu_{mn}{}^{+} \\ \mu_{mn}{}^{-} \end{pmatrix} \qquad (A.1)$$

in τ space and using their definition (2.16) and the definition (2.6), (2.7), and (2.13) of the operator $\hat{P}$, we find that far from the boundaries

$$\hat{\mu}_{mn} \equiv \hat{P}^{-1}\hat{\mu}_{m-1,n}\hat{P} = \sum_{n'}\hat{g}(n-n')\,\hat{\mu}_{m-1,n'}, \qquad (A.2)$$

where $\hat{g}(n)$ is an operator in τ space, i.e., a 2×2 matrix. It differs from zero only when its argument $n = 0, \pm 1$. Taking the Fourier transform of this matrix, we obtain in τ space the operator $\hat{g}(p)$ whose eigenvalues are equal to $\exp(\pm\gamma(p))$, where

$$\operatorname{ch}\gamma(p) = \operatorname{ch}2K^*\operatorname{ch}2K' - \operatorname{sh}2K^*\operatorname{sh}2K'\cos p. \qquad (A.3)$$

Let us represent the operator $\hat{g}(p)$ in the form of a sum of the operators of projection unto the eigenstates $\hat{g}_+(p)$ and $\hat{g}_-(p)$:

$$\hat{g}(p) = \exp(\gamma(p))\hat{g}_+ + \exp(-\gamma(p))\hat{g}_-, \qquad (A.4)$$

where the projection operators are equal to

$$\hat{g}_\pm(p) = \frac{1}{2} \pm \frac{1}{2\operatorname{sh}\gamma}[\hat{\tau}_x\operatorname{sh}2K'\operatorname{ch}2K^*\sin p + \hat{\tau}_y(\operatorname{sh}2K'\operatorname{ch}2K^*\cos p - \operatorname{ch}2K'\operatorname{sh}2K^*) - i\hat{\tau}_z\operatorname{sh}2K'\operatorname{sh}2K^*] \qquad (A.5)$$

(τ_x, τ_y, and τ_z are the ordinary Pauli operators in τ space).

Using now (A.2), we find the equation for the Green functions (2.17) considered here as matrices in τ space:

$$\hat{\mathfrak{G}}(m,n) = \sum_{n'}\hat{g}(n-n')\hat{\mathfrak{G}}(m-1,n') + \delta_{0m}. \qquad (A.6)$$

Solving this equation with the aid of the Fourier transformation, then using the decomposition (A.4) of $\hat{g}(p)$ in terms of the projection operators, and then carrying out the inverse Fourier transformation with respect to the first argument, we obtain for the Fourier transform of the Green function in the second argument:

$$\hat{\mathfrak{G}}(m,p) = \begin{cases} \exp(-\gamma(p)m)\hat{g}_-(p), & m \geqslant 0 \\ -\exp(-\gamma(p)|m|)\hat{g}_+(p), & m < 0. \end{cases} \qquad (A.7)$$

Using the values (A.5) of the projection operators $\hat{g}_\pm(p)$ and performing the inverse Fourier transformation, we find the diagonal components of the Green-function matrix:

$$\mathfrak{G}^{\pm\pm}(m,n) = \frac{1}{2}\frac{m \mp n}{|m|}\int_{-\pi}^{+\pi}\frac{dp}{2\pi}e^{-\gamma|m|+ipn} \qquad (A.8)$$

and the off-diagonal components:

$$\mathfrak{G}^{\pm\mp}(m,n) = \mp\frac{i}{2}\left[\operatorname{ch}2K'\operatorname{sh}2K^*\int_{-\pi}^{+\pi}\frac{dp}{2\pi}\frac{e^{-|\gamma|m+ipn}}{\operatorname{sh}\gamma} - \operatorname{sh}2K'\operatorname{ch}2K^*\int_{-\pi}^{\pm\pi}\frac{dp}{2\pi}\frac{e^{-\gamma|m|+ip(n\pm1)}}{\operatorname{sh}\gamma}\right]. \qquad (A.9)$$

These expressions are exact (far from the lattice boundaries). From them, it is easy to obtain, in particular, expressions for the free energy. In fact, using (2.12) and taking (2.8) into account, we find

$$\langle\mathscr{E}\rangle = (\operatorname{sh}2K)^{-1}(\operatorname{ch}2K - 2i\mathfrak{G}^{+-}(0,0)) = \frac{1}{2}\frac{\partial}{\partial K}\left[\ln\operatorname{sh}2K + \int_{-\pi}^{+\pi}\frac{dp}{2\pi}\gamma(p)\right], \qquad (A.10)$$

$$\langle\mathscr{E}'\rangle = -2i\mathfrak{G}^{+-}(0,-1) = \frac{1}{2}\frac{\partial}{\partial K'}\int_{-\pi}^{+\pi}\frac{dp}{2\pi}\gamma(p). \qquad (A.11)$$

Integrating these expressions, we immediately obtain Onsager's celebrated formula:

$$\ln z = \frac{1}{2}\left(\ln 2 + \ln\operatorname{sh}2K + \int_{-\pi}^{+\pi}\frac{dp}{2\pi}\gamma(p)\right). \qquad (A.12)$$

Near the transition temperature, when $K' \to K^*$, we have for small p and ϵ (see (3.1))

$$\gamma(p) \approx (\varepsilon^2 + p^2)^{1/2}\operatorname{sh}2K_c'. \qquad (A.13)$$

Then at large (in comparison with the dimensions of the unit cell) distances

$$\mathfrak{G}^{\pm\pm}(m,n) = \frac{\operatorname{sh}2K_c'}{2\pi R}(m \mp n)|\varepsilon|K_1(|\varepsilon|R), \qquad (A.14)$$

$$\mathfrak{G}^{\pm\mp}(m,n) = \frac{i}{2\pi}\left[\pm\varepsilon K_0(|\varepsilon|R) - \operatorname{ch}2K_c'\frac{n|\varepsilon|}{R}K_1(|\varepsilon|R)\right] \qquad (A.15)$$

$$R = (n^2 + m^2\operatorname{sh}^2 2K_c)^{1/2} \gg 1. \qquad (A.16)$$

We have replaced K' by $K'_c = K^*_c$. Here K_0 and K_1 are Macdonald functions. Let us remind ourselves of their behavior at large and small values of the argument:

$$K_0(x) = \begin{cases} -\ln x \\ (\pi/2x)^{1/2}e^{-x}, \end{cases} \quad K_1(x) = \begin{cases} 1/x, & x \ll 1. \\ (\pi/2x)^{1/2}e^{-x} & x \gg 1. \end{cases} \tag{A.17}$$

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[1] L. P. Kadanoff, Phys. Rev. **188**, 859 (1969).

[2] M. A. Mikulinskiĭ, Zh. Eksp. Teor. Fiz. **58**, 1848 (1970) [Sov. Phys.-JETP **31**, 991 (1970)].

[3] A. Z. Patashinskiĭ and V. L. Pokrovskiĭ, Zh. Eksp. Teor Teor. Fiz. **50**, 439 (1966) [Sov. Phys.-JETP **23**, 292 (1966)].

[4] L. P. Kadanoff, Physica **2**, 263 (1966).

[5] A. M. Polyakov, ZhETF Pis. Red. **12**, 538 (1970) [JETP Lett. **12**, 381 (1970)].

[6] H. S. Green and C. A. Hurst, Order-Disorder Phenomena, Interscience Publishers, New York, 1964.

[7] L. P. Kadanoff, Nuovo Cimento **40**, 276 (1966).

[8] R. Hecht, Phys. Rev. **158**, 557 (1967).

[9] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Methods of Quantum Field Theory in Statistical Physics), Fizmatgiz, 1962 (Eng. Transl., Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963).

[10] J. Stephenson, J. Math Phys. **7**, 1123 (1966).

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