Electron-electron collisions and the electric conductivity of metals at low temperatures

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The contribution of electron-electron collisions to the resistance of pure metals at low temperatures is determined. It is shown that due to compensation of Coulomb repulsion of electrons and their attraction due to virtual phonon exchange, the effective interaction between the electrons is appreciably decreased. Consequently, the contribution of electron-electron collisions to the resistance of a number of metals is proportional to T^6 in a broad temperature range (and not to T^2 as predicted by the Landau–Pomeranchuk theory; T is the metal temperature).

1. As first shown by Bloch[1], the resistance of a metal, due to electron interaction with the lattice vibration, is proportional to T^5 (T is the temperature). On the other hand, as shown by Landau and Pomeranchuk[2], the resistance due to the scattering of electrons by electrons, should be proportional to T^2. The principal role at low temperatures should therefore be played not by electron-phonon interactions, but by electron-electron interaction. Yet the T^2 law is not confirmed as a rule in experiments on pure metals at low temperatures[3].

The Landau-Pomeranchuk approach is quite general and makes use only of the circumstance that the electrons of the metal constitute a degenerate Fermi system, and the effective interaction between them is described by a potential that depends on the coordinate. It is therefore clear that there should exist a mechanism that suppresses the T^2 law for the resitance of metals. We show in this paper that this mechanism can be the mutual cancellation of the Coulomb repulsion of the electrons by their attraction due to phonon exchange.

It is known that total cancellation of the Coulomb repulsion of the electrons by their attraction due to phonon exchange takes place at low energy transfers in the jellium model. The jellium model therefore results, as will be shown below, in a T^6 law for metal resistance due to electron-electron interaction. In the more realistic pseudopotential interaction there is only partial cancellation. In this model the T^6 law should hold for temperatures down to $T \sim 10^{-2} \Theta_D$ (Θ_D is the Debye temperature), and the Landau-Pomeranchuk law for resistance should hold at lower temperatures.

2. The effective matrix element of the electron-electron interaction can be represented in the form

$$M(1.2; 3.4) = 4\pi e^2/q^2 \varepsilon(\mathbf{q}, \omega), \qquad (1)$$

where $\varepsilon(\mathbf{q}, \omega)$ is the dielectric constant of the metal as a function of the momentum transfer $\hbar\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_3 = \mathbf{p}_4 - \mathbf{p}_2$ and of the energy transfer $\hbar\omega = \varepsilon_1 - \varepsilon_3$ ($\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3, \mathbf{p}_4$ are the electron momenta before and after scattering, $\varepsilon_i \equiv \varepsilon(\mathbf{p}_i)$ is the electron energy).

Knowing the matrix element, we can write down the electron-electron collision integral:

$$\left(\frac{\partial n_1}{\partial t}\right)_{coll} = \frac{2\pi}{\hbar} \sum_{2,3,4} |M(1.2; 3.4)|^2 \{(1 - n_1)(1 - n_2) n_3 n_4$$
$$- n_1 n_2 (1 - n_3)(1 - n_4)\} \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4), \qquad (2)$$

where $n_i \equiv n(\mathbf{p}_i)$ is the electron distribution function.

The temperature dependence of the collision integral, and hence of the resistance, is determined essentially by the character of the matrix element, and the T^2 law for the resistance corresponds to the assumption that the matrix element $M(1, 2; 3, 4)$ does not depend on the energy transfer $\hbar\omega$. It is just this situation which obtains obviously, in the case of an interaction described by a potential that depends only on the coordinates.

Allowance for the possibility of phonon exchange between the electrons leads to a dependence of $M(1, 2; 3, 4)$ on the energy transfer, and this alters significantly the temperature dependence of the collision integral. To elucidate this circumstance, we consider first the simplest model, the jellium model[4], in which $\varepsilon(\mathbf{q}, \omega)$ is given by

$$\varepsilon(\mathbf{q}, \omega) = 1 + 1/(aq)^2 - \Omega_i^2/\omega^2, \qquad (3)$$

where a is the screening radius and Ω_i is the ion plasma frequency,

$$a^2 = \varepsilon_F / 6\pi e^2 n_e, \qquad \Omega_i^2 = 4\pi e^2 Z n_c / M_i$$

(ε_F is the Fermi energy, n_e is the electron density, Z is the valence, and M_i is the ion mass). This formula is valid of $\omega \ll q v_F$ and $\hbar\omega \ll \varepsilon_F$ (v_F is the limiting Fermi velocity).

Substituting (3) in (1), we represent the matrix element of the electron-electron interaction in the form

$$M(1.2; 3.4) = M_0(1.2; 3.4) F(q, \omega),$$

where

$$M_0(1.2; 3.4) = \frac{4\pi e^2}{q^2 + a^{-2}}, \qquad F(q, \omega) = \frac{\omega^2}{\omega^2 - \omega_s^2(q)}, \qquad (4)$$

$\omega_s(\mathbf{q})$ is the frequency of a phonon with wave vector $\mathbf{q}$ and $\hbar\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_3$ is the momentum transfer.

The quantity $M_0(1, 2; 3, 4)$ coincides with the matrix element describing screened Coulomb interaction, while the factor F describes the attenuation of the effective interaction between the electrons at low energy transfers, an attenuation due to cancellation of the Coulomb repulsion by the attraction due to virtual-phonon exchange.

To calculate the collision integral it is necessary to substitute (4) in (2) and assume that only small energy transfers, $\hbar\omega \sim T$, are significant. If the function F were equal to unity, we would obtain the well known T^2 law for the collision integral, and hence also for the resistance. But at small energy transfers the function F is significantly different from unity. Assuming typical

momentum transfer values $\hbar q \approx 2p_F$ (p_F is the Fermi momentum), we obtain for the function F at $\hbar\omega \ll \Theta_D$:

$$F(q, \omega) = -(\hbar\omega/\Theta_D)^2.$$

Thus, at low energy transfers the effective electron-electron interaction is attraction (and not repulsion as without allowance for phonon exchange), and the interaction intensity is decreased by a factor $(\Theta_D/\hbar\omega)^2$. As a result, an additional factor $(T/\Theta_D)^4$ appears in the expression for the collision integral.

Solving the kinetic equation

$$e\mathbf{E}\mathbf{v}\partial n/\partial\varepsilon = (\partial n/\partial t)_c$$

with the collision integral (2) (E is the electric field), we easily obtain the electron current and calculate the electric conductivity σ of the metal, due to the electron-electron collisions. As a result we obtain

$$\sigma = \xi \frac{e^2 n_e}{m}\frac{\hbar\varepsilon_F}{T^2}\left(\frac{\Theta_D}{T}\right)^4, \tag{5}$$

where ξ is a numerical factor of the order of unity.

We see that the contribution of the electron-electron collisions to the resistance of a normal model should decrease in the jellium model at low temperatures not like T^2 but like T^6.

3. The T^6 law obtained in the preceding section for the resistance is a result of the fact that the dielectric constant behaves in the jellium model like ω^{-2} as $\omega \to 0$. This behavior of the dielectric constant is in turn the consequence of the assumption, on which the jellium model is based, that all three types of elementary interactions in the system (electron-electron-ion, and ion-ion) can be regarded as classical Coulomb interactions. Actually, however, allowance for quantum effects causes the matrix elements of the elementary interactions to differ from one another and to differ from $V_c(q) = 4\pi e^2/q^2$ [5,6]. As a result, the dielectric constant has a pole not at $\omega^2 = 0$ but at $\omega^2 = \omega_A^2(q)$, where $\omega_A(q)$ is a certain frequency different from zero. The effective matrix element of the electron-electron interaction is determined as before by formula (4), in which the factor $F(q, \omega)$ must be substituted in the form

$$F(q, \omega) = (\omega^2 - \omega_A^2(q))/(\omega^2 - \omega_s^2(q)). \tag{6}$$

An important role in the calculation of the resistance is played by the behavior of the factor F at $\hbar q \sim 2p_F$ and $\hbar\omega \sim T$. Introducing the temperature $\Theta_A \equiv \hbar\omega_A(\hbar^{-1}2p_F)$, we see that two types of metals are possible in principle:

1) Metals with $\Theta_A \ll \Theta_D$. For these metals we can disregard ω_A^2 in the numerator of (6) at temperatures $\Theta_A \ll T \ll \Theta_D$. The effective electron-electron collisions should therefore make a contribution proportional to T^6 to the resistance of the metal in this temperature region. The main effect governing the resistance of such metals at $\Theta_A \ll T \ll \Theta_D$ (of course, for very pure samples), is the electron scattering by phonons, which leads to Bloch's law $\sigma^{-1} \sim T^5$. At $T \ll \Theta_A$, the Landau-Pomeranchuk law $\sigma^{-1} \sim T^2$ should hold, but even in this

	Li	Be	Na	Mg	Al	K	Ca	Cu	Zn
Y	$4\cdot10^{-3}$	10^{-2}	$2.5\cdot10^{-3}$	$7.2\cdot10^{-3}$	10^{-2}	$2\cdot10^{-3}$	$6.5\cdot10^{-3}$	10^{-4}	$6\cdot10^{-3}$

temperature region the resistance should be smaller by a factor $(\Theta_D/\Theta_A)^4$ than called for by the Landau-Pomeranchuk theory.

2) Metals with $\Theta_A \sim \Theta_D$. In these metals, the effective matrix element of the electron-electron interaction is of the same order as that of the Coulomb interaction at all temperatures. The resistance of a pure sample should behave at low temperatures in accord with the Landau-Pomeranchuk law.

4. We shall show now that the inequality $\Theta_A \ll \Theta_D$ does indeed hold for a number of metals in the pseudopotential model, which (for metals with ion dimensions small in comparison with the lattice constant) is a more realistic model than the jellium model. We use for this purpose the following formulas for the frequencies $\omega_S(q)$ and $\omega_A(q)$ [4]:

$$\omega_A^2(q) = \Omega_i^2 - V_c^{-1}(q)|v(q)|^2,$$
$$\omega_s^2(q) = \omega_A^2(q) + |v(q)|^2 / V_c(q)\varepsilon_e(q, 0), \tag{7}$$

where $v(q)$ is the amplitude of the electron-phonon interaction and $\varepsilon_e(q, 0) = 1 + (aq)^{-2}$ (we disregard the influence of Umklapp processes). In the pseudopotential model, in which one starts with an electron-ion interaction in the form

$$V(r) = -e^2 Z/r + \beta\delta(\mathbf{r}), \tag{8}$$

the quantity $v(q)$ is given by [5]

$$v(q) = -\frac{4\pi i e^2 Z}{q}\left(\frac{Zn_e}{M_i}\right)^{1/2}\left(1 - \frac{\beta q^2}{4\pi e^2 Z}\right). \tag{9}$$

Substituting (9) in (7) and assuming that $\hbar q = 2p_F$, we obtain

$$\Theta_A^2 = 2Y\Theta_D^2/Z, \tag{10}$$

where $\Theta_A = \hbar\Omega_i$ and $Y = \beta p_F^2/\pi e^2\hbar^2$. The values of Y for a number of metals are listed in the table (the data for the parameters β and p_F were taken from [5]).

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[1] F. Bloch, Zs. Phys., 59, 208 (1930).
[2] L. D. Landau and I. Ya. Pomeranchuk, Zh. Eksp. Teor. Fiz. 7, 379 (1937).
[3] J. Ziman, Electrons and Phonons, Oxford, 1960.
[4] D. Pines, Elementary Excitations in Solids, Benjamin, 1963, Chap. 5.
[5] W. Harrison, Pseudopotentials in Metal Theory, Benjamin, 1966.
[6] E. G. Brovman and Yu. Kagan, Zh. Eksp. Teor. Fiz. 52, 557 (1967); 57, 1329 (1969) [Sov. Phys.-JETP 25, 365 (1967); 30, 721 (1970)].

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38