On the theory of elementary-excitation spectrum in liquid He^4

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A theory of bound excitation states in a Bose system with a condensate is developed. The theory explains, in particular, the spectral branch below the excitation two-roton decay threshold observed in He^4 light-scattering experiments[2] but not observed in neutron experiments.[1] The existence of a singular continuation of the one-particle branch beyond the two-roton decay threshold is established; the main results of Cowley and Woods' experiments [1] can be interpreted if this circumstance is taken into account.

1. INTRODUCTION

We consider here the question of the origin of new branches of the spectrum of elementary excitations in liquid He^4, obtained in recent experiments on neutron scattering[1] and on Raman scattering of light[2]. The existence and the properties of different branches of the spectrum of He^4 are predicted here on the basis of the microscopic theory of a Bose system with condensate, a theory that establishes the distinguishing features of this system: 1) a distinct relation between the many-particle excitations ("bound states of quasiparticles") and the poles of the many-particle Green's functions (Sec. 2); 2) The existence of a singular continuation of the single-particle branch beyond the threshold of decay into rotons (Sec. 3).

It is shown in Sec. 2 that any excitation that has a bearing on neutron experiments, whether single-particle or many-particle, is described by the first Green's function $G^{(1)}$, the poles of which coincides with the poles of the response function to perturbations of the density χ; it is shown further that there exist excitations that do not appear in neutron experiments; on the other hand, the entire set of possible excitations is accounted for by the poles of the Green's function $G^{(2)}$ (which include also the poles of $G^{(1)}$). These circumstances make it possible to explain why experiments on scattering of light[2] reveal a branch with energy below the threshold for decay into rotons, a branch not observed in neutron experiments (this is discussed briefly in a paper by one of the authors[3]).

In Sec. 3 we explain the properties of the branch beyond the threshold, which were revealed by neutron scattering[1].

The theory developed below makes use only of the assumption that a condensate exists, and also of the main experimental data on the photon-roton form of the single-particle branch and its termination at the threshold for decay into rotons[1,4].

2. MANY-PARTICLE EXCITATIONS IN A BOSE SYSTEM WITH CONDENSATE

1. One of the causes of the appearance of new branches in the spectrum of a Bose system with condensate is the formation of bound states of excitations[5] (another cause is discussed in Sec. 3)[1)]. We consider first excitations that can be observed in experiments with neutron scattering. Obviously, they correspond to poles of the response to perturbations of the density χ. Graphically, χ is represented by an aggregate of dia-

FIG. 1

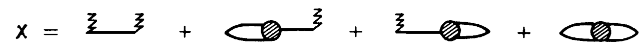


FIG. 2

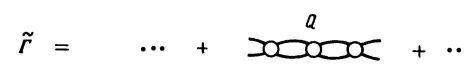


FIG. 3

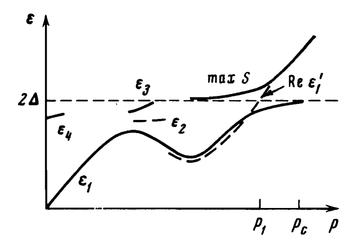


FIG. 4

grams to the two ends of which one can connect potential lines. In the case of a Bose system with condensate, there are two possibilities for such ends (Fig. 1; the solid line describes the particles in excess of the condensate, and the wavy line the condensate particles). This enables us to assume that the set of poles of χ should include also the energies of the single-particle excitations and of the bound pairs[2)] (Fig. 2).

An indication of the formation of bound pairs is the presence of extrema in the spectrum of the single-particle excitations[5], namely, integration with respect to the momenta near the extrema in each link of the latter diagrams corresponding to the scattering of two particles (Fig. 3) leads to a logarithmic singularity, so that the denominator $\tilde{\Gamma}$, which is equal to $1 + Q\ \ln(\ldots)$, can vanish at an arbitrarily weak effective interaction of quasiparticles Q of suitable sign[3)]. For example, for a pair of rotons, the equation

$$\tilde{\Gamma}^{-1} \sim \left[1 + Q(p)\ln\frac{\alpha}{2\Delta - \varepsilon}\right] \tag{1}$$

has a solution $\epsilon = \epsilon_2$ (Fig. 4) at arbitrary[4)] $Q < 0$.

Being interested in the poles of χ, it is natural for us to confine ourselves to only those poles of $\tilde{\Gamma}$, which are conserved upon "closing of the ends" of $\tilde{\Gamma}$ (Fig. 5). These poles $\tilde{\Gamma}$ are automatically poles of the self-energy

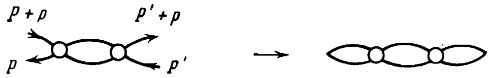


FIG. 5

FIG. 6

part Σ, which includes diagrams of the type of Fig. 6. Thus, these poles cannot belong to $G^{(1)}$. It may turn out that, describing bound pairs, they supplement the poles of $G^{(1)}$ in the set of the poles of χ. But this is not so, and the indicated poles do not correspond to any excitations in the system at all (see subsection 3 below), and the set of poles of χ is limited to the poles of $G^{(1)}$. At the same time, certain branches in the set of χ must be regarded as corresponding to bound pairs, and the aforementioned poles of $\tilde{\Gamma}$ serve as an indirect indication of the existence of these branches.

To prove the foregoing statements, we consider an aggregate of disgrams whose true ends can be independently either particle lines or potential lines (Fig. 7); the thin line denotes zero-order Green's function, and the thick one the exact functions; the circle represents the sum of all the permissible diagrams, including also the absence of a diagram. The indices i, k = 1, 2 in Fig. 7 correspond to the two possible directions of the arrows (Fig. 7, bottom). The diagrams for the function G_i and $\tilde{G}_i$ go over into one another when the directions of the arrows are reversed for all the intermediate Green's functions without change of momentum, so that $G_i \equiv \tilde{G}_i$.

Let us prove first that the poles of all the Green's functions of Fig. 7 coincide. This would automatically mean that the poles of $G^{(1)}_{ik}$ and χ coincide, since the poles of χ and Γ are known to coincide:

$$\Gamma = \nu + \nu\chi\nu.$$

We write down the exact equations for the functions of Fig. 7, using the irreducible self-energy parts $\tilde{\Sigma}_{ik}$, $\tilde{\Pi}$, and K_i, at the entrance and exit of which there are lines of either particles or potential; irreducibility means here that the diagrams cannot be separated vertically by crossing one line of the extra-condensate particles G^0 or potential ν (see Fig. 8; A and B are equivalent expressions).

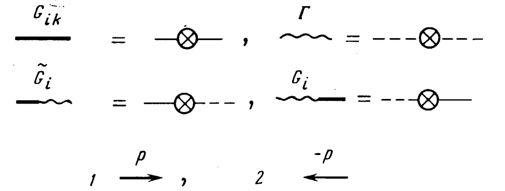


FIG. 7

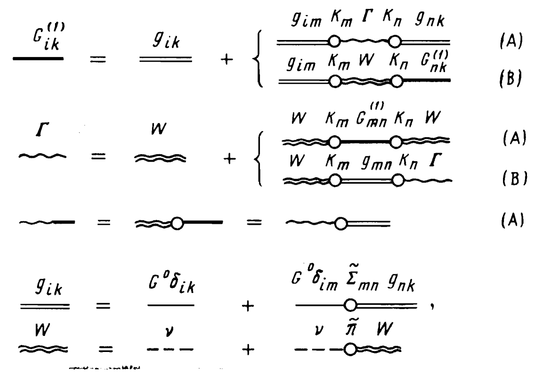


FIG. 8

It is seen from the equations of Fig. 8 that all the poles of $G^{(1)}_{ik}$, Γ, and G_i coincide (if we exclude the practically unlikely possibility of accidental vanishing of the vertex K at the poles of these functions). We note that, as is clear from Fig. 8B, at the pole W we have

$$G^{(1)}_{ik} = 0; \quad \Gamma, \Gamma^{-1} \neq 0;$$

and at the pole G_{ik} we have

$$\Gamma = 0; \quad G^{(1)}_{ik}, \quad [G^{(1)}_{ik}]^{-1} \neq 0.$$

The physical meaning of the foregoing proof is obvious: in a Bose system with condensate there takes place hybridization of the single-particle and paired excitations, the "bare" energies of which are characterized by the poles g_{ik} and W (the latter coincide with the aforementioned poles of $\tilde{\Gamma}$, see subsection 3); the hybridization constitutes an interaction between the excitations, which leads not only to a shift of the unperturbed (bare) energies, but also their duplication in each of the Green's functions of the excitations: if

$$H_0 = \varepsilon_1^0 a_1^+ a_1 + \varepsilon_2^0 a_2^+ a_2, \quad H_{int} \sim a_1^+ a_2 + a_1 a_2^+,$$

then the renormalized energies ϵ_1 and ϵ_2 enter as pole values in each of the functions $\langle T\tilde{a}_1\tilde{a}_1^+\rangle$ and $\langle T\tilde{a}_2\tilde{a}_2^+\rangle$.

Turning by way of example to the case of roton bound states, we have $\epsilon = \epsilon_2$ as the pole of W (see (1)). The appearance of this pole, which belongs also to the complete self-energy part

$$\Sigma = \tilde{\Sigma} + KWK = P(p)\ln[\alpha/(2\Delta - \varepsilon)] / \{1 + Q(p)\ln[\alpha/(2\Delta - \varepsilon)]\},$$

leads (under the condition $P/|Q| < \Delta$) to the appearance of an additional branch of $G^{(1)}$, $\epsilon = \epsilon_3$ (see [5]). In accordance with the preceding, this branch characterizes the bound state of a pair of rotons, which could be observed in principle in neutron experiments.

It is easy to see that the spectrum of the density oscillations in a Bose system with condensate (poles of χ) can contain, in principle, branches corresponding to excitations of a structure more complicated than described above (bound states of three and more particles), and the existence of the latter can be predicted in analogy with the case of the pair excitations (within the framework of the simple model of the "compact" Bose system [7], such branches can also be calculated directly). Indeed, since the branches of the pair excitations belong to $G^{(1)}$, the extrema of these branches serve as an indication of the existence of new bound states, in which the given pair excitation is contained as a separate component (see Fig. 3); obviously, similar reasoning can be extended also for the succeeding branches without limit, i.e., the functions $G^{(1)}$ and χ can contain, in principle, excitations with arbitrary numbers of particles. We note, however, that in practice the formation of complex excitations has low probability, since the conditions for the presence of extrema and for the proper sign of the irreducible vertices become cumulative, and the damping increases rapidly.

2. Although $G^{(1)}$ and χ can contain branches of complicated many-particle excitations, one cannot state that these functions contain the entire set of elementary excitations, for even among the bound pairs one can indicate such that certainly do not correspond to poles of χ.

The presence of pair excitations not connected with poles of χ is not a distinguishing feature of a Bose system with condensate. Let us consider, for example, in the case of a Fermi system. Here the poles of χ certainly do not include particle-hole pairs, for which the "internal" wave function $\varphi(\mathbf{x}-\mathbf{y})$ vanishes at the point $\mathbf{x}=\mathbf{y}$; indeed, in this case a state with excitation of the type

$$\int dx\,dy\,\Phi(\mathbf{x},\mathbf{y})\,\psi^+(\mathbf{x})\,\psi(\mathbf{y})\,|0\rangle,$$
$$\Phi(\mathbf{x},\mathbf{y}) = \exp\{ip(\mathbf{x}+\mathbf{y})/2\}\,\varphi(\mathbf{x}-\mathbf{y})$$

does not include density oscillations

$$\rho_{\mathbf{p}}|0\rangle \equiv \int dx\,dy\,\delta(\mathbf{x}-\mathbf{y})\,\psi^+(\mathbf{x})\,\psi(\mathbf{y})\,e^{ip(\mathbf{x}+\mathbf{y})/2}|0\rangle. \tag{2}$$

Such pairs include excitations with nonzero helicity $m = \mathbf{l}\cdot\mathbf{p}/|\mathbf{p}|$. We note that by virtue of the cylindrical symmetry of the system with excitation, all the excitations have a definite helicity; in particular, pair excitations, which are obviously characterized by poles of the exact four-prong vertex (Fig. 9) can be classified by helicity in accordance with the diagonality of Γ^* with respect to m:

$$\Gamma^*(|\mathbf{P}|, l, m;\ |\mathbf{P}'|, l', m';\ p) = \Gamma^{*(m)}(|\mathbf{P}|, l, |\mathbf{P}'|, l';\ p)\,\delta_{mm'}.$$

The fact that χ does not contain poles of $\Gamma^{*(m\neq 0)}$ is also clear from the fact that when the ends of $\Gamma^{*(m\neq 0)}$ are closed, so that diagrams of χ are produced (i.e., when Γ^* with $G^0(P+p)G^0(P)G^0(P'+p)G^0(P')$ is differentiated with respect to d^4p), the contribution vanishes from symmetry considerations.

For a Bose system with condensate, by virtue of the indistinguishability of particle-particle and particle-hole pairs, and also by virtue of hybridization with single-particle states, the wave function of the excitation containing density oscillations can have a form different from (2), but, in analogy with the foregoing, the poles of $\Gamma^{*(m\neq 0)}$ certainly do not belong to χ (they vanish when the "ends are closed"), as follows also from symmetry considerations.

The criterion for the appearance of the poles of $\Gamma^{*(m\neq 0)}$ follows from the analysis presented above. Indeed, since the three-prong vertex vanishes if $m\neq 0$, i.e., excitations with $m\neq 0$ cannot take part in the hybridization with single-particle excitations (the law of helicity conservation), the diagrams of Fig. 3 with $m\neq 0$ account for all the possible

$$\Gamma^{*(m\neq 0)} = \tilde{\Gamma}^{(m\neq 0)}.$$

For a pair of rotons, in particular, a bound state with $m\neq 0$ ($\epsilon=\epsilon_4$, Fig. 4) occurs for any $Q^{(m)}<0$:

$$\Gamma^{*-1} \sim \left[1 + Q^{(m)}(p)\ln\frac{\alpha}{2\Delta-\varepsilon}\right] = 0.$$

We see that although the poles of the vertex $\tilde{\Gamma}$ are only of auxiliary significance when $m=0$ (the bare energies of two-particle excitations described by $G^{(1)}$ and χ), when $m\neq 0$ the poles of $\tilde{\Gamma}\,(=\Gamma^*)$ characterize independent excitations that do not include density oscillations.

Branches with $m\neq 0$ cannot be observed in neutron experiments, where the scattering of the particles is single, but they can appear in effective second-order in the external perturbation, particularly in Raman scattering[2] (Fig. 10; in the experiment $p\to 0$ and $l=2$).

3. We have proved above that the bare poles of W and g_{ik} are not included among the poles of χ and $G^{(1)}$. They do belong, however, to the irreducible vertices $\tilde{\Gamma}$ and $\bar{\Gamma}$ (see (5) and (6), respectively, which represent terms the

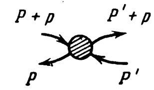
$\Gamma^*(\mathbf{P},\mathbf{P}';p) =$ 

FIG. 9

expressions (3) and (4) for the exact vertex Γ^*):

$$\Gamma^* = \tilde{\Gamma} + \left(\gamma + \int \tilde{\Gamma} G^{(1)} G^{(1)} \gamma\right) G^{(1)} \left(\gamma + \int \gamma G^{(1)} G^{(1)} \tilde{\Gamma}\right), \tag{3}$$

$$\Gamma^* = \bar{\Gamma} + \left(1 + \int \bar{\Gamma} G^{(1)} G^{(1)}\right) \Gamma \left(1 + \int G^{(1)} G^{(1)} \bar{\Gamma}\right), \tag{4}$$

$$\tilde{\Gamma} = \Gamma_0 + \left(1 + \int \Gamma_0 G^{(1)} G^{(1)}\right) W \left(1 + \int G^{(1)} G^{(1)} \Gamma_0\right), \tag{5}$$

$$\bar{\Gamma} = \Gamma_0 + \left(\gamma + \int \Gamma_0 G^{(1)} G^{(1)} \gamma\right) g \left(\gamma + \int \gamma G^{(1)} G^{(1)} \Gamma_0\right) \tag{6}$$

(the vertex $\tilde{\Gamma}$ is irreducible "with respect to one line of particles" G^0, see above, and $\bar{\Gamma}$ is irreducible "with respect to one line of the potential" ν, Γ_0 is irreducible in G^0 and ν, and the three-prong vertex γ is irreducible in G^0, ν, and G^0G^0). This raises the question whether the indicated "bare" poles are not included among the poles of $\Gamma^{*(m=0)}$ (as supplementary to the poles of χ and $G^{(1)}$), i.e., whether they characterize certain singular excitations in the system.

By representing Γ^* with the aid of (3) and (5) in the form

$$\Gamma^* = \Gamma_0 + \left(1 + \int \Gamma_0 G^{(1)} G^{(1)}\right) W \left(1 + \int G^{(1)} G^{(1)} \Gamma_0\right)$$
$$+ \left(\gamma + \int \tilde{\Gamma} G^{(1)} G^{(1)} \gamma\right) G^{(1)} \left(\gamma + \int \gamma G^{(1)} G^{(1)} \tilde{\Gamma}\right),$$

we verify that at a pole of g all the terms of Γ^* are finite (see subsection 1); it follows analogously from (4) and (6) that Γ^* is finite at a pole of W. Thus, the second terms in the right-hand side of (3) and (4) cancel exactly the divergence of the first terms in the poles of W and g (this cancellation can be intuitively explained by the fact that the "bare" functions W and g in $\tilde{\Gamma}$ and $\bar{\Gamma}$ are complemented to Γ and $G^{(1)}$, respectively, by the second terms of (3) and (4).

Let us see now whether the set of poles of Γ^* includes the poles of Γ_0. They certainly do not belong to the functions χ and $G^{(1)}$, namely, as $\Gamma_0\to\infty$ we have

$$\tilde{\Pi} = \Pi_0 + \int G^{(1)} G^{(1)} \Gamma_0 G^{(1)} G^{(1)} \to \infty, \quad W\to 0, \quad \Gamma\to 0 \text{ and}$$
$$\tilde{\Sigma}\to\infty, \quad g\to 0, \quad G^{(1)}\to 0.$$

We express Γ^* in terms of Γ_0:

$$\Gamma^* = \Gamma_0 + \left(\gamma + \int \Gamma_0 G^{(1)} G^{(1)} \gamma\right) g \left(\gamma + \int \gamma G^{(1)} G^{(1)} \Gamma_0\right)$$
$$+ \left(1 + \int \Gamma_0 G^{(1)} G^{(1)} + \gamma g \gamma + \left(\int \Gamma_0 G^{(1)} G^{(1)} \gamma\right) g\gamma\right) \Gamma \left(1 + \int G^{(1)} G^{(1)} \Gamma_0\right.$$
$$\left. + \gamma g \gamma + \gamma g \int \gamma G^{(1)} G^{(1)} \Gamma_0\right), \tag{7A}$$

$$\Gamma = \nu + \nu\left[\int G^{(1)} G^{(1)} + \gamma g \gamma + \left(\int G^{(1)} G^{(1)} \gamma\right) g\gamma + \gamma g \int \gamma G^{(1)} G^{(1)}\right.$$
$$+ \gamma g \int \gamma G^{(1)} G^{(1)} \Gamma_0 G^{(1)} G^{(1)} + \left(\int G^{(1)} G^{(1)} \Gamma_0 G^{(1)} G^{(1)} \gamma\right) g\gamma$$
$$\left. + \gamma g \left(\int \gamma G^{(1)} G^{(1)} \Gamma_0 G^{(1)} G^{(1)} \gamma\right) g\gamma\right]\Gamma,$$

$$g = G^0 + G^0\left(\int \gamma G^{(1)} G^{(1)} \gamma + \int \gamma G^{(1)} G^{(1)} \Gamma_0 G^{(1)} G^{(1)} \gamma\right) g. \tag{7B}$$

We note that in (3)–(7) we are interested everywhere only in the case $m=0$; for $m\neq 0$ we have $\Gamma^* = \tilde{\Gamma} = \bar{\Gamma} = \Gamma_0$.

It is important in what follows that the residue of the function $\Gamma^*(\mathbf{P},\mathbf{P}';\,\mathbf{p},\epsilon)$ at any of its poles $\epsilon=\epsilon_0(\mathbf{p}$ is

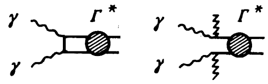


FIG. 10

fixed) has the multiplicative form:

$$\Gamma^* \sim f(\mathbf{P},\mathbf{P}')\,/\,(\varepsilon-\varepsilon_0),\quad f(\mathbf{P},\mathbf{P}') = \varphi(\mathbf{P})\varphi^*(\mathbf{P}')$$

(in this case $\varphi(\mathbf{P})$ plays the role of the internal wave function of the bound state of the pair, corresponding to the pole $\epsilon = \epsilon_0$; a similar property is possessed by any Green's function in a representation in which it is not diagonal). In the case of a real pole, the indicated multiplicativity follows directly from the representation

$$\Gamma^*(\mathbf{P},\mathbf{P}';\ p) = \sum_{\varepsilon_s} \langle a_{\mathbf{P}/2+\mathbf{p}}\, a^+_{\mathbf{P}/2-\mathbf{p}}\rangle_{0s} \langle a_{\mathbf{P}'/2+\mathbf{p}}\, a^+_{\mathbf{P}'/2-\mathbf{p}}\rangle_{0s} \times\left\{\frac{1}{\varepsilon-\varepsilon_s+i\delta} - \frac{1}{\varepsilon+\varepsilon_s-i\delta}\right\}. \tag{8}$$

(We do not consider here the little-likely possibility of random degeneracy of the levels ϵ_s.)

In the case of complex ϵ_0, the multiplicativity is evident from a comparison of the approximate formula for Γ^* near ϵ_0,

$$\Gamma^*(\mathbf{P},\mathbf{P}';\ p) \sim \frac{f(\mathbf{P},\mathbf{P}')}{\varepsilon - \operatorname{Re}\varepsilon_0 + i\operatorname{Im}\varepsilon_0} \sim \int d\varepsilon_s \frac{f(\mathbf{P},\mathbf{P}')\operatorname{Im}\varepsilon_0}{(\varepsilon_s-\operatorname{Re}\varepsilon_0)^2+\operatorname{Im}\varepsilon_0^2}\,\frac{1}{\varepsilon-\varepsilon_s+i\delta},$$

with the exact expression (8).

From the multiplicativity of

$$f \equiv \lim_{\varepsilon\to\varepsilon_0}(\varepsilon-\varepsilon_0)\Gamma^*$$

follows an analogous property for

$$f_0 = \lim_{\varepsilon\to\varepsilon_0}(\varepsilon-\varepsilon_0)\Gamma_0$$

since the second and third terms in the right-hand side of (7A) make a patently multiplicative contribution. Substituting in (7)

$$\Gamma_0 = f_0\,/\,(\varepsilon-\varepsilon_0) = \varphi_0(\mathbf{P})\varphi_0(\mathbf{P}')\,/\,(\varepsilon-\varepsilon_0),$$

we find that

$$f(\mathbf{P},\mathbf{P}';p) \equiv 0,$$

i.e., the poles of Γ_0 cannot belong to Γ^*.

The fact that the poles of the exact vertex $\Gamma^*(m=0)$ do not include poles of the irreducible vertices is a common property of all quantum systems; indeed, for a system without a condensate the result can be easily established from the equations

$$\Gamma^* = \Gamma_0 + \left(1+\int\Gamma_0 G^{(1)}G^{(1)}\right)W\left(1+\int G^{(1)}G^{(1)}\Gamma_0\right),$$

$$W = v + v\left(\int G^{(1)}G^{(1)} + \int G^{(1)}G^{(1)}\Gamma_0 G^{(1)}G^{(1)}\right)W.$$

By analogy, the absence of the poles of Γ_0 from $\tilde{\Gamma}$ and Γ follows from (5) and (6).

The result obtained for four-prong vertices can be generalized to the case of vertices with an arbitrary number of prongs: poles of any irreducible vertex do not belong to the exact vertex and do not correspond to any physical excitations whatever. For a Bose system with condensate, this leads to an important consequence: if the hybridization, which decreases the number of particles, is not forbidden by the helicity conservation law, all the poles of the many-pronged vertex are contained (are duplicated) in the set of poles of the vertex with smaller number of prongs. In other words, only the helicity conservation law prevents $G^{(1)}$ and χ from containing the poles of all the possible excitations of a Bose system with condensate (the single-particle states are limited by the condition $m = 0$). There is no analogous limitation for two-particle excitations, so that the set of poles of $G^{(2)}$ includes all the possible excitations.

3. SINGLE-PARTICLE EXCITATIONS BEYOND THE THRESHOLD OF DECAY INTO ROTONS

The branches complementing the phonon-roton branch (ϵ_1) in the spectrum of a Bose system with condensate do not necessarily correspond to bound states of the excitations. We consider here a new single-particle branch (ϵ_1'), which constitutes an "unphysical satellite" of branch ϵ_1 ahead of the threshold of decay into rotons, and a peculiar continuation of this branch in the region beyond the threshold. The appearance of ϵ_1' is the result of a break produced in the single-particle branch by a branch point of the Green's function $G^{(1)}$, characterizing the threshold of decay into rotons. We shall show that the existence of the new branch explains qualitatively the behavior of the dynamic form factor $S(p, \epsilon)$, which was established in experiments on neutron scattering in He^4[1].

According to the preceding, the linear response function χ has no poles that differ from the poles of the single-particle Green's function, so that the bound states of rotons with $m \neq 0$, which do not enter in the set of the poles of $G^{(1)}$, can certainly not be observed in experiments with neutron scattering. The analysis that follows shows that to explain the neutron experiments[1] it is likewise not necessary to assume the existence of the bound states contained in χ.

We start from the equation

$$[G^{(1)}]^{-1} = A^{-1}\left\{\varepsilon - \varepsilon_p^0 + P(p)\left[\ln\frac{\alpha}{2\Delta-\varepsilon} + iF(p)\theta(\varepsilon-\varepsilon_1)\right]\Big/\left[1+Q(p)\left[\ln\frac{\alpha}{2\Delta-\varepsilon}+iF(p)\theta(\varepsilon-\varepsilon_1)\right]\right]\right\}. \tag{9}$$

We have explicitly separated here from the total self-energy part Σ the component connected with the decay into two excitations; the logarithmic singularity characterizes the two-roton decay, virtual or real[4]; the remaining diagrams are taken into account in the regular term ϵ_p^0, which is real if ϵ is in the region where a real decay into three and more excitations is impossible. The functions P, Q, α, and F are real if the excitations of the branch ϵ_1 are stable up to the threshold for the decay into rotons. The function α includes the non-pole contribution in the integration of the product of two Green's functions. The term with F characterizes all processes of real decay into two excitations for energies $\epsilon - \epsilon_1$, including two-roton decay; for small $\epsilon > \epsilon_1$ we have

$$F \sim (\varepsilon-\varepsilon_1)^2.$$

As shown by Pitaevskiĭ[4], decay into rotons (the logarithmic singularity in (9)) leads to a termination of the branch ϵ_1: when $p > p_c$ the equation $[G^{(1)}]^{-1} = 0$ has no solutions whatever in the vicinity of the point $\epsilon = 2\Delta$, and has no solutions at all if a cut along the line $2\Delta < \epsilon < \infty$ is implied in the logarithm of the expression (9). On the other hand, it is physically obvious that in the region of large momenta there should exist at least damped excitations, and furthermore with energy close to $p^2/2m$ (there is no time for the fast particle to acquire a cloud of virtual excitation; this agrees also with the Feynman formula $\epsilon(p) = p^2/2mS(p)$, $S(p \to \infty) \to 1$). The need for a definite continuation of the single-particle branch into the region $p < p_c$ becomes particularly obvious for the case of a Bose system with arbitrarily "weak coupling": P/Δ, $Q \sim \eta \to 0$[5)[7]]; it is hardly likely that the corrections could lead to a vanishing of the entire section of the spectrum with $p > p_c$ for all $\eta \to 0$.

We shall show that the termination of the branch ϵ_1 does not contradict the existence of single-particle excitations with $p > p_c$. We turn to the arbitrary case of the decay of excitations of the real branch. The presence of a decay threshold $\epsilon = \epsilon_c$, i.e., of a branch point for the Green's function on the real axis, denotes the existence of not one but of two different analytical continuations of the Green's function from the positive semiaxis to the lower half-plane (or from the negative one to the upper half-plane): from the sections $\epsilon < \epsilon_c$ ($G^{I}_{R,A}$) and $\epsilon > \epsilon_c$ ($G^{II}_{R,A}$); adjacent to these sections are different sheets of the Riemann surface of the function G_R (G_A). The poles of each of the indicated analytic continuations have the physical meaning of frequencies of weakly-d damped excitations, provided only these poles are near the boundary of the analyticity region of G_R (G_A), namely the physical real axis. The poles of $G^{II}_{R,A}$ near the semiaxis $\epsilon < \epsilon_c$ or of $G^{I}_{R,A}$ near $\epsilon > \epsilon_c$ do not correspond to weakly-damped excitations (they lie in the unphysical region of the Riemann surface G). The pole branches of the neighboring analytic continuations of $G^{I}_{R,A}$ and $G^{II}_{R,A}$ can join into a single curve (for example, in the case of decay with emission of phonons, when the pole beyond the threshold moves off smoothly from the real axis), but can also diverge, namely, one of the branches (real) terminates in this case at the branch point, and the other (complex) goes past this point and goes over to the unphysical sheet of the Riemann surface G on going into the momentum region where the real branch exists. It is precisely the latter case which includes decay into rotons, when the non-analytic term is represented by a logarithm with a coefficient that does not vanish at the branch point. The phonon-roton branch ϵ_1, bounded by the interval $0 < p < p_c$, is the pole curve of the analytic continuation of G from the section $0 < \epsilon < 2\Delta$ (G^{I}_{R}); it can be easily verified that the analytic continuation of G from the section $2\Delta < \epsilon < \infty$ (see (9) with the substitution

$$\left.\ln\frac{\alpha}{2\Delta-\varepsilon} \to \ln\frac{\alpha}{\varepsilon-2\Delta} + i\pi\right)$$

has a complex pole ϵ_1' for all p.[6)]

Let us describe in detail the case P/Δ, $Q \ll 1$[7)]. If we exclude the region near the threshold, then the assembly of the physical branches $\epsilon_1 < 2\Delta$ and ϵ_1' ($\mathrm{Re}\,\epsilon_1' > 2\Delta$) "imitate" closely a single unperturbed branch ϵ^0_p, and differs from the latter only by a small correction (which is real up to the threshold and complex beyond it); the residue at the poles ϵ_1 and ϵ_1' is close to unity, and the pole contribution upon integration of the Green's functions with respect to energy is large in comparison with the contribution of the cut. The specific feature of the branch point $\epsilon = 2\Delta$ becomes manifest, however, in the unphysical continuation of the branch ϵ_1' under the threshold ($\mathrm{Re}\,\epsilon_1' < 2\Delta$) where, as before, $\epsilon_1' \approx \epsilon^0_p$, and in the character of the "near-threshold" section $p_1 \lesssim p < p_c$ of the branch ϵ_1 (p_1 and p_c are defined by the equations $\epsilon^0_{p_1} = 2\Delta$, $\epsilon_1(p_c) = \epsilon^0_{pc} - P/Q = 2\Delta$; see Fig. 4; the unphysical pole is shown dashed). In the near-threshold region, the branch ϵ_1 moves appreciably away from ϵ^0_p, and the residue at the pole ϵ_1 decreases to zero ($\sim (2\Delta - \epsilon)/\alpha$); it is remarkable that here ϵ_1 coexists with another physical pole ϵ_1' ($\mathrm{Re}\,\epsilon_1' > 2\Delta$), at which the residue is always of the order of unity, since, being complex, it never comes too close to the point $\epsilon = 2\Delta$. It is important that the region $p_1 \lesssim p < p_c$ is not small if P/Δ and Q are arbitrarily small (provided that $P/\Delta Q \sim 1$).

Let us verify that the theoretical picture of the behavior of the dynamic form factor $S(\mathbf{p}, \epsilon) \sim \mathrm{Im}\,\chi$, constructed with allowance for the poles ϵ_1 and ϵ_1' of χ, accounts for the principal features of the neutron experiments[1], even if we confine ourselves in the comparison to the model in which P/Δ, $Q \ll 1$. The function $S(\mathbf{p}, \omega)$ is the sum of three terms, two of which are connected with the poles ϵ_1 and ϵ_1' and the third (the many-particle term proper) is connected with the function F from (9).

In the region $p < p_1$, where the residue at ϵ_1 is close to unity and ϵ_1' is an unphysical satellite of ϵ_1, the principal contribution is the δ-like contribution of the pole ϵ_1; adjacent to the δ-peak is a "hill" generated by the function F (the function F is equal to zero at the point $\epsilon = \epsilon_1$, as shown above, and at the threshold $\epsilon = 2\Delta$ the form factor $S(\mathbf{p}, \epsilon)$ vanishes vigorously because of the divergence of the logarithm in (9)). Beyond the threshold, the contribution to S is made by the tail of the peak generated by the pole ϵ_1', the maximum of which is in the unphysical section of the real axis $\epsilon = \mathrm{Re}\,\epsilon_1'$; near the threshold $\epsilon = 2\Delta$, this tail is distorted by the presence of a branch point, so that when ϵ increases from the threshold 2Δ, the form factor $S(\mathbf{p}, \epsilon)$ first increases from zero and then decreases in proportion to

$$\frac{\mathrm{Im}\,\varepsilon_1'}{(\varepsilon-\mathrm{Re}\,\varepsilon_1')^2+(\mathrm{Im}\,\varepsilon_1')^2}$$

(i.e., it forms a blurred peak beyond the threshold).

In the region $p > p_1$, the height of the peak beyond the threshold increases sharply, reaching the maximum value; this corresponds to the transition of the branch ϵ_1' beyond the threshold (to its conversion into a physical branch); at the same time, the contribution of the δ-peak decreases to zero (the residue at the pole ϵ_1 tends to zero as $p \to p_c$); both processes correspond to satisfaction, for all p, of the sum rule

$$\frac{2m}{p^2}\int \varepsilon S(\mathbf{p},\varepsilon)\,d\varepsilon = 1$$

(see Fig. 4; above the threshold, the curve $\mathrm{Re}\,\epsilon_1'$ approaches the beyond-threshold peak of the form factor max S and merges with it). In the region $p > p_c$ we are left only with the peak of the pole ϵ_1' (if we neglect the small contribution of F) and its maximum $\epsilon_m \approx \mathrm{Re}\,\epsilon_1' \approx \epsilon^0_p$ approaches $p^2/2m$.

The described picture corresponds to the result of[1] (see Figs. 3, 5, 7 of[1]; $p \approx 2.4$ Å; $p_c \approx 3.6$ Å). In particular, at the point $p = 2.45$ Å^{-1} (Figs. 3 and 7 of[1]), one can see clearly two peaks of approximately equal height, corresponding to two physical poles ϵ_1 and ϵ_1' that coexist simultaneously with residues of the order of unity. In comparison with the experiment, it is necessary only to take into account the fact that the finite temperature smears out the δ-peak, and furthermore in such a way that for the experimentally investigated momenta (for which the δ-peak is not far enough from the threshold) the maximum connected with F is not observed

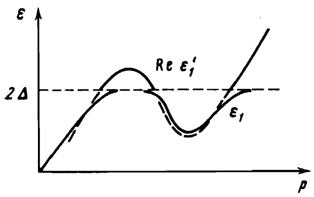


FIG. 11

separately from the δ-peak, all that is seen is a dip near the thresholds separating the peaks ϵ_1 and ϵ_1' (Figs. 3 and 5 of [1]). The single-particle origin of the peak beyond the threshold explains also the oscillatory variation of its contribution to

$$\frac{2m}{p^2}\int \varepsilon S(\mathbf{p}, \varepsilon)\, d\varepsilon$$

(Fig. 13 of [1]), which duplicates the course of the ϵ_1 curve. The indicated contribution is connected with the tail of the peak generated in the "unphysical" region by the pole ϵ_1', which is close to ϵ_1.

In conclusion, we mention a case that could arise if it were experimentally possible to obtain sufficiently strongly supercooled liquid He^4 at high pressure, such that the doubled roton minimum would turn out to be less than the pre-roton maximum (such a situation is of fundamental interest in the study of coherent crystallization [7]). It can be shown that in this case the ϵ_1 curve would consist of topologically unconnected pieces (Fig. 11).

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1)Jackson [6] pointed out one other possibility of explaining the maxima of the form factor of a Bose system with condensate, namely the use of a kinematic approach.

2)When speaking of pairs in the case of a Bose system with condensate, we cannot distinguish between particle pairs and "particle-hole" pairs. The two possibilities combine in a certain sense, since the "thick" lines can reverse direction.

3)The vertex Q is irreducible in the sense that its diagrams cannot be separated vertically by intersecting one or two lines of the extra-condensate particles (briefly speaking, Q is irreducible in G^0 and G^0G^0).

4)No account is taken in (1) of the damping which is obligatory for branches lying above the phonon-roton branch ϵ_1—see (9); this, however, is immaterial for the analysis in the present section.

5)In this model ("compact" Bose system) the potential energy of the pair is assumed to be small in comparison with the kinetic energy: $\eta = mp_0 |\nu_{p_0}| \ll 1$, but by virtue of the compactness ($p_0 \ll n^{1/3}$), the potential energy per unit volume is of the order of or larger than the kinetic energy $mn|\nu_{p_0}|p_0^{-2} \gtrsim 1$; in this case P/Δ and

$$Q \sim \eta \ll 1,\ \varepsilon_p{}^0 = \left[\frac{p^2}{2m}\left(\frac{p^2}{2m} + 2n\nu_p\right)\right]^{1/2}.$$

6)It can be shown that a similar situation obtains in the case of decay into excitations with parallel momenta (although the concrete character of the distortions introduced by the threshold non-analyticity is different here).

7)$P > 0$, for otherwise, as can easily be verified, the branch $\epsilon_1'(p)$ lies in the upper half-plane, which is forbidden by the Lehmann relations; $Q > 0$ near the threshold point, for in the region where $Q < 0$ the exact spectral curve $\epsilon_1(p)$ cannot approach the threshold even if the curve $(\epsilon_p^0 - P/Q)$ crosses the threshold.

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