# One-Dimensional motion of a low density plasma in a magnetic field

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A kinetic equation is derived which describes the drift motion of a collisionless plasma in a magnetic field. Its solution for the case of flow of a plasma into vacuum is investigated. It is shown that discontinuities are formed in a quasi-neutral flow. The oscillatory structure of a weak discontinuity is determined.

## INTRODUCTION

The purpose of the present paper is to find the exact nonlinear nonstationary solutions of the kinetic equation for a collisionless plasma in conjunction with Maxwell's equation. As is well known, in view of the complexity of the kinetic equation, it is quite difficult to find its solution, even a numerical one. In the absence of a magnetic field, a decisive simplification can be made if the plasma is quasineutral

$$N_i = N_e. \tag{1}$$

This condition corresponds to the limit

$$D \to 0 \tag{2}$$

(D is the Debye radius). In this limit the system of equations has no parameter with the dimension of length, making it possible in certain problems of physical interest to decrease the number of variables and find a number of solutions of the self-similar type and of the simple-wave type[1,2].

We emphasize that the transition to equations that do not contain a length parameter is of great fundamental significance. In kinetics such a limit is the analog of the transition to the case of an ideal liquid in hydrodynamics. The problem of formation of strong and weak discontinuities in a flow, and of other singularities, can be formulated distinctly only in this limit. In equations where the length parameter is preserved, all such singularities become smeared out and no longer have an exact meaning. Yet the presence or absence of discontinuities is a most important qualitative characteristic of motion. (For an investigation of the singularity connected with the breaking of the front of a simple wave see[3].)

It is clear from the foregoing that great interest attaches to such simplifications in the general case of a plasma that moves in an arbitrary magnetic field. In a magnetic field, however, there are also parameters of the dimension of length, namely the Larmor radii $\rho_{Hi}$ and $\rho_{He}$ of the ions and electrons. It is clear that the dimensional situation of "an ideal liquid" arises only when the magnetic field is sufficiently strong, i.e., when all the quantities vary little over the Larmor radius of the ions. Then we can expect[1)]

$$\rho_{Hi} \sim \bar{v}_i Mc / eB \to 0, \quad \rho_{He} \to 0. \tag{3}$$

These limits correspond to the so-called drift approximation. The kinetic equations in this approximation were obtained by Rudakov and Sagdeev[4]. The system of equations, however, turns out to be quite complicated, and, so far as we know, not even one nonlinear nonstationary kinetic problem was solved with its aid to date. By suitable joint utilization of conditions (2) and (3), we carry out further simplification of the equations for the case of one-dimensional nonstationary motion and apply these equations to the solution of the particular problem of expansion from a plasma-filled half space into vacuum.

## 2. THE BASIC EQUATIONS

In the drift approximation, the ion distribution function f depends on the projection $v_\parallel$ of the velocity on the direction of the magnetic field **B**, on the absolute value of the projection $v_\perp$ of the velocity on the plane perpendicular to the magnetic field, and on the coordinates **r** of the particle. In place of $v_\perp$ we introduce the variable $\mu = v_\perp^2/2B$. The volume element in velocity space can be represented in the form

$$d^3v = 2\pi B dv_\parallel d\mu.$$

According to Rudakov and Sagdeev[4], the equation for f is

$$\frac{\partial f}{\partial t} + v_\parallel \boldsymbol{\varepsilon} \nabla f + \mathbf{w} \nabla f + \left\{ \boldsymbol{\varepsilon} \left( \frac{e}{M} \mathbf{E} - \frac{\mu e}{Mc} \nabla B - \frac{d^0 \mathbf{w}}{dt} \right) \right\} \frac{df}{dv_\parallel} = 0;$$
$$\mathbf{w} = c \frac{[\mathbf{EB}]}{B^2}, \quad \frac{d^0}{dt} = \frac{\partial}{\partial t} + (v_\parallel \boldsymbol{\varepsilon} + \mathbf{w}) \nabla, \quad \boldsymbol{\varepsilon} = \frac{\mathbf{B}}{B}. \tag{4*}$$

The ions flux density is given by the formula[2)]

$$\mathbf{j}_i = -\operatorname{rot}\left( \int \frac{\mu c}{e} f \boldsymbol{\varepsilon}\, d^3v \right) + \int f \left\{ v_\parallel \boldsymbol{\varepsilon} + \frac{Mc}{eB} \left[ \boldsymbol{\varepsilon}, -\frac{e}{M} \mathbf{E} + v_\parallel \frac{d^0 \boldsymbol{\varepsilon}}{dt} + \frac{d^0 \mathbf{w}}{dt} \right] \right\} d^3v. \tag{5}$$

Analogous equations with the substitutions e → −e and M → m hold also for electrons.

We note that here, as before[1-3], we shall consider only motions of a plasma with hydrodynamic velocities $v \ll (T_e/m)^{1/2}$. In this case the electron distribution function differs little from the equilibrium distribution, which is naturally assumed to be of the Maxwell-Boltzmann type. As a result of this, in the calculation of the electron current, the even powers of v that enter in (5), namely $v_\parallel^2$ and $v_\perp^2$, can be averaged over the Maxwell-Boltzmann distribution. On the other hand, the integral $\int f_e v_\parallel d^3v$, which is determined by the corrections to the equilibrium function, can be obtained from general considerations. Indeed, by virtue of the quasineutrality condition (1), we can assume that

$$\operatorname{div} \mathbf{j} = e \operatorname{div}(\mathbf{j}_i - \mathbf{j}_e) = -e \frac{\partial}{\partial t}(N_i - N_e). \tag{6}$$

The integral $\int f_e v_\parallel d^3v$ can be determined from this condition without accurately calculating the correction to the electron distribution function.

Thus, Eqs. (4) together with relations (5) and (6), supplemented with Maxwell's equations, constitute a complete system describing the quasineutral drift motion of a magnetized plasma. We note that in Maxwell's equations it is necessary, in addition, to neglect the displacement current, since $v \ll c$, and to omit div $\mathbf{E}$ (by virtue of the quasineutrality condition (1)).

## 3. ONE-DIMENSIONAL MOTION

In this paper we confine ourselves to the case of one-dimensional motion of a plasma, when all the quantities depend only on one spatial coordinate x and the time t. The equation div $\mathbf{B} = 0$ yields in this case

$$B_x = \text{const.} \tag{7}$$

On the other hand, Eq. (6) reduces to $\partial j_x/\partial x = 0$ or, assuming that there is no current at infinity, we have

$$j_x = 0, \tag{8}$$

i.e., the current is perpendicular to the x axis.

We consider only planar motion in the sense that the magnetic field always lies in one plane, which we assume to be the (x, y) plane. In this case the current is directed along the z axis. It then follows from (8) that

$$\int f_e v_\parallel d^3v = \int f v_\parallel d^3 v. \tag{9}$$

In other words, the longitudinal currents of the electrons and ions fully cancel each other.

The electric field $\mathbf{E}$ has two components—potential, directed along x, and solenoidal, directed along z.

$$\mathbf{E} = \mathbf{E}_z - \nabla\varphi.$$

Since $\mathbf{E}_z$ is perpendicular to the magnetic field, its influence on the distribution of the particles reduces primarily to the production of drifts. It is easy to show that in order of magnitude

$$E_z \sim B(2T_e/M)^{1/2}/c$$

(see (14) below), so that the velocity of the produced drift is of the order of $(2T_e/M)^{1/2}$, i.e., much less than the thermal velocity of the electrons $(2T_e/m)^{1/2}$. Therefore the influence of $E_z$ on the distribution of the electrons can be neglected, assuming a Maxwell-Boltzmann electron distribution in the potential field $\varphi$. Taking the quasineutrality condition (1) into account, this yields

$$\varphi = \frac{T_e}{e}\ln\frac{N}{N_0}. \tag{10}$$

Omitting straightforward but cumbersome calculations, we write down the complete system of equations. The kinetic equation is

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\left(\frac{B_x}{B}v_\parallel - c\frac{E_z B_y}{B^2}\right) + \frac{\partial f}{\partial v_\parallel}\frac{B_x}{B}\left\{-\frac{T_e}{M}\frac{\partial}{\partial x}\ln\frac{N}{N_0}\right.$$
$$\left. + \left(-\mu\frac{B_y}{B} + v_\parallel\frac{cE_z B_x}{B^3} - \frac{c^2E_z^2B_y}{B^4}\right)\frac{\partial B_y}{\partial x} + \frac{cE_z}{B^2}\frac{\partial B_y}{\partial t}\right\} = 0; \tag{11}$$
$$N = 2\pi B\int f\,dv_\parallel\,d\mu, \qquad B^2 = B_x^2 + B_y^2.$$

To this equation we add Maxwell's equation

$$\frac{\partial E_z}{\partial x} = \frac{1}{c}\frac{\partial B_y}{\partial t}. \tag{12}$$

On the other hand, the equation

$$\frac{\partial B_y}{\partial x} = \frac{4\pi}{c}j_z$$

must be transformed, since expression (5) itself contains derivatives of the fields. As a result we obtain

$$\left\{\left[1 + \frac{4\pi NM}{B}\left(\bar\mu - \overline{v_\parallel^2}\frac{B_x^2}{B^3} + 2\bar v_\parallel c\frac{E_zB_xB_y}{B^4} - c^2\frac{E_z^2B_y^2}{B^5}\right)\right]\frac{\partial B_y}{\partial x}\right.$$
$$\left. + \frac{8\pi NM}{B^3}\left[c\frac{E_zB_y}{B} - \bar v_\parallel B_x\right]\frac{\partial B_y}{\partial t} - \frac{4\pi NMc}{B^2}\frac{\partial E_z}{\partial t}\right\} \tag{13}$$
$$= -\frac{4\pi NMB_y}{B}\left[\frac{\partial\bar\mu}{\partial x} + \left(\bar\mu + \frac{T_e}{MB}\right)\frac{\partial}{\partial x}\ln\frac{N}{N_0}\right].$$

The superior bars over v, $\mu$, etc. denote averaging

$$\bar\mu = \frac{2\pi B}{N}\int \mu f\,dv_\parallel\,d\mu, \qquad \bar v_\parallel = \frac{2\pi B}{N}\int v_\parallel f\,dv_\parallel\,d\mu \text{ etc.}$$

In expression (13) we have neglected the terms $\sim(m/M)^{1/2}$ and higher, and also the terms with the extra power of the Larmor radius. We note that those terms in the current, which contain the electric field itself and not its derivatives, would cancel out when the electron and ion currents were added.

Equations (11)—(13) constitute a complete system of equations for f, $B_y$, and $E_z$. (We recall that $B_x$ is a constant.) These equations are not altered by the simultaneous substitutions $x \to Ax$ and $t \to At$, meaning that there is no parameter with the dimension of length in these equations.

## 4. EQUATIONS OF SELF-SIMILAR MOTION

The dimensional properties of the system (11)—(13) are the reason why, in the case when the initial or the boundary conditions of the problem do not contain a characteristic dimension, the sought functions f, $E_z$, and $B_y$ depend on x and t only in the combination x/t. In other words, the motion is self-similar in this case. Self-similar motion is obtained in problems involving the decay of an arbitrary initial discontinuity.

Let us transform the system (11)—(13) under the assumption that the motion is self-similar. Let the unperturbed density and temperature of the ions as $x \to \infty$ be $N_0$ and $T_i$, and let the unperturbed magnetic field be $\mathbf{B}_0$. We produce new dimensionless variables:

$$f = N_0\left(\frac{M}{2\pi T_i}\right)^{3/2} g, \quad v_\parallel = \left(\frac{2T_e}{M}\right)^{1/2} u, \quad \frac{x}{t} = \left(\frac{2T_e}{M}\right)^{1/2}\tau, \quad \mathbf{B} = B_0\mathbf{b},$$
$$E_z = \frac{B_0}{c}\left(\frac{2T_e}{M}\right)^{1/2} a, \quad \mu = \frac{2T_e}{MB_0}\varkappa, \quad N = nN_0, \quad \beta = T_e/T_i. \tag{14}$$

The dimensionless concentration of the particles and other mean values are expressed in terms of g by the formula

$$n = 2\pi^{-1/2}b\beta^{3/2}\int g\,du\,d\varkappa, \qquad \bar u = 2\pi^{-1/2}b\beta^{3/2}\int\frac{ug}{n}\,du\,d\varkappa, \ldots. \tag{14a}$$

Assuming that $g = g(\tau, u, \kappa)$, we obtain from (11)

$$\chi\frac{\partial g}{\partial\tau} + F\frac{\partial g}{\partial u} = 0, \qquad b = (b_y^2 + \cos^2\alpha)^{1/2},$$
$$\chi = \frac{u}{b}\cos\alpha - \tau - \frac{b_y a}{b^2}, \tag{15}$$
$$F = \cos\alpha\left[-\frac{1}{2}\frac{d\ln n}{d\tau} + \left(\chi\frac{a}{b^2} - \varkappa\frac{b_y}{b}\right)\frac{db_y}{d\tau}\right].$$

Here $\alpha$ is the angle between the direction of the x axis and the direction of the magnetic field $B_0$ at $-\infty$, and $\cos\alpha = b_x$ as the component of the magnetic field along the x axis (it is constant, as given by (7)).

Equations (12) and (13) reduce to the form

$$\frac{da}{d\tau} = -\tau\frac{db_y}{d\tau}, \qquad \gamma = \frac{8\pi N_0T_e}{B_0^2}, \tag{16}$$
$$\frac{db_y}{d\tau} = -\gamma\frac{b_yn}{Pb}\left[\frac{d\bar\varkappa}{d\tau} + \left(\bar\varkappa + \frac{1}{2b}\right)\frac{d\ln n}{d\tau}\right],$$

$$P=1-\gamma\frac{n}{b^2}\left[-\bar{\varkappa}b+\overline{u^2}\frac{\cos^2\alpha}{b^2}-\frac{2\bar{u}\cos\alpha}{b}\left(\tau+\frac{ab_y}{b^2}\right)+\left(\tau+\frac{ab_y}{b^2}\right)^2\right]. \tag{17}$$

The parameter $\gamma$ determines the ratio of the electron pressure to the magnetic pressure in the unperturbed plasma.

The boundary conditions for g are specified at $\tau \to -\infty$ and at $\tau \to +\infty$ (see[1])

$$g\to g_1=\exp[-\beta(u^2+2\varkappa)],\ \text{as}\ \tau\to-\infty,$$
$$g\to g_2(u,\varkappa),\ \text{as}\ \tau\to+\infty.$$

The function $g_1$ is assumed here, for concreteness, to be Maxwellian. The boundary conditions for b and a are

$$b\to 1,\ b_y\to\sin\alpha,\ a\to 0\ \text{as}\ \tau\to-\infty.$$

As $\tau \to +\infty$, the magnetic and electric fields are connected with their values at $-\infty$ and with the initial current distribution (i.e., with the initial ion distribution function) by Maxwell's equations. It can be assumed, however, that the fields are also specified at $\tau \to \infty$

$$b_y\to b_0,\ b\to(b_0^2+\cos^2\alpha)^{1/2},\ a\to a_0\ \text{as}\ \tau\to+\infty. \tag{18}$$

In this case, definite conditions are imposed on the initial distribution of the ions. These conditions are specified on the discontinuities which, as will be shown subsequently, always occur in the case of self-similar motion of a plasma in a magnetic field.

An important role in the investigation of a self-similar equation in the absence of a magnetic field is played by an identity of the type [1,3]

$$\int g\,du=\frac{1}{2}\int\frac{\partial g}{\partial u}\frac{du}{u-\tau} \tag{19}$$

which is imposed on the solution. A similar but more complicated identity is obtained also in the case of a magnetized plasma. Indeed, dividing (15) by $\chi$ and multiplying by b, we integrate with respect to $dud\kappa$. We obtain an equation connecting

$$\frac{dn}{d\tau},\quad \frac{d(n\bar{\varkappa})}{d\tau}\ \text{and}\ \frac{db_y}{d\tau}.$$

Another equation is obtained by using the same procedure, but first multiplying (15) by $\kappa$. The obtained two equations together with (17) form a system of three nuclear homogeneous algebraic equations for the indicated three quantities. The determinant of this system should be equal to zero, and this yields the sought identity:

$$1-\frac{Q_0}{2n}+\frac{\gamma b_y^2}{Pb^2}\left[\left(Q_2+\frac{\bar{\varkappa}}{b}\right)\left(1-\frac{Q_0}{2n}\right)+\frac{1}{2n}\left(Q_1+\frac{n}{b}\right)^2\right]=0,$$
$$Q_k=\frac{2\cos\alpha}{\sqrt{\pi}}\int\frac{\varkappa^k}{\chi}\frac{\partial g}{\partial u}du\,d\varkappa. \tag{20}$$

Here p is the denominator in the right-hand side of (17). In the limit as $b_y \to 0$, Eq. (20) goes over, as it should, into the identity (19). We note that relation (20), like (19), is in the absence of a magnetic field the dispersion equation for small plasma oscillations (see[3]).

## 5. ESCAPE OF PLASMA INTO A VACUUM. WEAK DISCONTINUITY

By way of example, we consider the problem of escape of a plasma into a vacuum. A plasma situated in a magnetic field $\mathbf{B}_0$ occupies a half-space $x < 0$. Its free expansion begins at the instant $t = 0$. The problem is self-similar. The problem of flow of rapid stream of a magnetized plasma around a half-plane also reduces to the same problem (in the latter case, t takes the role of the coordinate along the direction of motion).

When solving (13)—(17), it is natural to use the method of characteristics (for details see[1]). The function g is constant on the characteristics $u(\tau)$:

$$\frac{du}{d\tau}=-\frac{F}{\chi},\quad \varkappa=\text{const},$$

where the functions F and $\chi$ are defined in accordance with (15). The results of the numerical integration of the equations of the characteristics, together with Eqs. (16) and (17) for $b_y$ and a, are shown by way of example in Fig. 1, which shows the course of the average quantity[3)] (for $\gamma = 4$, $\beta = 1$, $\cos\alpha = 2^{-1/2}$). We see that the ion concentration and the magnetic field decrease monotonically with increasing $\tau$. As $\tau \to \tau_0 = 0.82$, the magnetic-field component $b_y$ vanishes.

It is easy to verify that $\tau_0$ is a singular point. Indeed, we rewrite (17) in the form

$$\frac{1}{b_y}\frac{db_y}{d\tau}=-\frac{A}{P}.$$

We see that at the point $\tau_0$, where $b_y$ vanishes, the denominator P should also vanish. The singular point $\tau_0$ exists at any value of the perimeters $\gamma$ and $\cos\alpha$. The dependence of $\tau_0$ on $\gamma$ at $\cos\alpha = 2^{-1/2}$ is shown in Fig. 2.

Let us investigate the behavior of $b_y$ near the singularity $\tau \approx \tau_0$. At small $\tau' = \tau - \tau_0$ and $b_y$ it is necessary to expand the right-hand side of (17) in powers of $\tau'$ and $b_y$. The dependence of g on these quantities is determined from Eq. (15). In particular, in first order in $b_y$ we get from (15)

$$g\approx g(\tau_0)-\frac{\partial g(\tau_0)}{\partial u}b_y.$$

From this we readily obtain that there are no terms $\sim b_y$ in n or $\bar{\kappa}$, and that these terms also cancel out in

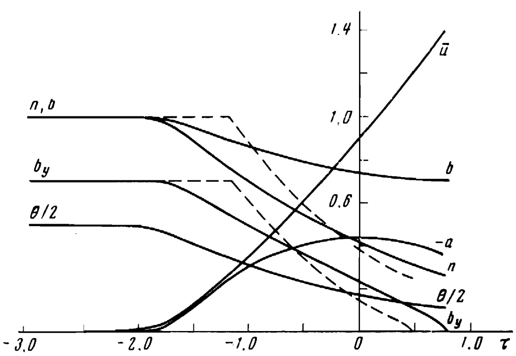


FIG. 1. Average quantities as functions of $\tau = x/t(2T_e/M)^{1/2}$ when plasma escapes into vacuum at $T_e = T_i$($\beta = 1$, $\gamma = 4$, $\cos\alpha = 2^{-1/2}$). Dashed—the same for cold ions ($T_i \to 0$).

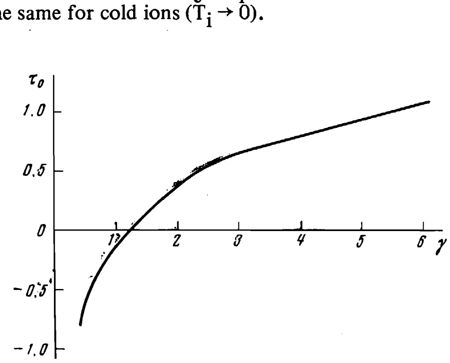


FIG. 2. Positions of the singular point as a function of $\gamma = 8\pi N_0T_e/B_0^2$.

the expression for P, i.e., the $b_y$ corrections in the right-hand side of (17) are of the order of $b_y^2$. Under the condition that $b_y$ vanishes more rapidly than $b_y \sim |\tau'|^{1/2}$, we can neglect the corrections $\sim b_y^2$. We then have from (17)

$$\frac{db_y}{d\tau'}=\alpha_1\frac{b_y}{\tau'},\quad \alpha_1=-\frac{\gamma}{b}\left[\frac{d(n\bar{\varkappa})/d\tau+{}^1/_2\,dn/d\tau}{dP/d\tau}\right]_{\tau=\tau_0} \tag{21}$$

so that

$$b_y=\begin{cases} C_-|\tau'|^{\alpha_1}, & \tau'<0,\\ C_+\tau'^{\alpha_1}, & \tau'>0.\end{cases} \tag{22}$$

The formulas are valid if $\alpha_1 > 1/2$.

Thus, if $\alpha_1 > 1/2$, then $|b_y|$ at the singular point vanishes in accordance with the power law, and then increases again in accordance with the same law. The constant $C^-$ is determined by the course of the solution at $\tau < \tau_0$. On the other hand, the constant $C_+$ is in this sense arbitrary and by choosing it it is possible to satisfy the boundary condition (18) on $b_y$ as $\tau \to +\infty$.

Let now $\alpha_1 > 1/2$. We then can no longer neglect the terms $\sim b_y^2$, and Eq. (17) assumes at small $|\tau'|$ and $b_y$ the form

$$\frac{db_y}{d\tau'}=\alpha_1 b_y\left(1+\delta\frac{db_y^2}{d\tau'}\right)\Big/(\tau'+\nu b_y^2),$$

where $\alpha_1$ has the same meaning as before, and $\delta$ and $\nu$ are the coefficients of the expansion with respect to $b_y^2$, which can be determined from (15). This equation can be rewritten in the form

$$\frac{d\tau'}{db_y^2}-\frac{\tau'}{2\alpha_1 b_y^2}=-(2\alpha_1\delta-\nu),$$

whence

$$|\tau'|=C|b_y|^{1/\alpha_1}-\frac{\nu-2\alpha_1\delta}{1-2\alpha_1}2\alpha_1 b_y^2. \tag{23}$$

At $\alpha_1 > 1/2$, relation (23) is equivalent to (22), as it should, and at $\alpha_1 < 1/2$ the principal term in $b_y$ is given by[4)]

$$b_y\approx\left[-\tau'\frac{1-2\alpha_1}{\nu-2\alpha_1\delta}\right]^{1/2}. \tag{24}$$

It follows therefore that at $\alpha_1 < 1/2$ the component $b_y$, regardless of the value of $\alpha_1$, vanishes like $\sim(\tau_0-\tau)^{1/2}$, beyond which it is impossible to continue the solution. A unique solution in this case, at $\tau > \tau_0$, is the solution with $b_y \equiv 0$, $\partial g/\partial\tau \equiv 0$.

Let us now transform expression (21) for $\alpha_1$. We have

$$\left.\frac{dP}{d\tau}\right|_{b_y=0,\ \tau=\tau_0}=\frac{\gamma}{b^2}\left[b\frac{d(n\bar{\varkappa})}{d\tau}-\frac{d(n\overline{u^2})}{d\tau}+2\tau_0\frac{d(n\bar{u})}{d\tau}-\tau_0^2\frac{dn}{d\tau}+2n(\bar{u}-\tau_0)\right]. \tag{25}$$

Equation (15) takes at $b_y = 0$ the form

$$\frac{\partial g}{\partial\tau}(u-\tau)-\frac{1}{2}\frac{\partial g}{\partial u}\frac{d\ln n}{d\tau}=0.$$

Integrating it once with respect to $2\pi b d\kappa du$, and then again with respect to $u 2\pi b d\kappa du$, we get[5)]

$$\frac{d(n\bar{u})}{d\tau}=\tau_0\frac{dn}{d\tau},\quad \frac{d(n\overline{u^2})}{d\tau}=\tau_0\frac{d(n\bar{u})}{d\tau}-\frac{1}{2}\frac{dn}{d\tau}=\left(\tau_0^2-\frac{1}{2}\right)\frac{dn}{d\tau}. \tag{26}$$

Substituting in expression (21) for $\alpha_1$, we obtain ultimately

$$\alpha_1=-\left[1+2n(\bar{u}-\tau_0)\Big/\left(\cos\alpha\frac{d(n\bar{\varkappa})}{d\tau}+\frac{1}{2}\frac{dn}{d\tau}\right)\right]^{-1}. \tag{27}$$

We note that at small $\gamma$ the discontinuity occurs in a region of remote negative values of $\tau$ (see Fig. 2). In this case $dn/d\tau$ and $d(n\bar{\kappa})/d\tau$ are small and, in accordance with (27), $\alpha_1$ is also small, so that the discontinuity has a "non-continuable" form (24). The "continuable" discontinuity of the type (22) takes place at large values of $\gamma$. The quasineutral solution can in this case be continued into the region behind the singularity, in accordance with formula (22). At the point $\tau = \tau_0$, it has a weak discontinuity, the structure of which will be investigated in the next section.

The very occurrence of a "non-continuable" discontinuity is of considerable interest. In this case the quasineutral drift equations (15)–(17) have no solutions satisfying the boundary solutions of the problem, even though the thermal motion of the ions has been fully taken into account. The situation here is qualitatively different from the situation in a plasma without a magnetic field, where the quasineutral equation always has a solution at $\beta \sim 1$.

## 6. STRUCTURE OF WEIGHT DISCONTINUITY

So far, the discontinuity was investigated within the framework of the system (15)–(17) corresponding to the first nonvanishing approximation in the parameters D and $\rho_{Hi}$. Allowance for the finite character of D and $\rho_{Hi}$ leads to a smearing of the discontinuity, and to the appearance of an oscillatory structure of the discontinuity. To be able to investigate this structure, it is necessary to write down in place of Eq. (17) for $b_y$ a more accurate equation that takes into account the terms $\sim D^2$ or $\sim\rho_{Hi}^2$. In actual fact, we usually have $\rho_{Hi} \gg D$. We shall therefore take into account just the corrections $\sim\rho_{Hi}^2$. We assume that $\alpha_1 > 1/2$ and we therefore disregard the terms $\sim b_y^2$.

Calculation of the current density in the next higher order in $\rho_{Hi}^2$, i.e., in the approximation that follows the drift approximation, is given in the Appendix. Substituting formula (A.9) in Maxwell's equation (A.2), we obtain an equation for $b_y$, which is valid in the limit of small $b_y$ and $\tau' = \tau - \tau_0$:

$$\frac{\varepsilon}{t^2}\frac{\partial^3 b_y}{\partial\tau'^3}-\tau'\frac{\partial b_y}{\partial\tau'}+\alpha_1 b_y+\frac{1+\alpha_1}{2}t\frac{\partial b_y}{\partial t}=0. \tag{28}$$

In (28), $\alpha_1$ has the same meaning as before, and

$$\varepsilon=\frac{1}{(b\omega_H)^2}n[3b^2\overline{\varkappa(u-\tau_0)^2}-\overline{(u-\tau_0)^4}]\times\left[b\frac{d(n\bar{\varkappa})}{d\tau}+\frac{1}{2}\frac{dn}{d\tau}+2n(\bar{u}-\tau_0)\right]^{-1}\sim\frac{1}{\omega_H^2}. \tag{29}$$

The term with the third derivative takes into account effects of order $\rho_{Hi}^2$. Naturally, the coefficient in this term is small: $\epsilon/t^2 \sim (\omega_H t)^{-2} \ll 1$, since the entire theory is valid only at times $t \gg 1/\omega_H$ when the self-similar motion has already become established.

The boundary conditions of (28) are determined by the requirement that the sought function $b_y$ assume the form of the self-similar motion has already become established.

The boundary conditions of (28) are determined by the requirement that the sought function $b_y$ assume the form of the self-similar solution (22) outside the discontinuity, i.e., at $\tau' \to \pm\infty$. A function satisfying these conditions can be constructed by putting

$$b_y = t^{-2\alpha_1/3} Z(\eta), \quad \eta = t^{2/3}\tau' \frac{(2-\alpha_1)^{1/3}}{(3\varepsilon)^{1/3}}. \tag{30}$$

Equation (28) assumes, in terms of the variables (30), the form

$$\frac{d^3 Z}{d\eta^3} - \eta \frac{dZ}{d\eta} + \alpha_1 Z = 0. \tag{31}$$

The solution (31) can be obtained by the Laplace method (see, e.g., [7], p. 681). Two linearly-independent solutions of Eq. (31), which vanish at infinity, can be represented in the form of contour integrals:

$$Z_{1,2} = \Gamma(1+\alpha_1) \int_{I,II} \exp\left(p\eta - \frac{p^3}{3}\right) \frac{dp}{p^{1+\alpha_1}}. \tag{32}$$

The integration contours are shown in Fig. 3: contour I bypasses the point p = 0 from the left, and contour II from the right. As $\eta \to +\infty$, we have $Z_2 \to \eta^{\alpha_1}$, and $Z_1$ decreases exponentially. As $\eta \to -\infty$ we have

$$Z_2 \approx \frac{2\pi^{1/2}\Gamma(1+\alpha_1)}{\eta^{(3+2\alpha_1)/4}} \sin\left(\frac{2}{3}|\eta|^{3/2} + \frac{3+2\alpha_1}{4}\pi\right). \tag{33}$$

The calculations are analogous here to those carried out in the determination of the asymptotic forms of Airy functions (see [7], pp. 684 and 685).

A solution of (28), satisfying the boundary conditions (22), can be finally written in the form

$$b_y = \left[\frac{3\varepsilon}{(2-\alpha_1)t^2}\right]^{1/3} \left\{ C_- Z_1\left[\tau'\left(t^2\frac{2-\alpha_1}{3\varepsilon}\right)^{1/3}\right] + C_+ Z_2\left[\tau'\left(t^2\frac{2-\alpha_1}{3\varepsilon}\right)^{1/3}\right]\right\}. \tag{34}$$

Thus, a region of fast oscillations of the magnetic field is established behind a weak discontinuity[6)]. They attenuate more slowly the smaller $\epsilon$.

## 7. EXPANSION OF A STRONGLY NON-ISOTHERMIC PLASMA INTO VACUUM

We consider now the motion of a strongly non-isothermal plasma $\beta = T_e/T_i \to \infty$. In this case the thermal scatter of the ion velocities is immaterial, so that the ion distribution function $g(\tau, u, \kappa)$ can be written in the form

$$g(\tau, u, \varkappa) = \frac{\pi^{1/2} n}{2b} \delta(u - \bar{u}) \delta(\varkappa). \tag{35}$$

Substituting the distribution function (35) in (15) and integrating it with respect to the velocities, we arrive at the following equations for the concentration n and the average velocity $\bar{u}$:

$$\bar{\chi}\frac{dn}{d\tau} + \frac{n}{b}\cos\alpha\frac{d\bar{u}}{d\tau} - \frac{an}{b^4}\cos^2\alpha\frac{db_y}{d\tau} - \bar{\chi}\frac{b_y n}{b^2}\frac{db_y}{d\tau} = 0, \tag{36}$$

$$\bar{\chi}\frac{d\bar{u}}{d\tau} + \frac{\cos\alpha}{2bn}\frac{dn}{d\tau} - \frac{a\cos\alpha}{b^3}\bar{\chi}\frac{db_y}{d\tau} = 0,$$

$$\bar{\chi} = \frac{\bar{u}\cos\alpha}{b} - \tau - \frac{ab_y}{b^2}. \tag{37}$$

Maxwell's equations (16) and (17), with allowance for the fact that $\bar{\kappa} = 0$ and $\overline{u^2} = \bar{u}^2$, are rewritten in the form

$$\frac{da}{d\tau} = -\tau\frac{db_y}{d\tau}, \quad \frac{db_y}{d\tau} = -\frac{\gamma b_y}{2[b^2 - \gamma n\bar{\chi}^2]}\frac{dn}{d\tau}. \tag{38}$$

It is possible to arrive at Eqs. (36)–(38) also by starting directly from the equations of magnetohydrodynamics with addition of $P = NT_e$ (it must be born in mind that the presence of a field **E** perpendicular to B is equivalent in hydrodynamics to the presence of a velocity $\mathbf{v} = c\mathbf{E}\times\mathbf{B}/B^2$). This is perfectly natural, since the magnetohydrodynamic equations at $P = NT_e$, as

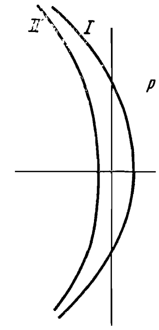


FIG. 3

shown by Klimontovich and Silin [8], describe quasineutral motion of a collisionless plasma with cold ions[7)].

Equating to zero the determinant of the system (36)–(13), we obtain its first integral in the form

$$\bar{\chi}^2\left[1 + \frac{\gamma n b_y^2}{2b^2(b^2 - \gamma n\bar{\chi}^2)}\right] = \frac{\cos^2\alpha}{2b^2}. \tag{39}$$

This relation coincides with (20), if we assume the function g in the form (35). On the other hand, we note that Eq. (39) with $\bar{\chi} = \omega/k(2T_e/M)^{1/2}$ coincides with the dispersion equation for the fast and slow magnetosonic waves in magnetohydrodynamics ([9], p. 250).

We consider now the problem of plasma escaping into a vacuum. In this case the boundary conditions for Eqs. (36)–(38) as $\tau \to -\infty$, i.e., in the region of the amplitude plasma, take the form

$$n = 1, \ b_y = \sin\alpha, \ a = 0, \ \bar{u} = 0, \ b = 1. \tag{40}$$

Relation (39) is not satisfied as $\tau \to -\infty$. Therefore in the region of large negative values of $\tau$ there is only a trivial solution of Eqs. (36)–(38), namely, all the quantities are constants independent of $\tau$ and determined by the boundary conditions (40). Substituting the values of (40) in (39) we obtain the singular point $\tau_1$, at which the relation (39) is satisfied for the first time:

$$\tau_1 = -\frac{1}{\sqrt{2}}\left\{\frac{1}{2} + \frac{1}{\gamma} + \left[\left(\frac{1}{2} + \frac{1}{\gamma}\right)^2 - \frac{2\cos^2\alpha}{\gamma}\right]^{1/2}\right\}^{1/2}. \tag{41}$$

At $\tau > \tau_1$ there exists already a nontrivial solution of Eqs. (36)–(38). At the point $\tau_1$, the solution has a weak discontinuity. At this point, the velocity x/t is equal to the velocity of the fast magnetosonic wave moving away in the direction of the unperturbed plasma.

The result of the numerical solution of equations (36)–(38) at $\gamma = 4$ and $\cos\alpha = 2^{-1/2}$ is shown dashed in Fig. 1. We see that in magnetohydrodynamics the plots of the plasma concentration and of the field $b_y$ differ noticeably from the results of the exact kinetic solution at $T_i = T_e$.

The component $b_y$ vanishes at $\tau = \tau_0 \approx 0.48$. The point $\tau_0$ is singular, since the denominator of Eq. (38) and of relation (39) vanishes at this point. Thus, in magnetohydrodynamics we obtain the same type of singularity as in the kinetic solution. An investigation of the behavior of the solution near the singularity leads to an analogous result, namely, the component $b_y$ at $\tau \approx \tau_0$ and $\alpha_1 > 1/2$ is described by formula (12), where the parameter $\alpha_1$ is equal to[8)]

$$\alpha_1 = -\left[1 + 4\sqrt{2}\cos^2\alpha/\gamma\left(\frac{dn}{d\tau}\right)_{\tau_0}\right]^{-1}. \tag{42}$$

The singularity has a "continuable" character if

$\alpha_1 > 1/2$, and "non-continuable" if $\alpha_1 < 1/2$. It is interesting that for the "continuable" singularity at the singular point $\tau_0$, the velocities of the fast and slow magnetosonic waves coincide, i.e., the singularity is connected with the crossing of these branches. The concentration of the plasma at the singular point is $n(\tau_0) = 2\cos^2\alpha/\gamma$. Taking this relation into account, formula (42) coincides with (27) at the required accuracy. When the solution is continued beyond the singular point, it is necessary to go over from the root $\chi$ in (39), corresponding to a fast magnetosonic wave, to a root corresponding to a slow wave. The results of the corresponding calculation are shown in Fig. 4 ($\cos\alpha = 0.5$; $\gamma = 4$) for different values of the ratio of the constants $C_+/C_-$ (22) indicated in this figure.

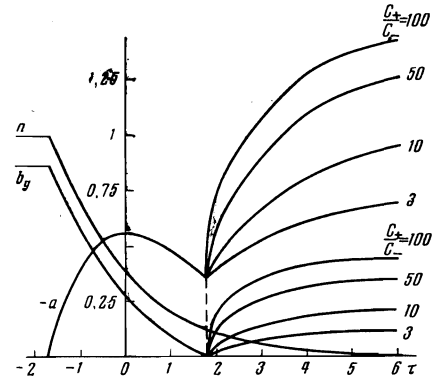


FIG. 4. Plots of $b_y$, n, and a as functions of $\tau$ when the solution is continued beyond the singularity ($T_i = 0$, $\gamma = 4$, $\cos\alpha = 0.5$); different curves correspond to different values of the ratio of the constants $C_+/C_-$ indicated in the figure.

## APPENDIX

### Current density in second approximation

We present here a derivation of formula (28), which describes the change of the magnetic field $b_y$ in the second approximation, i.e., the approximation that follows the drift approximation. We assume that we are near the point $\tau = \tau_0$ of a weak discontinuity, where $b_y$ is small, and obtain equations linearized in $b_y$. The results are suitable also for discontinuities when $\alpha_1 > 1/2$, in which the terms $\sim b_y^2$ can be neglected.

The ion-current density is

$$j_z = eN(2T_e/M)^{1/2}I, \quad I = \int w_z' g\, d^3w' = \int w_\perp' g \sin\varphi\, d^3w', \tag{A.1}$$

where we have introduced the dimensionless vector $\mathbf{w}' = \mathbf{v}(M/2T_e)^{1/2}$, $\mathbf{w}'_\perp$ is the production of $\mathbf{w}$ on the xy plane, and $\varphi$ is the angle between $\mathbf{w}_\perp$ and the y axis. Maxwell's equation for the magnetic field can be expressed in the form

$$\frac{\partial b_y}{\partial \xi} = \gamma\Omega_H I, \quad \Omega_H = \frac{eB_0}{Mc}. \tag{A.2}$$

We write out the kinetic equation for g in the ordinary rather than the drift variables:

$$\frac{\partial g}{\partial t} + w_x'\frac{\partial g}{\partial \xi} + a\omega_H\frac{\partial g}{\partial w_z'} - \frac{1}{2}\frac{\partial n}{\partial \xi}\frac{\partial g}{\partial w_x'} + \omega_H\frac{\partial g}{\partial \mathbf{w}'}[\mathbf{w}'\mathbf{b}] = 0, \quad x = (2T_e/M)^{1/2}\xi. \tag{A.3}$$

According to (13), the electric field $\mathbf{a}$ can be represented in the form

$$\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_1,$$

where $\mathbf{a}_0$ is a constant vector and $a_1 \sim b_y$. Separating from $\mathbf{w}$ the drift velocity in the constant fields $\mathbf{a}_0$ and $b_x$:

$$\mathbf{w}' = \mathbf{w} + [\mathbf{a}_0\mathbf{b}_x]/b_x^2,$$

we obtain

$$\frac{\partial g}{\partial t} + w_x\frac{\partial g}{\partial \xi} + \omega_H a_1\frac{\partial g}{\partial w_z} - \frac{1}{2}\frac{\partial n}{\partial \xi}\frac{\partial g}{\partial w_x} + \omega_H\frac{\partial g}{\partial \mathbf{w}}[\mathbf{wb}] = 0. \tag{A.4}$$

We now represent g in the form

$$g = g_0(\tau, w_x, |\mathbf{w}_\perp|) + g_1, \tag{A.5}$$

where $g_1 \sim b_y$, and $g_0$ satisfies the equation

$$(w_x - \tau)\frac{\partial g_0}{\partial \tau} = \frac{1}{2}\frac{\partial g_0}{\partial w_x}\frac{d}{d\tau}\ln\left(\int g_0\, d^3w\right)$$

i.e., the equation (16) with $b_y = 0$. We substitute (A.5) in (A.4) and neglect the terms $\sim b_y^2$. In the linear equation for $g_1$, we can also neglect the longitudinal electric field (along the x axis). It leads only to motion in x direction, and makes to contribution to the current along the z axis. Ultimately we have

$$\frac{\partial g_1}{\partial \varphi} - \hat{D}g_1 = A\sin\varphi, \tag{A.6}$$

where we have introduced the differential operator

$$\hat{D} = \frac{1}{b\Omega_H}\left(\frac{\partial}{\partial t} + w_x\frac{\partial}{\partial \xi}\right),$$

and the right-hand side is given by

$$A = \frac{1}{b}\frac{\partial g_0}{\partial w_\perp}(w_x b_y + a_1) - \frac{1}{b}\frac{\partial g_0}{\partial w_x}w_\perp b_y.$$

(We have put, with the required accuracy, $b_x = b$.)

The formal solution of (A.6) can be written in the form[9]

$$g_1 = -\int_0^\infty \exp(-z\hat{D})\sin(z+\varphi)A = -\frac{\sin\varphi\hat{D}}{1+\hat{D}^2}A. \tag{A.7}$$

Expanding this expression in powers of $\hat{D}$, we can write out the equation for $g_1$ with allowance for derivatives of any order. The corresponding expression for the current, accurate to derivatives of third order, is

$$I \approx -\frac{1}{2}\int w_\perp[(\hat{D}-\hat{D}^3)A]d^3w = \frac{1}{2b}\int\Big[2(\hat{D}-\hat{D}^3)(w_x b_y + a_1)g_0 - \frac{w_\perp^2}{b\Omega_H}(1-3\hat{D}^2)\frac{\partial}{\partial \xi}(b_y g_0)\Big]d^3w. \tag{A.8}$$

We now change over from the variables $\xi$, t to the variables $\tau = \xi/t$, t. Then

$$\left.\frac{\partial}{\partial \xi}\right|_t = \frac{1}{t}\left.\frac{\partial}{\partial \tau}\right|_t, \quad \left.\frac{\partial}{\partial t}\right|_\xi = \left.\frac{\partial}{\partial t}\right|_\tau - \frac{\tau}{t}\left.\frac{\partial}{\partial \tau}\right|_t$$

In the drift approximation, $b_y$ is a function of $\tau$ only. Now $b_y$ is a function of $\tau$ and t, but the derivatives with respect to t will be small. The terms with third-order derivatives contain an extra small coefficient $1/\Omega_H^2$. We can therefore neglect in them the derivative with respect to t and differentiate only the function $b_y$, which varies more readily. The derivatives of first order with respect to t, on the other hand, must be retained, and we must put in these terms $\tau = \tau_0$ and $a_1 = -b_y\tau_0$. As a result we obtain ultimately

$$I = \frac{1}{b\Omega_H t}\left\{n[\overline{(u-\tau)^2} - \bar{\varkappa}b]\frac{\partial b_y}{\partial \tau} - \left(\frac{1}{2}\frac{dn}{d\tau} + \frac{d(n\bar{\varkappa})}{d\tau}\right)b_y + n(\bar{u}-\tau_0)t\frac{\partial b_y}{\partial t} + \frac{n}{(b\Omega_H t)^2}[3\overline{\varkappa(u-\tau_0)^2}b - \overline{(u-\tau_0)^4}]\frac{\partial^3 b_y}{\partial \tau^3}\right\}. \tag{A.9}$$

(We have made here, with required accuracy, the substitution $w_x \to \mathbf{bw}/b \equiv u$.) The terms $\sim b_y$ and $\partial b_y/\partial\tau$, when substituted in (A.2), yield Eq. (17) in the limit as $b_y \to 0$ (in these terms, of course, it is necessary to take into account the electron contribution which is of no significance in other places). Substituting (A.9) in (A.2) and expanding the coefficient $\partial b_y/\partial\tau$ in powers of $\tau' = \tau - \tau_0$ with allowance for formulas (25) and (26), we obtain Eq. (28).

*[EB] ≡ E × B.

1)We note that formally the limits in (2) and (3) correspond to the electron charge e tending to infinity.

2)Rudakov and Sagdeev's formula in [4] contains errors, which have been corrected here.

3)The ion distribution function with respect to the longitudinal velocity u is analogous qualitatively to that obtained earlier [1, 3].

4)It is assumed that a solution exists at $\tau' < 0$. Then $\nu > 2\alpha_1\delta$.

5)We do not write out the terms $\sim b_y$. As already mentioned, they cancel out.

6)For concreteness we assume that $\epsilon > 0$. At negative $\epsilon$ the solution oscillates at $x > 0$.

7)Certain self-similar solutions of unique type were obtained by Korobeinikov[5] for the equations of magnetohydrodynamics, in which $\mathbf{v} = \mathbf{v}(r/t)$ and $\mathbf{B} = \mathbf{b}(r/t)/r$, and were investigated by Gintsburg[6]. The presence of these solutions is not connected with the drift approximation, and they admit of generalization also to the case of kinetics. The physical meaning of these solutions, however, is not perfectly clear.

8)This investigation was carried out by V. M. Atrazhev[10].

9)We have left out from (A.7) terms proportional to cos $\varphi$, since they make no contribution to the current density (A.1).


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