Nonlinear ion surface oscillations in a semi-restricted current-carrying plasma

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A nonlinear theory is developed for the instability of surface ion-acoustic oscillations of a nonisothermal ($T_e \gg T_i$) plasma with a current. The excitation threshold for short-wave surface waves is lower than that for volume waves. It is shown that in a single-mode regime, the major mechanism limiting the growth of the unstable wave amplitude is the nonlinear shift of its frequency. The frequency and amplitude of a stationary nonlinear wave are found near the instability threshold.

1. INTRODUCTION

A nonlinear theory of ion-acoustic instabilities of a spatially unbounded, nonisothermal ($T_e \gg T_i$) plasma with a current has been developed previously[1-3] for conditions near the instability threshold, when only a single mode with maximum increment turns out to be increasing. It has been shown that the principal mechanism that limits the growth of the unstable wave amplitude in the long-wave limit is the nonlinear pumping of energy from the fundamental excitation mode to higher damped harmonics,[1,2] while the mechanism for short-wave instability, which determines the saturation of the ion oscillations, is the nonlinear shift of the frequency of the excited mode.[3]

In the present work, the theory noted above is generalized to the case of a plasma that is bounded in space. In a bounded plasma, along with the volume waves, excitation of surface ion-acoustic waves is possible. The increment of growth of these can, under certain conditions, exceed the increment of volume oscillations, or the thresholds of their excitation can turn out to be lower, and then the development of the ion-acoustic instability for the surface modes will be more probable than for the volume modes.

As is known,[4] the surface ion-acoustic oscillations in a nonisothermal plasma ($T_e \gg T_i$) are weakly damped only in the limit of sufficiently short wavelengths, where $\lambda/r_{De} \ll (M/m)^{1/4} \lesssim 10$. We shall therefore limit ourselves below to the study of the short-wave limit $\lambda \ll r_{De}$, in which the ion-acoustic waves degenerate into ionic oscillations. It is shown that the growth of the amplitudes of such surface oscillations in the development of the instability is limited in the same way as in the volume case by the nonlinear shift in the frequency of the excited mode.

2. STATEMENT OF THE PROBLEM. THE LINEAR APPROXIMATION

We consider a nonisothermal ($T_e \gg T_i$) plasma with a current, occupying the half-space $x \geq 0$, in which the electrons drift relative to the ions parallel to the surface of the plasma, $U \parallel z$. We are interested in the instability of such a plasma relative to short-wave surface excitations that travel along the plasma-vacuum interface and attenuate on both sides of it. As shown in[3], for the description of short-wave oscillations of a nonisothermal plasma, in both the linear and the nonlinear approximations, we can use the following set of equations:

$$\begin{gathered}\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{r}}-\frac{M}{m}\mathbf{E}\frac{\partial f_0}{\partial \mathbf{v}}=\frac{\nu_e}{N_0}\frac{\partial f_0}{\partial \mathbf{v}}\int \mathbf{v}f\,d\mathbf{v},\\ \frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\nabla)\mathbf{V}+\nu_i\mathbf{V}=\mathbf{E},\quad \frac{\partial N}{\partial t}+\operatorname{div}N\mathbf{V}=0,\\ \operatorname{div}\mathbf{E}=\frac{\omega_{Li}^2}{N_0}(N-N_e),\quad N_e=N_0+\int f\,d\mathbf{V}.\end{gathered}\tag{2.1}$$

All the notation above is in accord with that used in[3].

For the short-wave surface excitations of interest to us, the set (2.1) can be conveniently reduced to the two equations:[1)]

$$\begin{gathered}\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\nabla)\mathbf{V}+\nu_i\mathbf{V}=-\nabla\Phi,\\ \operatorname{div}\left(\frac{\partial^3}{\partial t^3}-\nu_i\omega_{Li}^2+\omega_{Li}^2\frac{\partial}{\partial t}\right)\nabla\Phi-\frac{1}{r_{De}^2}\frac{\partial^3\Phi}{\partial t^3}\\ +\frac{1}{r_{De}^2}\int d\mathbf{r}'Q(\mathbf{r}-\mathbf{r}')\left(\frac{\partial}{\partial t}+\mathbf{u}\nabla\right)\frac{\partial^3\Phi}{\partial t^3}\\ +\operatorname{div}\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}(\mathbf{V}\Delta\Phi)+\omega_{Li}^2(\mathbf{V}\nabla)\mathbf{V}\right]=0,\end{gathered}\tag{2.2}$$

where $\mathbf{E} = -\nabla\Phi$ and the quantity $Q(\mathbf{R})$ is determined in[3]:

$$Q(\mathbf{R})=\frac{1}{v_{Te}}\left(\sqrt{\frac{\pi}{2}}\frac{1}{(2\pi)^3}\int\frac{d\mathbf{k}}{k}e^{i\mathbf{k}\mathbf{R}}+\frac{\nu_e}{4\pi R v_{Te}}\right),$$

Equations (2.2) describe the behavior of the field of oscillations not only in a plasma, but also in a vacuum. Actually, as $\omega_{Li}^2 \to 0$, the second equation of (2.2) transforms into the Laplace equation

$$\Delta\Phi=0.\tag{2.3}$$

Furthermore, this set is obtained under the assumption of arbitrary inhomogeneity of the plasma density $N_0(\mathbf{r})$, and therefore it is valid over all space. This permits us to obtain the boundary conditions directly from Eqs. (2.2) by integrating them over a physically infinitely narrow (in comparison with the wavelength of the ion-acoustic oscillations) transition layer near the plasma-vacuum boundary.

In the linear approximation, we get from (2.2)

$$\begin{gathered}\operatorname{div}\left(\frac{\partial^3}{\partial t^3}-\nu_i\omega_{Li}^2+\omega_{Li}^2\frac{\partial}{\partial t}\right)\nabla\Phi\\ +\frac{1}{r_{De}^2}\int dr'Q(r-r')\left(\frac{\partial}{\partial t}+\mathbf{u}\nabla\right)\frac{\partial^3\Phi}{\partial t^3}=0.\end{gathered}\tag{2.4}$$

The boundary conditions here are written in the form

$$\left\{\left(\frac{\partial^3}{\partial t^3}-\nu_i\omega_{Li}^2+\omega_{Li}^2\frac{\partial}{\partial t}\right)\frac{\partial\Phi}{\partial x}\right\}_{x=0}=0,\quad \{\Phi\}_{x=0}=0.\tag{2.5}$$

Here the symbol $\{\}$ denotes the discontinuity of the corresponding quantity at the plasma boundary.

Now, when the boundary conditions are obtained, we can solve Eq. (2.4) separately in the vacuum ($x < 0$) and in the plasma ($x > 0$), assuming the plasma to be homogeneous in this case. For a solution of the type of surface waves $\Phi = \Phi(x)\exp(-i\omega t + ik_z z)$, we find

$$\Phi(x) = \begin{cases} C_{1v}\exp(|k_z|x), & x<0 \\ C_{1p}\exp(-\varkappa_1 x), & x>0 \end{cases} \qquad (2.6)$$

where

$$\varkappa_1^2 = k_z^2 + i\frac{\omega - k_z u}{r_{De}^2\varepsilon_i(\omega)}\frac{1}{k_0 v_{Te}}\left(\sqrt{\frac{\pi}{2}} + \frac{\nu_e}{k_0 v_{Te}}\right),$$
$$k_0^2 = -\frac{1}{r_{De}^2\varepsilon_i(\omega)}, \quad \varepsilon_i(\omega) = 1 - \frac{\omega_{Li}^2}{\omega^2}\left(1 + i\frac{\nu_i}{\omega}\right). \qquad (2.7)$$

Substituting the solution (2.6) in the boundary condition (2.5), we obtain a set of two homogeneous algebraic equations for the coefficients C_{1v} and C_{1p}, the condition for the solution of which leads to a dispersion relation for the determination of the spectrum of the surface waves under consideration:

$$\varepsilon_i(\omega)\varkappa_1 = -|k_z|. \qquad (2.8)$$

When the smallness of the imaginary components in Eqs. (2.7), is taken into account, it follows that the surface ion oscillations are important only in the range of frequencies $\omega < \omega_{Li}$, and the ir spectrum is determined by the formulas ($\omega \to \omega + i\gamma$)

$$\omega = \frac{\omega_{Li}}{\sqrt{2}}, \quad \gamma = -\frac{\nu_i}{2} - \frac{(\omega - k_z u)}{8\sqrt{2}k_z^2 r_{De}^2}\left(\sqrt{\frac{\pi}{2}} + \frac{\nu_e}{\omega_{Le}}\right). \qquad (2.9)$$

So far as the quantities ν_i are concerned, $\nu_i = 3.2\,\nu_{ii}T_i/T_e$ in a fully ionized plasma[2] (here $\nu_e = \nu_{ei}$), and $\nu_i = \nu_{i0}$ in a weakly ionized one (here $\nu_e = \nu_{e0}$).

Equating the expression for the increment of oscillation growth γ to zero and minimizing the thus-obtained drift velocity of the electrons $u(k_z)$ with respect to k_z, we obtain the excitation threshold for the surface modes of oscillation in a nonisothermal plasma with a current:

$$\frac{u_{\rm thr}}{v_s} = 4\left[\sqrt{\frac{M}{m}}\frac{\nu_i}{\omega_{Li}}\left(\sqrt{\frac{\pi}{2}} + \frac{\nu_e}{\omega_{Le}}\right)^{-1}\right]^{1/2}, \quad k_{z\,\rm thr}r_{De} = \frac{\sqrt{2}v_s}{u_{\rm thr}}. \qquad (2.10)$$

We note that in a fully ionized plasma, ion-acoustic waves waves are significant only under the condition $\nu_e \ll \omega_{Le}$, when collisions of electrons with ions can be neglected in comparison with the Cerenkov dissipation of the field by the electrons.

It follows from Eq. (2.10) that short-wave surface oscillations can be excited (just as in the volume case) at relatively small drift velocities of the electrons $u \ll v_s$, but satisfaction of the relation

$$\sqrt{\frac{\pi}{2}} + \frac{\nu_e}{\omega_{Le}} > \frac{\nu_i}{\omega_{Li}}\sqrt{\frac{M}{m}} \qquad (2.11)$$

is necessary in this case. This condition is assumed satisfied in what follows.

3. NONLINEAR EVOLUTION OF THE SURFACE ION-ACOUSTIC WAVE

In the study of the nonlinear stage of development of the ion-acoustic instability of the surface modes, it is convenient to introduce the above-threshold parameter

$$\varepsilon = u/u_{\rm thr} - 1. \qquad (3.1)$$

Here the oscillation growth increment (2.9) is written in the form $\gamma = \nu_i\epsilon$. Near threshold, when $\epsilon \ll 1$, one must expect excitations of the first few harmonics of the fundamental mode with $k_z = k_{z\,\rm thr}$. Therefore, limiting ourselves to two harmonics, we have the solution of the nonlinear problem in the form

$$\Phi(x) = \Phi_1(x)\exp\{-i\omega t + ik_z z\} + \Phi_2(x)\exp\{-2i\omega t + 2ik_z z\}. \qquad (3.2)$$

This materially simplifies the analysis of the system (2.2), which is easily reduced to a single nonlinear equation in such an approximation (with accuracy to within terms of third order in Φ).[3] In view of its cumbersome form, we shall not write it out here. We only note that the solution is obtained from the second equation of (2.2) by substituting in it the approximate solution of the first equation:

$$\mathbf{V} = -\hat{t}^{-1}\nabla\Phi - \hat{t}^{-1}[(\hat{t}^{-1}\nabla\Phi)\nabla](\hat{t}^{-1}\nabla\Phi). \qquad (3.3)$$

Here the operator $\hat{t}^{-1}$ denotes integration with respect to time with account of the principle of causality and the adiabatic turning-on of the excitation field at some time in the infinitely distant past, i.e.,

$$\hat{t}^{-1}A(t) = \int_{-\infty}^{t} dt' A(t'). \qquad (3.4)$$

The set of equations (2.2) is supplemented by nonlinear boundary conditions obtained by integrating these equations over the narrow transition layer near the surface of the plasma, in the form:

$$\{\Phi\}_{x=0} = 0,$$
$$\left\{\left(\frac{\partial^3}{\partial t^3} - \nu_i\omega_{Li}^2 + \omega_{Li}^2\frac{\partial}{\partial t}\right)\frac{\partial\Phi}{\partial x} + \frac{\partial^2}{\partial t^2}\Delta\Phi\cdot V_x + \frac{\partial^3}{\partial z\,\partial t^2}\frac{\partial\Phi}{\partial x}V_z - \omega_{Li}^2\frac{\partial}{\partial t}(\mathbf{V}\nabla)V_x\right\}_{x=0} = 0, \qquad (3.5)$$

where $\mathbf{V}$ is determined by the expression (3.3).

We now substitute the solution in the form (3.2) in the nonlinear equation for Φ (the first equation of (2.2) with the substitution of (3.3)) and the boundary conditions (3.5), and equate the coefficients of like powers in the exponentials. As a result, we get a set of nonlinear equations for the functions $\Phi_1(x)$ and $\Phi_2(x)$ with nonlinear boundary conditions. These equations have been solved by the method of successive approximations under the assumption of the smallness of the nonlinear components, and the solution of the first (linear) approximation of the fundamental has been written in the form of (2.6), (2.7). The complete solution of the problem has the form

$$\Phi_1(x) = \begin{cases} C_{1v}e^{|k_z|x} & \text{for } x<0 \\ C_{1p}e^{-\varkappa_1 x} + C_{1p}^* C_{2p}\dfrac{2k_z^2}{\varepsilon_i(\omega)}\dfrac{\omega_{Li}^2}{\omega^4}e^{-3|k_z|x} & \text{for } x>0 \end{cases}, \qquad (3.6)$$

$$\Phi_2(x) = \begin{cases} C_{2v}e^{2|k_z|x} & \text{for } x<0 \\ C_{2p}e^{-\varkappa_2 x} & \text{for } x>0 \end{cases}, \qquad (3.7)$$

where

$$\varkappa_2^2 = 4k_z^2 + i\sqrt{2\frac{m}{M}}\frac{\omega_{Li}(\omega - uk_z)}{\varepsilon_i(2\omega)v_s^2}\left(\frac{1}{i}\sqrt{\frac{\pi}{2}} - \frac{\nu_e}{\sqrt{2}\omega_{Le}}\right). \qquad (3.8)$$

Substituting Eqs. (3.6) and (3.7) in the boundary conditions (3.5), we get the set of equations for the amplitudes of the first and second harmonics of the considered wave in the plasma (after simple transformations)

$$(C_1 = C_{1p}, \quad C_2 = C_{2p}):$$
$$(\varepsilon_i(\omega)\varkappa_1 + k_z)C_1 = 3\frac{k_z^3}{\omega^2}C_1^* C_2, \qquad (3.9)$$
$$(\varepsilon_i(2\omega)\varkappa_2 + 2k_z)C_2 = \frac{k_z^3}{\omega^2}C_1^2.$$

The set of equations (3.9) determines the stationary value of the amplitude of the surface ion-acoustic wave near the instability threshold and the nonlinear shift of its frequency:

$$|C_1|^2 = {}^3/_2\beta\varepsilon u_{\rm thr}^4,$$
$$\omega^2 = \frac{\omega_{Li}}{2}(1 + 12\varepsilon\beta), \qquad (3.10)$$

where

$$\beta = \left(\sqrt{\frac{\pi}{2}} + \frac{\nu_e}{\omega_{Le}}\right)\left(\sqrt{\frac{\pi}{2}} + 3\frac{\nu_e}{\omega_{Le}}\right)^{-1}.$$

Further, recognizing that $\mathbf{E} = e\mathscr{E}/M = -\nabla\Phi$, we then find the stationary value of the electric field intensity in the nonlinear surface wave:

$$\frac{|\mathscr{E}_z|^2}{4\pi N_0 T_e} = 3\varepsilon\beta \frac{u_{\text{thr}}^2}{v_s^2}, \quad (3.11)$$

where N_0 is the equilibrium plasma density.

4. DISCUSSION OF RESULTS

A number of conclusions can be drawn from the results obtained above.

1. Short-wave surface ion-acoustic waves are excited in a nonisothermal plasma (just as for the volume case) at relatively low current velocities of the electrons, $u < v_s$. A comparison of the expression (2.10) for the threshold velocity with the corresponding expressions in[3] (see Eqs. (2.3) and (2.4)) shows that under the conditions of excitation of the short-wave modes of oscillation, i.e., upon satisfaction of the inequality (2.11) in the plasma, surface waves are always excited earlier; the threshold velocities for their excitation are lower than for the excitation of the volume modes.

2. Short-wave surface waves can be excited in a plasma only upon satisfaction of the condition (2.11). In the opposite case, u_{thr} is greater than the sound velocity v_s and long-wave sound oscillations will be excited in the plasma. This conclusion also applied to volume oscillations.

3. Surface oscillations arise in a plasma at a depth of the order $\delta \approx 1/k_{z\,thr} \approx u_{thr}/\omega_{Li}$. This quantity is, on the one hand, rather larger, considerably exceeding the Debye radius, which is determined by the ions, $\delta \gg r_{Di}$, so that we can neglect the inhomogeneity of the boundary of the plasma and justify the assumption of an abrupt boundary; on the other hand, it is sufficiently small, $\delta \lesssim v_s/\omega_{Li} = r_{De}$, for us to consider the plasma as semi-unbounded for all practical purposes without loss of generality, in the study of the excitation of surface waves; in any real plasma, the curvature of the surface is considerably larger than δ.

4. The fundamental nonlinear mechanism that limits the growth of the amplitude of short-wave surface oscillations is, just as for the volume case, the nonlinear shift in the frequency of the excited mode, which is determined by Eq. (3.10). The level of oscillations that are established for a given superthreshold value ϵ in the case of excitation of short-wave surface oscillations is less than in the excitation of volume modes. This is also a consequence of the smallness of the ratio u_{thr}/v_s in the case of the surface instability.

Finally, we note that all the results obtained above are valid in a moving coordinate system which moves relative to the laboratory with constant velocity $u_0 = 3\epsilon\beta u_{thr}$, due to the action of the average force of the field of the wave. The frequency shift found by us in the transition to the laboratory system is compensated by the Doppler shift $k_{z\,thr}u_0$, which is in complete accord with the conclusion drawn in[5] that there is no nonlinear frequency shift for Langmuir oscillations of a cold plasma in the laboratory system of coordinates. This remark also applies to[3].

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1) We note that in the second equation of (2.2), there remain in the linear part small terms of order r_{De}^{-2}/λ^{-2}, which are necessary for the correct determination of the damping of ion-acoustic waves; in the nonlinear part, such components are omitted.

2) We note that we mean by ν_i in the set (2.1) and in what follows the operator $\nu_i = -(8/5)\nu_{ii}v_{Ti}^2\Delta/\omega^2$, and therefore $\nu_i = (8/5)\nu_{ii}k_0^2v_{Ti}^2/\omega^2 = (16/5)\nu_{ii}T_i/T_e$ in Eqs. (2.9) and (2.10).

3) When account is taken of terms of third order in Φ, it may be also necessary to include the third harmonic in the expansion (3.2). However, the calculations show that the allowance for the third harmonic leads to insifnificant corrections to the nonlinear phenomena investigated below.

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220