Thermal ionization wave in a current-carrying system

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(Submitted August 17, 1972)
Zh. Eksp. Teor. Fiz. **64**, 1245-1251 (April 1973)

The generation and propagation of a plasma column when a strong electric field is applied to a plane layer of neutral gas is discussed. It is shown in the one-dimensional case that if, at an initial instant of time, electrical breakdown in the neutral gas produces a layer of plasma with finite dimensions, its subsequent expansion may occur with a constant speed $u \ll c_S$, where c_S is the velocity of sound, i.e., without the formation of a strong shock wave. The rate of expansion of the plasma column is then controlled by diffusion and ionization processes, the combined effect of which leads to the possibility of a time-independent wave propagating through the gas. Ionization of the gas by epithermal electrons then occurs on the wave front.

Consider a plane capacitor with large plates, the space space between which is filled with a neutral gas of density n_0. A potential difference is applied between the plates, producing an electric field E_0 along the y axis which is at right-angles to them. We shall suppose that electrical breakdown in the gas initially produces a plane layer of plasma (for the sake of simplicity, we shall assume that this layer is restricted in the direction of the x axis but is unbounded in the direction of the z axis). Subsequent behavior of the plasma column depends on the magnitude of the field E_0. If this field is strong enough to accelerate the electrons to energies at which cascade ionization of the gas begins ($E > E_{crit}$), the expansion of the plasma column occurs very rapidly because the active electrons appear throughout the space as a result of diffusion, and subsequent plasma formation takes place in the form of a shower. In practice, however, the source has a restricted strength and we have the situation where, because of the presence of a high resistance in the external circuit, the field falls after breakdown to a value below E_{crit} so that the plasma column carries a current which can only lead to its heating to temperatures $T \sim 1-3$ eV.[1] It is clear that, in this case, the diffusion of plasma and the ionization of the neutral gas by epithermal electrons results in an increase in the width of the plasma layer but the rate of its expansion is such lower.

It is important to note that when $E_0 < E_{crit}$ it is possible to have a different situation, in which sufficient energy is liberated in a narrow plasma layer in a short period of time so that a strong shock wave is generated and the ionization of the gas takes place behind the shock wave front. This case was discussed in detail by D'yachenko and Imshennik[2] and we shall not consider it here in detail. We merely note that to ensure that the shock wave can ionize the gas behind the shock-wave front its intensity must be very high, for example, if the temperature in the region facing the shock-wave front is $T_0 \approx 300°K$ and behind the front $T_1 \approx 10^{4}°K$ (gas ionization occurs at precisely such temperatures) then the Mach number for the wave will have to be $M \approx 15$. A very strong gas-heating source would, of course, be necessary to produce such a wave. Moreover, since the velocity of this shock wave would be expected to be of the order of the velocity of sound in the heated gas, high currents would be necessary to maintain it.

We shall suppose below that a strong shock wave is not produced in the system, and that weak shock waves lead merely to slight heating of the gas, whereas the ionization waves are accompanied by the emission of radiation behind the front.

Thus, suppose that the gas to which the electric field E_0 has been applied contains a plane layer of plasma with electron temperature $T_e \gtrsim T_i \gg T_n$, where T_i, T_n are, respectively, the ion and neutral-particle temperatures. The behavior of this layer can be described by the following set of equations of two-fluid hydrodynamics:

$$\begin{aligned} 0 &= -eE - T_e\frac{\partial \ln n}{\partial x} - \nu_{en} m V_e, \\ 0 &= eE - T_i\frac{\partial \ln n}{\partial x} - \nu_{in} M V_i \end{aligned} \tag{1}$$

and

$$\begin{aligned} &\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nV_e) \cong C\left(V_{Te}\frac{J}{T_e}e^{-J/T_e}n(n_0-n) - 10^{-14}\frac{J}{T}n^3\right), \\ &\frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\chi\frac{\partial T}{\partial x} = \sum_i P_i - \sum_i Q_i; \quad C \cong 10^{-17}\left[\frac{\text{cm}^2}{\text{eV}}\right]. \end{aligned} \tag{2}$$

In these equations ν_{en} and ν_{in} are the collision frequencies of electrons and ions with neutrals, respectively, J is the ionization potential of the atoms, $\chi = T_e/(\nu_{ee}m + \nu_{en}m)$ is the electron component of electrical conductivity, ν_{ee} is the frequency of electron-electron collisions, and P_i and Q_i are, respectively, the plasma heating and cooling sources due to the passage of current, ionization, excitation, recombination, emission, and so on (we have purposely avoided writing out the specific form of the functions P_i and Q_i for reasons discussed below).

The right-hand side of the continuity equation contains terms describing the rate of increase in the electron density due to the ionization of the gas by epithermal electrons[3] and their recombination [it is assumed that $T_e < J$, so that the gas ionization is produced only by electrons with velocities $V > (2J/m)^{1/2}$ and when their distribution is Maxwellian this contribution is exponentially small]. The equations of continuity and thermal conductivity are written out only for the electron component because it is clear that the ion and electron densities and velocities are equal, and the temperature transfer is determined largely by the electron thermal conductivity. Combining the two equations in Eq. (1) and substituting for V in the first equation in Eq. (2), we obtain the ambipolar diffusion equation, in which case Eq. (2) can be rewritten in the form

$$\frac{\partial n}{\partial t} - \frac{\partial}{\partial x}D\frac{\partial n}{\partial x} = \gamma(T)n(n_0-n) - \beta(T)n^3,$$

$$\frac{\partial T}{\partial t}-\frac{\partial}{\partial x}\chi\frac{\partial T}{\partial x}=\sum_i P_i-\sum_i Q_i. \tag{3}$$

In these expressions $D = (T_e + T_i)/(M\nu_{in} + m\nu_{en})$ is the ambipolar diffusion coefficient and

$$\gamma(T)=CV_{T_e}\frac{J}{T_e}e^{-J/T_e}\left[\frac{\mathrm{cm}^3}{\mathrm{sec}}\right],\quad \beta(T)=10^{-11}C\frac{J}{T_e}\left[\frac{\mathrm{cm}^6}{\mathrm{sec}}\right].$$

It has been pointed out above that the strength of the source is assumed to be sufficient to maintain the electron temperature at 1–5 eV in the region of effective heating. If we now compare the diffusion coefficient D and the thermal conductivity χ it turns out that their ratio is much less than unity:

$$\frac{D}{\chi}=\left(\frac{m}{M}\right)^{1/2}\frac{n}{n_0}\frac{\sigma_{ee}}{\sigma_{in}}\approx\left(\frac{m}{M}\right)^{1/2}\frac{n}{n_0}\frac{3\cdot 10^{-14}}{\sigma_{in}}\sim 10\left(\frac{m}{M}\right)^{1/2}\frac{n}{n_0}\ll 1.$$

In this expression $\sigma_{in} \approx 3 \times 10^{-15}$ is the resonance charge-transfer cross section. It follows from the foregoing calculation that temperature equalization occurs more rapidly than the density equalization, and we can assume henceforth that, whenever there are electrons, their temperature is constant and time-independent. This enables us to eliminate from the analysis the equation of thermal conductivity, and assume that the temperature in the diffusion equation is constant. Using the solution obtained below for the plasma-density profile, we can readily show that the change in the temperature due to thermal conduction across a characteristic inhomogeneity is, in fact, $\Delta T/T_0 \sim D/\chi \ll 1$. It is also important to note that the opposite limiting case, when the increase in the electron density due to ionization occurs more rapidly than the temperature equalization in the wave, has been investigated by Velikhov and Dykhne.[4] Without loss of generality, we can also neglect the recombination term in Eq. (3). This, in any case, is valid in hydrogen for $T_e \approx 3$ eV and $n \leq 10^{18}$ cm^{-3}.[1] Moreover, it is clear that even when the opposite inequalities are valid, the replacement of the term proportional to n^3 by a similar term proportional to n^2 can only lead to a small numerical change in the parameters characterizing the process but cannot, to any great extent, change the physics of the transport process. Bearing in mind the foregoing remarks, we shall rewrite Eq. (3) in the form

$$\frac{\partial n}{\partial t}-D\frac{\partial^2 n}{\partial x^2}=\alpha n\left(1-\frac{n_0}{n}\right), \tag{4}$$

where $\alpha = \gamma n_0$.

Equation (4) must satisfy the following boundary conditions: $n(x = -\infty) = n_0$, $n(x = +\infty) = 0$. As usual, in the case of problems involving the generation of a wave, we shall consider a half-space rather than a plasma layer. Dimensional analysis shows that Eq. (4) has a parameter with the dimensions of velocity $(\alpha D)^{1/2}$ and, therefore, we may assume that it has a traveling wave-type solution. Substituting the dimensionless variable $\xi = (x - ut)u/2D$, where u has the dimensions of velocity but, for the moment, is arbitrary, we have

$$y''+2y'+\beta y(1-y)=0. \tag{5}$$

In this expression, $y = n/n_0$, $\beta = 4\alpha D/u^2$ and the differentiation is with respect to ξ. Equation (5) can be satisfied by the following boundary conditions:

$$y(\xi=-\infty)=1,\quad y(\xi=+\infty)=0. \tag{5'}$$

Moreover, this equation describes the behavior of plasma density and, therefore, for any ξ,

$$0\leqslant y\leqslant 1. \tag{5''}$$

We have noted that u is the unknown wave velocity. It is intuitively obvious that this quantity must be unique, i.e., the wave velocity should be unambiguously determined by the plasma parameters. Analysis of Eq. (5) for $y \ll 1$ shows that, when $\beta > 1$, the function describing the relation between y and ξ changes sign [$y \propto \sin\sqrt{\beta-1}\,\xi$) and, consequently, it does not satisfy Eq. (5'')]. Therefore, if the wave exists, it can propagate through the gas only with velocity $u \geq (4\alpha D)^{1/2}$. It turns out, however,[5], that for an equation such as Eq. (5) there is no unique characteristic value of the parameter β for which the solution would satisfy the imposed boundary conditions, i.e., Eq. (4) has wave-type solutions with any $\beta \leq 1$.

Analytically, a solution of this kind can be obtained in approximate form for $\beta \ll 1$. In fact, if we neglect in Eq. (5) the term with the highest-order derivative and integrate, we obtain

$$y=Ce^{-\beta\xi/2}/(1+Ce^{-\beta\xi/2}). \tag{6}$$

It is clear that Eq. (6) satisfies all the above boundary conditions and the relation $y''/y' \sim \beta \ll 1$, so that the approximation which we have selected is correct. However, Kolmogorov et al.[6] have investigated an equation similar to Eq. (4) for a monotonic initial distribution of density, and showed that the solution of Eq. (4) tends asymptotically in time to the solution with the characteristic velocity $u_0 = (4\alpha D)^{1/2}$, i.e., $\beta = 1$. It has also been shown[6] that all the wave solutions with velocity $u > u_0$ ($\beta < 1$) tend asymptotically to zero, i.e., $y(x - ut, t) \to 0$ as $t \to \infty$, Since, on the other hand, there are no time-independent solutions with $\beta < 1$, the only time-independent solution to which the system will tend is a wave traveling with the speed $u_0 = (4\alpha D)^{1/2}$.

The physical meaning of this process can be elucidated as follows. Suppose that at time $t = 0$ the plasma density is described by Eq. (6) with $\beta \ll 1$. Since Eq. (6) satisfies Eq. (5) for any $t \geq 0$, it is clear that the only process which can lead to a distortion of the profile is its instability with respect to small perturbations. As is usual in studies of the stability of solutions, we shall write the solution in the form of the sum

$$y(\xi,t)=y_0(\xi)+\delta y_1(\xi,t),$$

where $y_0(\xi)$ satisfies the unperturbed solution of Eq. (5), with $\beta \ll 1$ and $\delta \ll \beta \ll 1$. The function $y_1(\xi, t)$ must satisfy the equation

$$\beta\frac{\partial y_1}{\partial\tau}-2y_1'-y_1''-\beta y_1(1-2y_0)=0 \tag{7}$$

which is obtained from Eq. (4) by introducing the variables ξ and $\tau = \alpha t$ followed by linearization and the imposition of the boundary conditions $y_1(\xi = \pm\infty) = 0$. It is readily seen that when $\beta \ll 1$, Eq. (7) has unstable solutions

$$y_1=e^{\lambda\tau}\frac{Ce^{-\beta(1-\lambda)\xi/2}}{(1+Ce^{-\beta\xi/2})^2} \tag{8}$$

for any $0 \leq \lambda < 1$.

Therefore, solutions such as Eq. (6) with $\beta \ll 1$ are unstable against small perturbations. To understand the consequences of this instability we must consider a system with weak nonlinearity when the perturbation amplitude can no longer be regarded as infinitesimal and we must retain in Eq. (4) terms which are quadratic in $\delta y_1(\xi, t)$. We shall now suppose that the constant C in the unperturbed density distribution $y_0(\xi)$, which satisfies Eq. (6), is a slowly varying function of time. In the second approximation, it then follows from Eq. (4) that

$$\frac{\partial \ln C}{\partial \tau} = -\delta^2 e^{2\lambda\tau+\beta\lambda\xi}\frac{Ce^{-\beta\xi/2}}{(1+Ce^{-\beta\xi/2})^2} \tag{9}$$

Assuming for simplicity that $\lambda \ll 1$ and $\delta^2 e^{2\lambda\tau+\beta\lambda\xi} \approx \text{const} = \epsilon^2$, we have from Eq. (9)

$$\frac{\partial \ln C}{\partial \tau} = \frac{2\varepsilon^2}{\beta} y_0'. \tag{10}$$

Integrating this equation with respect to ξ within the limits of the front width $0 < \xi < 2/\beta$ (this corresponds, in fact, to averaging Eq. (10) over a time interval less than the reciprocal of the growth rate), we have

$$C = C_0 e^{-4\varepsilon^2\tau/\beta^2}. \tag{11}$$

Substituting for C from Eq. (11) into Eq. (6), we obtain

$$y_0 = \frac{C_0 e^{-\gamma\tau} e^{-\beta\xi/2}}{1 + C_0 e^{-\gamma\tau} e^{-\beta\xi/2}}, \tag{12}$$

where $\gamma = 4\epsilon^2/\beta^2$. Since we are interested in the value of y_0 on the wave front, where $\xi = (x - ut)u/2D \approx 0$, we can replace τ with $\alpha x/u$ in Eq. (12), and if we use the variables x and t, we obtain

$$y_0 = \frac{C_0 e^{\alpha t} e^{-\alpha(1+\gamma)x/u}}{1 + Ce^{\alpha t} e^{-\alpha(1+\gamma)x/u}}. \tag{13}$$

Let us now return to Eq. (6) and consider how it is modified when the wave velocity is slightly reduced. Substituting $u = u_0 - \Delta u$ ($\Delta u \ll u_0$) in Eq. (6), we obtain

$$y_0 = C_0 e^{\alpha t} \exp\left[-\frac{\alpha}{u_0}\left(1+\frac{\Delta u}{u_0}\right)x\right] \Big/ \left\{1 + C_0 e^{\alpha t}\exp\left[-\frac{\alpha}{u_0}\left(1+\frac{\Delta u}{u}\right)x\right]\right\}. \tag{14}$$

It is clear that the functions given by Eqs. (13) and (14) are the same if $\Delta\mu/\mu_0$ is replaced with γ in Eq. (14). It follows that the density-profile instability for $\beta \ll 1$ over the nonlinear stage leads to an effective reduction in the wave velocity. This continues until the wave velocity reaches its maximum possible value: $u = (4\alpha D)^{1/2}$ or $\beta = 1$. Subsequent reduction in the velocity cannot occur for reasons described above. It is important to emphasize once again that the foregoing considerations should be looked upon only as a physical interpretation of the wave process and not the strict mathematical theory which is given in[6]. Thus, for any monotonic initial distribution of density the solution of Eq. (4) tends asymptotically to a wave-type solution with the wave velocity given by $u = (4\alpha D)^{1/2}$.

Equation (4) was solved numerically to determine the rate at which the wave front is established. Substituting the dimensionless quantities $y = n/n_0$, $\tau = \alpha t$, $\eta = (\alpha/D)^{1/2}x$ in Eq. (4), we obtain

$$\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial \eta^2} = y(1-y). \tag{15}$$

The boundary conditions are $y(\eta, \tau)_{\eta=-\infty} = 1$, $y(\eta, \tau)_{\eta=+\infty} = 0$.

The implicit four-point approximation scheme and the sweep method were used to solve Eq. (15). In the finite difference form we used grids with a time interval $\Delta\tau$ and a coordinate interval h so that Eq. (15) assumed the form

$$Ay_{i-1}^{j+1} - Cy_i^{j+1} + By_{i+1}^{j+1} = -F_i.$$

In this expression,

$$y_i^j = y(\tau_0 + i\Delta\tau,\ \xi_0 + jh),\quad j = 1, 2, \ldots, N,\quad i = 1, 2, \ldots, M,$$
$$A = B = \frac{1}{h^2},\quad C = \frac{1}{\Delta\tau} + \frac{2}{h^2},\quad F_i = \frac{y_i^j}{\Delta\tau} + y_i^j(1 - y_i^j),$$

where N is the total number of steps along the τ axis and M is the number of steps along the η axis.

The principle of the sweep method is that the value of the function at a given point at some instant of time is determined in terms of the value of this function at a neighboring point as follows: $y_{i+1}^j = \alpha_{i+1} y_{i+}^j + \beta_{i+1}$, where the coefficients α and β are definite functions of A, B, C, and F_i.[7] The system is stable for any positive h and $\Delta\tau$.

Figure 1 shows the results of the calculations for an initial electron density distribution in the form of a step, and Fig. 2 shows the corresponding situation when the initial electron distribution is less steep. However, in both cases, the wave-front profile tends to the same form and is established in the same interval of time $t = 6/\alpha$. The wave-front width is $\Delta \sim 2(D/\alpha)^{1/2}$. The results shown in Figs. 1 and 2 enable us to determine more accurately the velocity of the wave, which is $u_0 = 1.8(\alpha D)^{1/2}$. This is somewhat less than the analytically derived upper limit $u_0 = 2(\alpha D)^{1/2}$. The discrepancy may be due to the following. We have already noted that the solutions of Eq. (6) with $\beta \ll 1$ are unstable against small perturbations, and this instability leads to an effective reduction in the wave velocity. Numerical calculations show that even solutions with $\beta = 1$ are unstable and, therefore, when the critical velocity is reached by the wave, it tends to reduce it still further, leading to the time-independent regime, i.e., it tends to return to the state with $u_0 = 2(\alpha D)^{1/2}$. On average, such oscillations may produce a net reduction in the velocity as compared with its calculated value.

Finally, let us briefly consider the effect of the magnetic field of the current on the expansion of the plasma column. We have assumed throughout that the magnetic field is zero in the system. However, when we consider the expansion of a plasma column or cylinder of finite dimensions (it is clear that our results are valid for these cases), the system is subjected to the intrinsic magnetic field of the current, which can be neglected only during the initial stage of the process when the radius of the plasma column is small enough. As it increases, the associated magnetic field increases in accordance with the expression $H_\varphi = 2I/R = 2\pi R\sigma E_0$, where I is the total current in the column, $I = \pi R^2 J = \pi R^2 \sigma E_0$, and J is the plasma conductivity. The current eventually reaches a value such that the magnetic pressure becomes comparable with the gas-kinetic pressure $H_\varphi^2/8\pi$

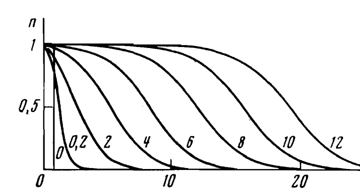

FIG. 1

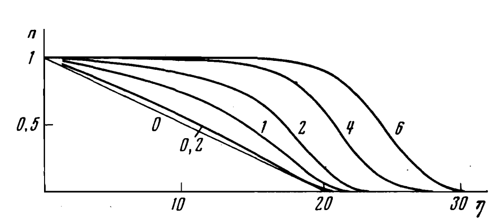

FIG. 2

$= nT$. The expansion of the plasma column then terminates and it begins to contract.

Thus, under certain definite conditions an initially narrow layer of plasma may expand at constant speed $u_0 = 2(\alpha D)^{1/2}$ and this expansion is not accompanied by the formation of strong shock waves but is controlled by diffusion and ionization processes. The characteristic width of the wave front over which the electron density is reduced by an amount comparable with the density itself is $\Delta \sim 2(D/\alpha)^{1/2}$. Any initial distribution of the plasma density is then found to tend asymptotically to the time-independent state described above, and the characteristic time necessary for this state to be reached is $t \sim 10\alpha^{-1}$ sec.

In conclusion, the authors wish to express their gratitude to R. Z. Sagdeev for valuable discussions.

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Translated by S. Chomet
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