On the theory of open resonators

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A general method is proposed for constructing the short wave asymptotic solutions of the integral equations of the theory of open resonators. It is shown that taking aberrations into account strongly perturbs the resonator modes in the case of frequency degeneracy. The first aberration correction for the zero mode of a stable resonator is evaluated. A formula is derived which relates diffraction losses to the amplitude of the proper oscillations at the edge of the mirror. The behavior of diffraction losses for resonators close to a confocal one is discussed.

INTRODUCTION

In order to obtain the short wave asymptotic behavior of the proper oscillations and the resonance frequencies of open resonators and of other quasioptical systems use is made basically of integral equations[1-4] and of the method of the parabolic equation[5-7].

Integral equations can be written down for a broader class of open resonators both stable and unstable, and they take into account (with an accuracy up to the Kirchhoff approximation) the finiteness of the resonator mirrors. However, an explicit solution of these equations has been obtained only for the simplest kernels which correspond to the paraxial approximation in the geometrooptical description of the resonators[2-4,8].

In this paper a method is proposed for constructing the short wave approximation to the solutions of integral equations of the theory of open resonators with kernels of a sufficiently general form which is based on applying the formula of the saddle point method. This method enables one in addition to the well-known results to discuss the effect of aberrations on the proper oscillations.

In Sec. 1 the problem is reduced to certain nonlinear functional equations which do not contain a large parameter. These equations play the same role as do the eikonal and transport equations in the WKB method for differential equations.

In Sec. 2 analytic methods are proposed for the solution of the functional equations thus obtained. It is shown that in the case of frequency degeneracy small changes in the shape of the mirror "destroy" proper oscillations. The first aberration correction to the zero mode of the resonator is obtained and the behavior of this correction is discussed as the resonator approaches a confocal one.

The problem of diffraction losses is discussed in Sec. 3. It is well known that, for example, in the case of a confocal resonator there are many unclear points in this problem. Experiments demonstrate a decrease in the quality factor for resonators close to a confocal one[9,10], while a theoretical investigation of an "ideal" confocal resonator, i.e., of a resonator described in the paraxial approximation, predicts minimum losses[8,11]. An explanation of this contradiction is possible only by invoking aberration corrections.

In a recent paper by Melekhin[12] a geometrical-optics description of stable resonators was proposed taking into account the nonlinearity in the transformation of the rays on reflection from mirrors. The results of the present paper are in good agreement with such a description.

Certain details of a mathematical nature, for example problems of convergence of the series being constructed, are discussed in another paper by the present author[13] and will be considered later.

1. Derivation of the basic functional equations

The integral equations for fields at the resonator mirrors have the following form[1,14]:

$$U_p(M_p)=\gamma_p\int\limits_{\sigma_p}\frac{\exp[ikR_{pq}(M_p,M_q)]}{R_{pq}(M_p,M_q)}U_q(M_q)\,dS_q. \tag{1}$$

Here (and everywhere below) p and q are subscripts, not equal to each other, which take on the values 1 and 2; $R_{pq}(M_p, M_q)$ is the distance between the points M_p and M_q on the one and on the other mirror of the resonator; k is the wave number; integration is taken over the surface of the mirrors. It is assumed that TE-oscillations with Neumann conditions at the mirrors are realized in the resonator, and that the resonator parameters satisfy the conditions

$$kl\gg1,\quad ka^2/l\gg1,\quad l/a\gg1,\quad \rho/a\gg1, \tag{2}$$

where l is the length of the optic axis of the resonator, a is a quantity characterizing the transverse dimension of the mirrors, ρ is the minimum radius of curvature of the mirror surface. In subsequent discussion we choose the length of the optic axis as the scale unit ($l = 1$), so that k and a will be dimensionless quantities.

In order to have the possibility of carrying out a sufficiently detailed investigation we go over from system (1) to the simplified one-dimensional system of equations

$$U_p(x)=\gamma_p\sqrt{\frac{k}{2\pi}}\int\limits_{-a_q}^{a_q}\exp[ikF_{pq}(x,x')]G_{pq}(x,x')U_q(x')\,dx'. \tag{3}$$

The system (3) can be obtained if we assume that it is possible in (1) to carry out a separation of variables in Cartesian coordinates on the mirrors. One is also led to equation (3) as a result of considering the problem of a resonator not in three-dimensional space, but on a plane. Although in the latter case the zero order Hankel function will be the kernel of the equations, by utilizing its asymptotic behavior and neglecting terms of order k^{-1} we obtain equation (3). It is just this last model that will be used for specific calculations.

We assume that the functions $F_{pq}(x, x')$ and $G_{pq}(x, x')$ can be represented in the form

$$F_{pq}(x,x')={}^1\!/_2g_px^2+{}^1\!/_2g_qx'^2-xx'+\tilde F_{pq}(x,x'), \tag{4}$$

$$G_{pq}(x, x') = 1 + \tilde{G}_{pq}(x, x'), \quad (5)$$

where we have separated the principal terms which define the paraxial approximation, and the terms $\tilde{F}_{pq}(x, x')$ and $\tilde{G}_{pq}(x, x')$ which are of higher order in x and x', and which take into account the aberration corrections and (phenomenologically) the properties of the mirror surfaces. The parameters g_p are determined by the radii of curvature of the mirrors on the optic axis R_p

$$g_p = 1 - R_p^{-1}. \quad (6)$$

We seek the proper oscillations $U_p^{(n)}(x)$ in the form

$$U_p^{(n)}(x) = e^{-kf(x)} \sum_{m=0}^{\infty} h_{pm}^{(n)}(x) k^{-m}. \quad (7)$$

The form of the solution (7) is suggested by the well known results for a stable resonator in the paraxial approximation with infinite limits of integration

$$U_p^{(n)}(x) = e^{-t^2/2} H_n(t), \qquad t = \sqrt{k}\sqrt{g_p/g_q - g_p^2}\,x, \quad (8)$$

where $H_n(t)$ are the Hermite polynomials[2,3]. Here one must introduce one clarification. The scale along the mirror and along the optic axis is chosen to be the same. In this case the domain of the oscillations of the first few natural oscillations of a stable resonator (the diameter of the light spot) will be of order $k^{-1/2}$. Comparing (8) and (7) one can conclude that in order to obtain within the domain of oscillations a uniform formula for the n-th mode one must obtain n terms of the expansion (7). In the present paper only the principal term of the expansion (7) will be investigated, so that the uniform formula will be obtained only for the zeroth natural oscillation. For the other proper oscillations the principal term of the asymptotic expansion determines the nature of the falling off of the field in the shadow zone beyond the caustics. In the case of unstable resonators the principal term of the asymptotic expansion gives a uniform approximation (up to a small neighborhood of the edges of the resonator mirrors) to the proper oscillation with smallest losses.

If one substitutes the expansions of $U_p^{(n)}(x)$ into the right hand sides of Eqs. (3), then the resulting integrals under certain quite general assumptions concerning the functions $F_{pq}(x, x')$, $G_{pq}(x, x')$, $f_p(x)$, $h_{pm}^{(n)}(x)$ (requirement of analyticity) can be evaluated by the saddle-point method, in which we assume that everywhere where it is required the contour of integration can be deformed from the real axis into the complex domain.

Utilizing the well-known formula of the saddle-point method in the first approximation[15]

$$\int_\alpha^\beta H(z) e^{k\Phi(z)} = \sqrt{\frac{2\pi}{k}} H(z_0) e^{k\Phi(z_0)} |\Phi''(z_0)|^{-1/2} \exp\left\{i \frac{\pi - \arg\Phi''(z_0)}{2}\right\},$$

where the saddle point z_0 is obtained from the condition $\Phi'(z_0) = 0$, and requiring that Eqs. (3) be satisfied with respect to the leading term in k, we obtain the system of functional equations

$$-f_p(x) = iF_{pq}(x, \varphi_p(x)) - f_q(\varphi_p(x)), \quad (9)$$

$$h_{p0}^{(n)}(x) = \gamma_p^{(n)} h_{q0}^{(n)}(\varphi_p(x)) T_{pq}(x, \varphi_p(x)). \quad (10)$$

The functions $\varphi_p(x)$ determine the saddle points and are obtained from the equations

$$i\frac{\partial F_{pq}(x, \varphi_p(x))}{\partial \varphi_p(x)} - f_q'(\varphi_p(x)) = 0, \quad (11)$$

while the functions $T_{pq}(x, \varphi_p(x))$ are equal to

$$T_{pq}(x, \varphi_p(x)) = G_{pq}(x, \varphi_p(x)) \,|\, ikF_{pq}(x, \varphi_p(x)) - f_q(\varphi_p(x)) \,|^{-1/2} \exp\{{}^1\!/\!{}_2 i[\pi - \arg(ikF_{pq}(x, \varphi_p(x)) - f_q(\varphi_p(x)))]\}. \quad (12)$$

In the derivation of Eqs. (9)–(11) the contribution from the end points of the integration was not taken into account. Having obtained the final results one can verify that both in the cases of stable and of unstable resonators it will be smaller than the contribution from the saddle point. Exceptions to this will be the plane and the concentric resonators. In general application of the proposed method to these cases leads to great difficulties, and this is connected, in all probability, with the fact that the edges of the mirrors in the case of these resonators play a considerably greater role in the formation of proper oscillations than in the other cases.

We differentiate Eqs. (9) with respect to x, utilize (11) and introduce a change of variable $x \to \varphi_p(x)$. We obtain a system of nonlinear functional equations for the functions $\varphi_p(x)$

$$\frac{\partial F_{pq}(x, \varphi_p(x))}{\partial \varphi_p(x)} + \frac{\partial F_{qp}(\varphi_p(x), \varphi_q(\varphi_p(x)))}{\partial \varphi_p(x)} = 0. \quad (13)$$

Equations (13) play the same role in the theory being developed here as do the Riccati equations in the one-dimensional variant of the WKB method or the eikonal equations in the multidimensional variant.

In the case of identical mirrors $F_{12}(x, x') = F_{21}(x, x')$ and $\varphi_1(x) = \varphi_2(x)$, so that the system (13) reduces to the one functional equation[15]

$$\frac{\partial F(x, \varphi(x))}{\partial \varphi(x)} + \frac{\partial F(\varphi(x), \varphi(\varphi(x)))}{\partial \varphi(x)} = 0. \quad (14)$$

In the paraxial approximation this equation has the following form

$$\varphi(\varphi(x)) - 2g\varphi(x) + x = 0. \quad (15)$$

We introduce, without reproducing the derivation, the analog of Eq. (14) for a real three-dimensional resonator with identical mirrors. Here the position of the saddle point will now be characterized by two functions: $\varphi(x, y)$ and $\psi(x, y)$, where x and y are the Cartesian coordinates on the mirrors

$$\begin{gathered}\frac{\partial F(x, y, s, t)}{\partial s} + \frac{\partial F(s, t, u, v)}{\partial s} = 0,\\ \frac{\partial F(x, y, s, t)}{\partial t} + \frac{\partial F(s, t, u, v)}{\partial t} = 0,\\ s = \varphi(x, y), \quad t = \psi(x, y), \quad u = \varphi(\varphi(x, y), \psi(x, y)),\\ v = \psi(\varphi(x, y), \psi(x, y)).\end{gathered} \quad (16)$$

After the functions $\varphi_p(x)$ have been obtained, Eqs. (11) are converted into differential equations, from which the functions $f_p(x)$ are obtained by integration

$$f_p(x) = -i\int_0^x \frac{\partial F_{pq}(x, \varphi_p(x))}{\partial x} dx. \quad (17)$$

Equations (10), on the other hand are converted into linear functional equations with respect to the function $h_{p0}^{(n)}(x)$, which are analogs of the transport equations of the WKB method.

2. Solution of the functional equations

Since we have assumed that the transverse dimensions of the mirrors are much smaller than the length of the optic axis, we need to obtain the solution of the functional equations (11) and (13) only for small values of x. Therefore, the apparatus of power series expansions turns out to be convenient.

We seek the functions $\varphi_p(x)$ in the form of the series

$$\varphi_p(x)=\sum_{m=1}^{\infty} C_{pm}x^m, \tag{18}$$

where the coefficients C_{p1} completely determine the solution in the paraxial approximation while the remaining coefficients take into account the aberration corrections of appropriate orders. Substituting the series (18) in Eqs. (13), carrying out all the required re-expansions, and requiring that (13) be satisfied with respect to all powers of x we obtain a recurrent system of pairs of equations

$$C_{p1}C_{q1}-2g_pC_{q1}+1=0, \tag{19}$$

$$C_{pm}(C_{q1})^m+C_{p1}C_{qm}-2g_pC_{qm}$$
$$+\vartheta_{qm}(C_{q1},C_{p1},\ldots C_{p\,m-1},C_{q\,m-1})=0,\quad m\geqslant 2. \tag{20}$$

The solution of the first pair of equations is

$$C_{p1}=g_p\pm\sqrt{g_p^2-g_p/g_q}. \tag{21}$$

If the condition for the stability of the resonator[2] is satisfied

$$0<g_1g_2<1, \tag{22}$$

the coefficients C_{p1} are complex and of unit modulus. Choosing in (21) the sign corresponding to a negative imaginary part we obtain from (7) and (17) natural proper oscillations that fall off exponentially towards the edge of the resonator. In the case of an unstable resonator the coefficients C_{p1} are real, the choice of sign in (21) is made as a result of the requirement that the saddle point should lie within the interval of integration. The cases $g_p = 0, \pm1$ require special investigation.

In order that it should be possible to obtain the remaining coefficients C_{pm} it is sufficient that the determinant of each pair of equations from (20) should differ from zero. In the case of unstable resonators this is fulfilled; in the case of stable resonators it is necessary that

$$\alpha=\pi^{-1}\arccos\sqrt{g_1g_2}\neq n_1/n_2, \tag{23}$$

where n_1 and n_2 are integers.

The parameter α and the condition (23) arise in the case of different approaches in the theory of open resonators. If one considers the ray geometry in resonators in the paraxial approximation then in the case of rational α the rays form closed cycles, and $2n_2$ is the number of traversals after which the ray closes in on itself. Taking aberration corrections into account corresponds to a nonlinear perturbation of such a dynamic system, but the use of perturbation theory leads to vanishing denominators[12,16]. Thus, the impossibility of constructing proper oscillations in the form (7) in the case of rational α corresponds to a destruction of the initial dynamical system of rays. Since, from a physical point of view, one can not prescribe α absolutely exactly, the matter reduces to the situation that for α close to such numbers as $1/2$, $1/3$, $2/3$ etc., the proper oscillations undergo large deformations compared with the paraxial approximation. We shall consider these deformations below when $\alpha \to 1/2$ (a confocal resonator). In solving the problem of proper oscillations in a closed resonator by the method of the parabolic equation condition (23) also arises[7].

We go over to the linear system of functional equations (10) in terms of the functions $h_{p0}^{(n)}(x)$. Equations of this type have been discussed in the literature[17]. We seek a solution in the form of the power series

$$h_{p0}^{(n)}(x)=\sum_{m=n}^{\infty} d_{pm}^{(n)}x^m. \tag{24}$$

The coefficients $d_{pm}^{(n)}$ satisfy the recurrent system of pairs of equations

$$d_{pn}^{(n)}=\gamma_p^{(n)}d_{qn}^{(n)}(C_{p1})^nT_{pq}(0), \tag{25}$$

$$d_{pm}^{(n)}=\gamma_p^{(n)}d_{qm}^{(n)}(C_{p1})^mT_{pq}(0)+\Omega(d_{qn}^{(n)},\ldots,d_{qm-1}^{(n)}),$$
$$m\geqslant n+1. \tag{26}$$

For a nontrivial solution of the homogeneous system (25) it is necessary that the determinant should vanish. From this we obtain the product of the eigenvalues

$$\gamma_1^{(n)}\gamma_2^{(n)}=C_{11}^{-n}C_{21}^{-n}T_{12}^{-1}(0)T_{21}^{-1}(0). \tag{27}$$

Utilizing formulas (12) and (21) we arrive in the case of a stable resonator at the well-known result[2,3]

$$\gamma_1^{(n)}\gamma_2^{(n)}=\exp\{i[(2n+1)\arccos\sqrt{g_1g_2}-\pi/2]\}, \tag{28}$$

from which we obtain the spectrum of the proper oscillations of a two-dimensional resonator (cf., for example[3,5])

$$k_{ln}=\pi l+(n+1/2)\arccos\sqrt{g_1g_2}, \tag{29}$$

where l is an integer, $l \gg 1$. The diffraction losses in this approximation are equal to zero, since we have neglected the contribution from the end points of the integration. Comparing formula (29) with the condition (23) we can easily see that the rational values of α correspond to frequency degeneracy of proper oscillations. For an unstable resonator we obtain

$$\gamma_1^{(n)}\gamma_2^{(n)}=\exp\{-1/2\,i\pi+(2n+1)\,\mathrm{Arch}\sqrt{g_1g_2}\}, \tag{30}$$

where the factor $M = \exp[(2n+1)\cosh^{-1}\sqrt{g_1g_2}]$ characterizes by how many fold does the cross section of the geometrooptical beam of rays increase in the passage from one mirror to the other and back[18]. From this we obtain that part of the diffraction losses in the course of a single passage which is determined by geometrical optics and does not depend on the mirror aperture.

The determinant of subsequent pairs of equations (26) does not vanish when condition (23) is satisfied and the coefficients $d_{pm}^{(n)}$ are expressed in terms of $d_{pn}^{(n)}$ in a recurrent fashion. The coefficients $d_{pn}^{(n)}$ are determined by the normalization of the proper oscillations $U_n^{(n)}(x)$.

We now proceed to the explicit calculation of the first aberration correction. We consider a two-dimensional resonator symmetric with respect to the optic axis which for the sake of simplicity is assumed to be formed by identical mirrors defined by equations

$$z=\frac{1-g}{2}x^2+\beta x^4,\qquad 1-z=\frac{1-g}{2}x^2+\beta x^4. \tag{31}$$

For parabolic mirros $\beta = 0$, while for mirrors of circular shape $\beta = (1-g)^3/8$, and for mirrors described by an ellipse, $\beta = (1-g)^2/4$. With an accuracy up to terms of the fourth order in x, x′ the function $F(x, x')$ is equal to

$$F(x,x')=-\frac{g}{2}(x^2+x'^2)-xx'+\left(\frac{1-2g}{8}-\beta\right)(x^4+x'^4)$$
$$+\left(\frac{-1-2g}{4}\right)x^2x'^2+\frac{g}{2}(x^3x'+x'^3x). \tag{32}$$

It can be seen that for no values of β and g is it possible to get rid of terms of the fourth order in x and x′, i.e., to obtain an "ideal" resonator.

Solving equation (14) with the appropriate degree of

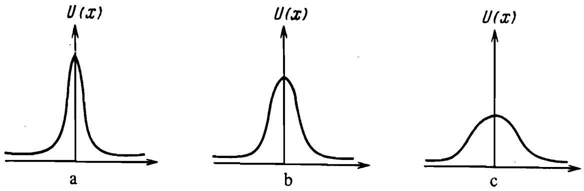

FIG. 1. Schematic behavior of the zeroth mode of the resonator for small g. a—$g > 0$, $\beta = 1/8$, the first correction for aberration has been taken into account; b — $g = 0$, paraxial approximation c — $g < 0$, $\beta = 1/8$, the first correction for aberration has been taken into account.

accuracy we obtain for $|g| < 1$

$$\varphi(x)=e^{i\arccos g}x+\frac{e^{2i\arccos g}\{(g-1)(2g^3-g-1)-4\beta\}x^3}{2ig\sqrt{1-g^2}}. \quad (33)$$

For $|g| > 1$ one must carry out the natural analytic continuation of g in formula (33). Knowing $\varphi(x)$ we obtain $f(x)$ from (17):

$$f(x)=-\sqrt{1-g^2}\frac{x^2}{2}+\frac{4\beta(1-2g^2)-(1-g)^2(2g+1)}{8g\sqrt{1-g^2}}x^4 \quad (34)$$

while in the case of mirrors of circular shape usually utilized in experiments

$$f(x)=-\sqrt{1-g^2}\frac{x^2}{2}+\frac{(1-g)\sqrt{1-g^2}(2g^2-4g-1)x^4}{16g}. \quad (35)$$

As can be seen from (7), the function $f(x)$ basically determines the character of the falling off of the proper oscillations at the edge of the mirror for a stable resonator, and the phase of the oscillations for an unstable resonator.

We examine the stable resonators in greater detail. The coefficient in front of x^4 in formula (34) has a singularity when $g = 0, \pm 1$. Thus, the paraxial approximation is, essentially, poorly applicable to resonators close to a confocal, a plane parallel and a concentric ones. Small aberrations strongly affect the distribution of the field over the mirrors, and, consequently, the diffraction losses and the divergence of the radiation emerging from the resonators. These facts are well confirmed in experimental practice[18]. The modes of other stable resonators can also be deformed as a result of aberrations (cf., the discussion of conduction (23)), but to a smaller degree, since these deformations manifest themselves in an essential manner in higher powers in x. We note that taking aberration corrections into account removes the equivalence of resonators which differ only by the sign of g[3,5]. Thus, in the case of circular mirrors the principal oscillation is concentrated closer to the optic axis when $g \to +0$ and, conversely, is smeared out when $g \to -0$ (cf. Fig. 1). In order that this effect should in a real way affect the size of the light spot the confocal condition must be satisfied with an accuracy up to a quantity of the order of a wavelength, while at the same time it affects the diffraction losses in a more significant manner.

It follows from the technique of solving Eqs. (20) and (26) and also from formula (34) that the aberration corrections in the case of unstable resonators play an insignificant role (cf.,[18]).

3. Diffraction losses in resonators

Although the finiteness of the dimensions of the resonator mirror has a relatively small effect on the formation of proper oscillations, nevertheless it is essential for the determination of diffraction losses. The effect of the mirror edge could be estimated within the framework of the proposed method taking into account in the asymptotic integration in addition to the contribution of the saddle point also the contribution from the end points of integration. However, we utilize another more graphic idea widely used in theoretical physics. In particular, in considering a problem with finite mirrors as one perturbed with respect to the problem with infinite mirrors it turns out to be possible to express the first correction to the eigenvalues of the integral equations in terms of the unperturbed eigenfunctions. This correction determines the diffraction losses, albeit with certain stipulations. The point is that the integral equations have been derived from the Kirchhoff principle which takes into account neither the generation of a diffraction wave on reflection from the edge of the resonator mirror, nor the penetration of the field to the back side of the mirror. Therefore, the results for the diffraction losses obtained with the aid of these equations are valid only with an accuracy up to a certain numerical factor which does not differ greatly from unity, and which takes these effects into account. The proposed method introduces another additional error of a somewhat higher order into the evaluation of the diffraction losses.

In order not to clutter up the paper with calculations we consider the case of a two-dimensional resonator with identical mirrors symmetric with respect to the optic axis, i.e., we start with the integral equation

$$U(x)=\gamma\int_{-a}^{a}K(x,x')U(x')\,dx', \quad (36)$$

where we shall not need the explicit form of the kernel, and it is only essential that the kernel be symmetric, i.e.,

$$K(x,x')=K(x',x),$$

and that the eigenfunctions of (36) change relatively little when the finite limits are replaced by infinite ones, and also have the property of being even.

We differentiate (36) with respect to a:

$$\begin{gathered}\dot U(x)=\gamma\int_{-a}^{a}K(x,x')\dot U(x')\,dx'+\frac{\dot\gamma}{\gamma}U(x)\\ +\gamma[K(x,a)U(a)-K(x,-a)U(-a)],\end{gathered} \quad (37)$$

where $\dot U(x) = dU(x)/da$, $\dot\gamma \equiv d\gamma/da$. Equation (37) is an inhomogeneous integral equation in $\dot U(a)$. In order for it to be soluble it is necessary that the term outside the integral should be orthogonal to the eigenfunctions of the equation adjoint to (36)[19]. After elementary transformations utilizing the symmetry of the kernel this condition can be written in the form

$$\frac{\dot\gamma_n}{\gamma_n}\int_{-a}^{a}U_n^2(x)\,dx+2U_n^2(a)=0. \quad (38)$$

Equation (38) can be regarded as a differential equation which determines the behavior of $\gamma_n(a)$, where as a limiting condition it is natural to require that as $a \to \infty$ γ_n should assume values in accordance with formulas (28) and (30) which we denote by $\gamma_n(\infty)$. Integrating (38) we obtain

$$\gamma_n(a)=\gamma_n(\infty)\exp\left[\int_a^\infty\frac{2U_n^2(a)\,da}{\int_{-a}^{a}U_n^2(x)\,dx}\right]. \quad (39)$$

Formula (39) relates the exact eigenvalues and eigenfunctions of (36). However, in accordance with the ideas of perturbation theory one can substitute into the right hand side of (39) functions constructed by the method described above, and to obtain a correction for the finite size of the mirrors to the eigenvalues in the first approximation.

We investigate to what results does this correction lead in the case of unstable resonators. Since aberrations have a small effect on the natural oscillations we utilize the paraxial approximation. In accordance with the calculations carried out earlier (cf., also[4]) the function $U_0(x)$ (the mode with the lowest geometrooptical losses) has the form

$$U_0(x) = \exp(ik\sqrt{g^2-1}\,x^2/2). \quad (40)$$

Substituting this expression into the right hand side of (39) and evaluating the integrals by asymptotic methods we obtain for values of g not too close to ±1,

$$\gamma_0(a)=\gamma_0(\infty)\left[1+\frac{\exp\{i[k\sqrt{g^2-1}\,a^2-3\pi/4]\}}{\sqrt{\pi}k(g^2-1)^{1/4}}\right]. \quad (41)$$

Thus, the diffraction losses in a single passage

$$\Lambda_0 = 1-|\gamma_0|^{-1} \quad (42)$$

oscillate around a mean geometrooptical value as the dimensions of the mirror are altered. From the physical point of view these oscillations arise as a result of the fact that the wave (40) propagated along the mirror reaches the edge of the mirror in a different phase and generates on reflection from the edge a diffraction wave with a different direction diagram. It is natural that this effect depends strongly on the shape of the mirror edge[4,18].

In the case of a stable resonator with circular mirrors we take the zeroth mode with the first aberration correction taken into account, viz.,

$$U_0(x)=\exp\left\{-k\left[\sqrt{1-g^2}\frac{x^2}{2}+\frac{(g-1)\sqrt{1-g^2}(2g^2-4g-1)x^4}{16g}\right]\right\}. \quad (43)$$

Carrying out the evaluation of the integrals in (30) we obtain for the diffraction losses in a single passage

$$\Lambda_0=\frac{\exp\{-k\sqrt{1-g^2}\cdot{}^1\!/_2\,a^2[1+(g-1)(2g^2-4g-1)a^2/8g]\}}{a\sqrt{\pi k}(1-g^2)^{1/4}}. \quad (44)$$

It can be seen that the aberration correction strongly alters the quality factor of resonators close to a confocal one, and either increases or reduces it depending on the sign of g. These results agree well with the experimental curves obtained in the paper of Manankov[10]. Manenkov[10] explains the reduction in the quality factor as the resonator approaches a confocal one by the pumping over of energy due to the mode becoming degenerate. An investigation of the confocal resonator in the paraxial approximation yields a minimum of diffraction losses[8,11], while formula (44) simply does not allow one to perform a limiting transition to the confocal case. The schematic behavior of losses for resonators close to a confocal one is shown in Fig. 2. Neither formula (44), nor formula (41) permits one to carry out a limiting transition to the plane parallel and the concentric resonators. This can be easily understood, since in these cases the diffraction at the edges of the mirrors is a fundamental reason for the formation of proper oscillations and can not be treated as a perturbation.

More exact expressions for the diffraction losses can be obtained taking into account the function $G(x, x')$ which appears in the integral equations (1) and (3). However, this leads only to the appearance of an additional

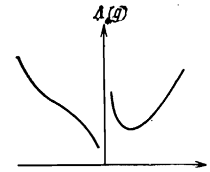

FIG. 2. Schematic behavior of diffraction losses Λ(g) for resonators close to a confocal one.

factor of the order of magnitude of unity. Qualitatively different results will be obtained only if $G(x, x')$ is a function taking on complex values.

A discussion of a three-dimensional resonator problem by the proposed method must lead to results analogous to those which have been obtained for a two-dimensional problem, but in this case new effects are possible and are associated, for example, with the question whether the aberration corrections preserve the separation of variables available in the paraxial approximation, or not.

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