Electron kinetics and stationary generation in semiconductor lasers

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Kinetic electronic processes in semiconductor lasers are considered under the assumption that collisions between electrons play the major role. An analytic solution linearized near the kinetic-equation threshold is obtained under some very general assumptions regarding the form of the collision integral. The electron energy distribution function has a minimum at the point corresponding to transition to the generated frequency. A result of this is that above the threshold the electron distribution may vary with increasing pumping. The solution obtained permits a consistent study of the problem of the appearance and stability of two-mode conditions by taking into account spatial burnout.

The present paper is devoted to kinetic processes of electrons in a semiconductor laser. It is assumed that the lasing is the result of direct transitions of the electrons from the conduction band into the valence band. The semiconductor is assumed to be pure enough for collisions between electrons to assume the principal role in the kinetics. It is also assumed that the pumping is optical. Near-threshold regimes are considered, so that the kinetic equations can be simplified by linearization near the threshold distribution function. Under rather general assumptions concerning the form of the collision integral, we consider in succession the kinetics of the electrons in the energy and coordinate spaces.

The obtained electron energy distribution function has a minimum (a well) at a point corresponding to a transition at the generated frequency. This result differs from the assumption made in most papers (see, for example,[1]), namely that the distribution function has a Fermi form. This difference is small, and therefore the cited papers describe the single-mode regime for the most part correctly. The difference, however, changes the qualitative picture and allows the electron distribution function above threshold to vary with increasing pumping. A definite peculiarity is introduced into this process by spatial diffusion of the electrons.

The obtained solution enables us to consider in detail the onset and stability of the two-mode regime. It turns out that the stability of the two-mode regime is determined by the competition between the spatial and energy diffusions, and depends on a number of parameters of the semiconductor. These parameters specify the minimum distance between modes in the two-mode regime, and this distance can be smaller than the width of the electronic levels.

The results of this paper are in qualitative agreement with the experiments. A detailed comparison is made difficult, however, by the vagueness of the experimental situation. It is undoubtedly desirable to obtain a quantitative comparison between the theory and the experimental data, particularly a direct observation of the well in the distribution function by determining the Raman emission spectrum.

Most assumptions made in the present paper are not fundamental limitations. The procedure developed can be used in a more general case.

FUNDAMENTAL EQUATIONS

We consider a semiconductor placed in an electromagnetic field that is a superposition of standing waves

$$\mathbf{E}(\mathbf{r},t)=\sum_n \mathbf{E}_n(t)\cos \mathbf{k}_n\mathbf{r}\sin(\omega_n t+\varphi_n(t)), \tag{1}$$

where $\mathbf{E}_n(t)$ and $\varphi_n(t)$ are slowly varying functions of the time. The state of the electron system of the semiconductor is described by four functions that are mean values of quadratic combinations of the operators a_p (electron annihilation in the conduction band) and b_p (hole annihilation in the valence band):

$$n_q^c(\mathbf{p})=\langle a_{\mathbf{p}}^+ a_{\mathbf{p}+\mathbf{q}}\rangle,\quad n_{\mathbf{q}}^v=\langle b_{\mathbf{p}-\mathbf{q}} b_{\mathbf{p}}^+\rangle,$$
$$\rho_{\mathbf{q}}(\mathbf{p})=\langle a_{\mathbf{p}}^+ b_{\mathbf{p}+\mathbf{q}}^+\rangle,\qquad \rho_{\mathbf{q}}^*(\mathbf{p})=\langle b_{\mathbf{p}+\mathbf{q}} a_{\mathbf{p}}\rangle, \tag{2}$$

The first two quantities are the Fourier Q-components of the spatial distribution function of the electrons in the conduction and valence bands, and $\rho_q(\mathbf{p})$ is the Fourier component of the off-diagonal element of the density matrix. The equations for these functions are

$$\left(\frac{\partial}{\partial t}+i\mathbf{q}\mathbf{v}\right)n_{\mathbf{q}}^c(\mathbf{p})+i\sum_n \lambda_n\{\exp[-i(\omega_n t+\varphi_n)][\rho_{\mathbf{q}-\mathbf{k}_n}(\mathbf{p})+\rho_{\mathbf{q}+\mathbf{k}_n}(\mathbf{p})]-\text{c.c.}\}=-\frac{n_q^c(\mathbf{p})}{\tau_R}+\left(\frac{\partial}{\partial t}n_{\mathbf{q}}^c\right)_{st}+Q_{\mathbf{q}}^c(\mathbf{p}), \tag{3}$$

$$\left[\frac{\partial}{\partial t}-i(E_{\mathbf{p}}^c+E_{\mathbf{p}+\mathbf{q}}^v)\right]\rho_{\mathbf{q}}(\mathbf{p})+i\sum_n \lambda_n\exp[i(\omega_n t+\varphi_n)]\{n_{\mathbf{q}+\mathbf{k}_n}^c(\mathbf{p})+n_{\mathbf{q}-\mathbf{k}_n}^c(\mathbf{p})-n_{\mathbf{q}+\mathbf{k}_n}^v(\mathbf{p}+\mathbf{q})-n_{\mathbf{p}-\mathbf{k}_n}^v(\mathbf{p}+\mathbf{q})\}=\left(\frac{\partial}{\partial t}\rho_{\mathbf{q}}(\mathbf{p})\right), \tag{4}$$

where $E_p^{c,v}$ stands for the energy of the electrons and the holes, respectively, $\lambda_n=e\mathbf{v}_{cv}\cdot\mathbf{E}_n/4\omega\hbar$ is a quantity proportional to the amplitude of the transition from the conduction band to the valence band, and the matrix element $\mathbf{v}_{cv}$ of this transition depends little on $\mathbf{p}$ (for small $\mathbf{p}$). The left-hand sides of (3) and (4) are the dynamic equations of the motion. In the right-hand sides, the term $Q_q(\mathbf{p})$ describes the pumping by the external source, the term n_q/τ_R describes recombination with a lifetime τ_R, and $(\partial n/\partial t)_{st}$ and $(\partial\rho/\partial t)_{st}$ describes collisions of the electrons with one another and with the phonons. The equations for n_q^v and ρ_q^* are similar in form.

To obtain a closed system it is necessary to add to equations (3) and (4) the abbreviatted equations for the amplitude $\mathbf{E}_n(t)$:

$$\frac{\partial\mathbf{E}_n(t)}{\partial t}+\frac{1}{\tau_0}\mathbf{E}_n(t)=-\pi(\mathbf{j}_{\mathbf{k}_n}+\mathbf{j}_{-\mathbf{k}_n}), \tag{5}$$

where

$$\frac{1}{\tau_0}=\frac{c}{\eta}\left(\beta-\frac{\ln R}{l}\right), \tag{6}$$

η is the refractive index, R is the reflection coefficient,

and l is the resonator length. In these equations, j_k are the Fourier components of the current and are expressed in terms of the matrix $\rho_q(\mathbf{p})$ in the following manner:

$$j_k = -\frac{e}{V}\sum_{\mathbf{p}} v_{cv}\{\rho_k(\mathbf{p}) + \rho_{-k}^*(\mathbf{p})\}_s. \tag{7}$$

The subscript s signifies that we take in the current the part proportional to $\sin(\omega_n t + \varphi_n)$.

To simplify the subsequent calculations, we assume full equivalence of the electrons and holes, so that

$$E_p^v = E_p^c \equiv E_p, \quad Q^v = Q^c, \quad n_q^v(\mathbf{p}) = \delta_{q,0} - n_q^c(p). \tag{8}$$

In the case $\lambda_n \tau \ll 1$ considered by us, the collision integrals describe the processes of electron collisions in the absence of a field. Then the collision integral in (4) should lead to damping of $\rho_q(\mathbf{p})$, since this quantity is equal to zero in the equilibrium state. We express it in the simplest form that ensures this damping

$$\left(\frac{\partial \rho}{\partial t}\right)_{st} = -\gamma \rho_q(\mathbf{p}), \tag{9}$$

where γ is of the order of the reciprocal time of the electron collision[2]. The collision integral in (3) has the usual form, since the function ρ, as will be shown below, differs from zero in a narrow energy region near $E_p = \omega_n/2$. The deviation of E_p from $\omega_n/2$ will be denoted by

$$\xi_n = E_p - \omega_n/2. \tag{10}$$

We consider the processes near the generation threshold, expanding all quantities near the threshold values. At the generation threshold, all λ_n are equal to zero, the pump is equal to a certain threshold value $Q^0(\mathbf{p})$, and the distribution function of the electrons in this state is $n^0(\mathbf{p})$. This function can be obtained by solving Eq. (3) with $\lambda_n = 0$ and with $Q^0(\mathbf{p})$ in the stationary state:

$$-\frac{n^0(\mathbf{p})}{\tau_R} + \left(\frac{\partial n^0}{\partial t}\right)_{st} + Q^0(\mathbf{p}) = 0. \tag{11}$$

(We have put $\mathbf{q} = 0$, assuming that the source Q^0 is homogeneous in space.)

The excess of pump over threshold will be denoted $Q(\mathbf{p}) = Q^c(\mathbf{p}) - Q^0(\mathbf{p})$. The deviation of the distribution function from $n^0(\mathbf{p})$, due to the excess of the pump over threshold and to coherent emission, will be described by the function $n_q(\mathbf{p})$:

$$n_q^c(\mathbf{p}) - \delta_{q,0} n^0(\mathbf{p}) = n_q(\mathbf{p}). \tag{12}$$

Assuming this quantity to be small, the collision integral of Eq. (3) can be linearized and reduced to a linear integral operator.

We consider a model in which the kernel of the integral operator depends only on the energy difference, i.e.,

$$\left(\frac{\partial n_q}{\partial t}\right)_{st} = \frac{1}{\tau}\left[-n_q(\mathbf{p}) + \int K(\xi - \xi') n_q(\mathbf{p}) d\xi' \frac{do'}{4\pi}\right]. \tag{13}$$

This model retains the essential features of the general case and admits of a transition to the case of weak collisions and to the inverse limiting case. Such a model was considered in the theory of gas lasers and has been well justified.[3]

We note certain properties of the collision integral (13). The main contribution to the relaxation of a small fraction of electrons is made by the fastest electron-electron collisions. Therefore the quantity τ in (13) is the time of electron-electron collisions. From the requirement of conservation of the number of particles we can easily find the zeroth moment of the function K:

$$\int K(\xi)\, d\xi = 1. \tag{14}$$

To estimate the first moment, we multiply (13) by ξ and integrate:

$$\begin{aligned}\int \frac{d\xi\, do}{4\pi}\xi\left(\frac{\partial n}{\partial t}\right)_{st} &= \frac{1}{\tau}\left\{-\int n(\mathbf{p})\xi\, d\xi \frac{do}{4\pi}\right. \\ &\left. + \int d\xi\, d\xi' \frac{do\, do'}{(4\pi)^2}\xi K(\xi - \xi') n(\mathbf{p}')\right\} \\ &= \frac{1}{\tau}\int \frac{d\xi' do'}{4\pi} n' \int d\xi \xi K(\xi).\end{aligned} \tag{15}$$

For the case of pure electron-electron collisions, owing to the energy conservation in collisions, this integral should be equal to zero.

Thus, the first moment is determined only by the electron-phonon collisions and its relative order of magnitude is τ/τ_{ef}. The second moment of the function K is of the order of the square of the energy transferred in one collision, $-\Delta$. The value of Δ is defined by the equation

$$\Delta^2 = {}^1/_2 \int \xi^2 K(\xi)\, d\xi. \tag{16}$$

SINGLE-MODE REGIME

The values of the density matrix at the threshold are determined from Eq. (4), in which it is necessary to substitute $n^0(\mathbf{p})$ for $n^{v,c}$. The nonzero Fourier components, with Eqs. (8) and (10) taken into account, are given by

$$\rho^0_{\pm k_n}(\mathbf{p}) = \frac{\lambda_n \exp(i\omega_n t + i\varphi_n)}{i\gamma + {}^1/_2(\xi_{n,\mathbf{p}} + \xi_{n,\mathbf{p}\pm\mathbf{k}_n})}[n^0(\mathbf{p}) + n^0(\mathbf{p}\pm\mathbf{k}_n) - 1]. \tag{17}$$

Substituting them into the expression (7) for the current and using (5), we obtain the following expression for the mode gain coefficients:

$$\alpha_n^0 = \frac{\pi e^2 v_{cv}^2}{\hbar\omega_n V}\sum_{\mathbf{p}} \frac{\gamma[2n^0(\mathbf{p}) - 1]}{\xi_n^2 + \gamma^2} - \frac{1}{\tau_0}. \tag{18}$$

(We neglect the quantity k_n in the argument of the distribution function, assuming $\mathbf{v}\cdot\mathbf{k}_n$ to be small in comparison with γ.) By definition, the gain $\alpha_n^{(0)}$ is equal to zero for a certain mode n = 1 and is negative for all the other modes:

$$\alpha_1^0 = 0. \tag{19}$$

We proceed to consider the stationary regime of single-mode generation. The deviation of the electron distribution function from $n^0(\mathbf{p})$ is determined by the excess of the pump over the threshold value $Q(\mathbf{p})$ and by the coherent emission. Since the values of λ_1 near threshold are small, the coherent emission can be taken into account by merely substituting in (3) the density matrix in the zeroth approximation (17). It is easily seen then that the three Fourier components $n_0(\mathbf{p})$ and $n_{\pm 2k_1}(\mathbf{p})$ of the distribution function differ from zero. The equation for the first of these functions, taking (13) into account, is

$$\begin{aligned}\frac{n_0(\mathbf{p})}{\tau_R} + \frac{1}{\tau}\left[n_0(\mathbf{p}) - \int K(\xi - \xi') n_0(\mathbf{p}') d\xi' \frac{do'}{4\pi}\right] \\ = Q(p) - \lambda_1^2 \frac{2\gamma[2n^0(p) - 1]2}{\xi_1^2 + \gamma^2}.\end{aligned} \tag{20}$$

When the analogous equation was solved in[1], it was proposed that the complete distribution function $n_0(\mathbf{p})$

+ n(p)) has a Fermi form and that the excess of the pump over threshold and the coherent emission influence only the magnitude of the chemical potential. We make here no assumptions concerning the explicit form of the function $n^0(\mathbf{p})$, and obtain $n^0(\mathbf{p})$ as by solving (20). In fact, as seen from (20), the function $n_0(\mathbf{p})$ does not depend on the angles of the vector p, i.e., is only a function of ξ. Equation (20) can be easily solved by taking the Fourier transform with respect to the variable ξ:

$$\begin{aligned} n_0(\xi) &= \int dg \frac{d\xi'}{2\pi} e^{ig(\xi-\xi')} \left[\frac{1}{\tau_R} + \frac{1}{\tau} - \frac{1}{\tau} \int d\eta K(\eta) e^{-ig\eta} \right]^{-1} \\ &\times \left\{ Q(\xi') - \frac{2\lambda_1^2 \cdot 2\gamma [2n^0(\xi') - 1]}{\xi'^2 + \gamma^2} \right\} \\ &= \frac{\tau\tau_R}{\tau+\tau_R} \left\{ Q(\xi) - 2\lambda_1^2 \frac{2\gamma}{\xi_1^2+\gamma^2} (2n^0(\xi) - 1] \right\} \qquad (21) \\ &+ \frac{\tau\tau_R}{\tau+\tau_R} \int \frac{dg\, d\xi'}{2\pi} \frac{\tau_R}{\tau+\tau_R} \exp(ig\xi - ig\xi') \\ &\times \left\{ \left[\int d\eta K(\eta) e^{-ig\eta} \right]^{-1} - 1 \right\}^{-1} \left[Q(\xi') - \frac{4\lambda_1^2\gamma}{\xi_1'^2+\gamma^2} (2n^0 - 1) \right]. \end{aligned}$$

The last integral with respect to g in (21) converges at small values of g, making it possible to expand the Fourier component of K. In light of the statements made above concerning the moments of this function, we obtain

$$\left[\int d\eta\, K(\eta) e^{-ig\eta} \right]^{-1} \approx 1 + g^2\Delta^2.$$

The integrals in (21) can now be calculated explicitly. Assuming that the function $n^0(\xi)$ varies little at an energy on the order of γ, and that $Q(\xi)$ differs from zero near a certain value ξ_0, the function $n_0(\xi)$ takes the form

$$\begin{aligned} n_0(\xi) &= \frac{\tau\tau_R}{\tau+\tau_R} \left[Q(\xi) + Q \frac{\tau_R}{\tau\Theta} \exp\left(-\frac{|\xi-\xi_0|}{\Theta} \right) \right] \\ &- \frac{2\lambda_1^2\tau\tau_R}{(\tau+\tau_R)\gamma} \left[\frac{2\gamma^2}{\xi_1^2+\gamma^2} + \frac{1}{s} \exp\left(-\frac{|\xi|}{\Theta} \right) \right] [2n^0(\xi_1 = 0) - 1], \end{aligned} \qquad (22)$$

$$Q = \int Q(\xi) d\xi. \qquad \Theta = \Delta \left(\frac{\tau+\tau_R}{\tau} \right)^{1/2}, \qquad s = \frac{\tau\Theta}{\pi\tau_R\gamma}. \qquad (23)$$

Each of the components in (22) is a sum of two terms, the first of which duplicates the form of the source (sink), and the second is a smooth function that varies over distances $\sim\Theta$. The quantity Θ is the distance, in energy space, over which the electron can diffuse within a lifetime τ_R:

$$\Theta^2 \approx \Delta^2\tau_R / \tau.$$

Let us consider the term proportional to λ_1^2. It constitutes the inhomogeneous solution of (20), governed by the sink, and thus describes the decrease in the number of electrons as a result of coherent emission. In the region near $\xi_1 = 0$, the principal relation is given by

$$-\lambda_1^2 \frac{\tau\tau_R}{\tau+\tau_R} 2 \frac{\gamma}{\xi_1^2+\gamma^2} [2n^0(\xi_1 = 0) - 1].$$

Consequently, a dip appears in the distribution of the electrons[1]. The ratio of the depth of the dip (well) to the value of the smooth function in this region is determined by the parameter s, which plays an important role in the theory. Its meaning can be easily understood by using the concept of holes in the electron distribution. Then the dip (well) describes production of holes at the location of the sources (sink for electrons). Their number in this region is determined by their escape through collision and is consequently proportional to the time τ between collisions. The escaping holes diffuse in energy space over a distance Θ. The distribution of the diffused holes is described by the smooth part of the function. The lifetime of these holes is τ_R, and their fraction at the location of the source is $-\gamma/\Theta$. Thus, the relative depth of the well of the order of τ is equal to $\tau_R\gamma/\Theta = \pi s$.

It is seen from the foregoing reasoning that our calculations assume that the diffusion length Θ is small in comparison with the distance from the generation point $\xi_1 = 0$ to the bottom of the band $\epsilon_0 = (\omega - E_g)/2$ and from the location ξ_0 of the electron pump source.

In real cases, the inverse inequalities $\Theta > \epsilon_0$ and $\Theta > \xi_0$ are more frequently realized. If only one of them is satisfied, then the coefficient of the smooth function is doubled. If both are satisfied, then the quantity Θ in the parameter s (23) is replaced by $(\epsilon_0 + \xi_0)/2$. This result can be obtained from an exact solution of the integral equation for the particular case of an exponential kernel K.

The spatial Fourier components $n_{\pm 2k_1}(\mathbf{p})$ characterize the spatial burning out of the excess populations. The equations for these functions are of the form

$$\begin{aligned} &iqvn_q(\mathbf{p}) + \frac{n_q(\mathbf{p})}{\tau_R} + \frac{1}{\tau} \left[n_q(\mathbf{p}) - \int K(\xi-\xi') n_q(\mathbf{p}') d\xi' \frac{do'}{4\pi} \right] \\ &= -\lambda_1^2 \frac{2\gamma}{\xi_1^2+\gamma^2} [2n^0(\mathbf{p}) - 1], \qquad \mathbf{q} = \pm 2\mathbf{k}_1. \end{aligned} \qquad (24)$$

We proceed from (24) to an equation for the function

$$n_q(\xi) = \int do\, n_q(\mathbf{p}).$$

To this end, we divide equation (24) by $\tau_R^{-1} + \tau^{-1} + i\mathbf{q}\cdot\mathbf{v}$, which is the coefficient of $n_q(\mathbf{p})$, and obtain after integrating with respect to the angles

$$\frac{n_q(\xi)}{\tau_{\text{eff}}} - \frac{1}{\tau} \int K(\xi-\xi') n_q(\xi') d\xi' = -\lambda_1^2 \frac{2\gamma}{\xi_1^2+\gamma^2} (2n^0(\mathbf{p}) - 1), \qquad (25)$$

where

$$\frac{1}{\tau_{\text{eff}}} = qv_0 \Big/ \operatorname{arctg} \frac{qv_0\tau_R\tau}{\tau+\tau_R} \approx \frac{1}{\tau} + \frac{1}{\tau_R} + \frac{4q^2v_0^2}{3}\tau$$

(In the integration we have set v equal to the value at $\xi_1 = 0$ and used the smallness of the electron mean free path $v_0\tau$ in comparison with the wavelength).

Equation (25) is similar in form to (20). Its solution can therefore be written down immediately:

$$\begin{aligned} n_{\pm 2k_1}(\xi) &= -\frac{\lambda_1^2\tau_{\text{eff}}}{\gamma} \left[\frac{2\gamma^2}{\xi_1^2+\gamma^2} + \frac{\pi\tau_{\text{eff}}\gamma}{(\tau-\tau_{\text{eff}})\Theta_{\text{eff}}} \exp\left(-\frac{|\xi_1|}{\Theta_{\text{eff}}} \right) \right] \\ &\times [2n^0(\xi_1 = 0) - 1] \approx -\lambda_1^2 \frac{\tau\tau_R}{\gamma(\tau+\tau_R)} \left[\frac{2\gamma^2}{\xi_1^2+\gamma^2} \right. \\ &\left. + \frac{\exp(-|\xi_1|/\Theta_{\text{eff}})}{s(1+4Dk^2\tau_R)^{1/2}} \right] [2n^0(\xi_1 = 0) - 1]. \end{aligned} \qquad (26)$$

Here $\Theta_{\text{eff}}^2 = \Theta^2/(1 + 4Dk^2\tau_R)$, and $D = v_0^2\tau/3$ is the spatial-diffusion coefficient[2].

The function $n_{\pm 2k_1}$ has a form similar to (22). The only difference is that the smooth part of the function is decreased by a factor $(1 + 4Dk^2\tau_R)^{1/2}$, whereas the "well" remains unchanged. This result is physically obvious. As already stated, the well in the distribution results from emission of electrons that had no time to experience collisions. Naturally, these electrons have no time to experience spatial diffusion. As to the second group of electrons, they diffuse within the lifetime τ_R over a distance $(D\tau_R)^{1/2}$, the ratio of which to the wavelength of the electromagnetic field does indeed determine the decrease in the inhomogeneity of the distribution.

The quantity Θ_{eff} has a simple physical meaning.

Since $Dk^2\tau_R \gg 1$ in real cases, we have

$$\Theta_{\text{eff}}^2 \approx \frac{\Delta^2}{\tau}\frac{1}{4Dk^2}.$$

As already mentioned, Δ^2/τ is the diffusion coefficient in energy space, and $1/4Dk^2$ is the damping time of the spatial inhomogeneity of the distribution. Thus, Θ_{eff} is the length of diffusion within the lifetime of the electrons in this state.

The coefficient of the smooth function in (26) can be obtained from the same simple considerations as in the case of (22). It is necessary only to replace the diffusion length Θ by Θ_{eff}, and the lifetime τ_R by $1/4Dk^2$. Since $Dk^2\tau_R \gg 1$, the damping time of the spatial inhomogeneity is much shorter than the recombination time τ_R, so that $\Theta_{\text{eff}} \ll \Theta$. In a real situation, Θ_{eff} can be either smaller or larger than ϵ_0 or ξ_0. In the former case, expression (26) remains the same. In the latter case it is necessary to replace the diffusion length Θ_{eff} in the coefficient of the smooth function in (26) by ϵ_0, so that it takes the form

$$1/s(1+4Dk^2\tau_R),$$

i.e., the diffusion smooths out the spatial inhomogeneity of the distribution much more strongly.

For the sake of simplicity, we use in the subsequent calculations expressions (22) and (26) for the functions $n_0(\xi)$ and $n_{\pm 2k_1}(\xi)$. It should be borne in mind here that in the case when the diffusion lengths Θ and Θ_{eff} exceed ϵ_0 and ξ_0, it is necessary to make the corresponding changes in the coefficients s.

With the aid of functions (23) and (26) we obtain from (4) the first-approximation density matrix $\rho^{(1)}_{\pm k_1}(\mathbf{p})$. Substituting it in the expression (7) for the current, we obtain an equation for the field amplitude

$$\frac{\partial \mathbf{E}_1}{\partial t} = \mathbf{E}_1(\alpha_1^0 - \beta_1\lambda_1^2 + \delta_1 Q), \tag{27}$$

where

$$\delta_1 = \frac{\pi e^2 v_{cv}}{\hbar\omega_1 VQ}\sum_{\mathbf{p}}\frac{2\gamma}{\xi_1^2+\gamma^2}2\frac{\tau\tau_R}{\tau+\tau_R}\left[Q(\xi)+Q\frac{\tau_R}{\tau\Theta}\exp\left(-\frac{|\xi-\xi_0|}{\Theta}\right)\right]$$
$$\approx 4\pi^2 N(0)\frac{e^2 v_{cv}^2}{\hbar\omega_1}\frac{\tau_R}{\Theta}\exp\left(-\frac{|\xi_0|}{\Theta}\right), \quad N(0)=\frac{m^2 v_0}{2\pi^2\hbar^3}, \tag{27'}$$

$$\beta_1 = \frac{\pi e^2 v_{cv}^2}{\hbar\omega_1}[2n^0(\xi_1=0)-1]\frac{1}{V}\sum_{\mathbf{p}}\frac{2\gamma}{\xi_1^2+\gamma^2}\left\{\frac{4\tau\tau_R}{\gamma(\tau+\tau_R)}\right.$$
$$\left.\times\left[\frac{3\gamma^2}{\xi_1^2+\gamma^2}+\frac{1}{s}\exp\left(-\frac{|\xi|}{\Theta}\right)+\frac{1}{2s(1+4Dk^2\tau_R)^{1/2}}\exp\left(-\frac{|\xi|}{\Theta_{\text{eff}}}\right)\right]\right\}$$
$$\approx \frac{e^2 v_{cv}^2\tau}{\hbar\omega_1\gamma}\left(\frac{3}{2}+\frac{1}{s}+\frac{1}{2s}(1+4Dk^2\tau_R)^{-1/2}\right)(2n^0-1)8\pi^2 N(0). \tag{27''}$$

In the stationary case we obtain

$$\lambda_1^2 = Q\frac{\delta_1}{\beta_1} \approx Q\exp\left(-\frac{\xi_0}{\Theta}\right)\left[2\pi(2n^0-1)\left(1+\frac{3s}{2}+\frac{1}{2(1+4Dk^2\tau_R)^{1/2}}\right)\right]^{-1}. \tag{28}$$

(We took into account here the fact that $\alpha_1^0 = 0$ for the mode of interest to us.)

It is convenient to rewrite (28) by using the number of photons N_{ph} and their lifetime in the resonator τ_0:

$$\frac{N_{\text{ph}}}{\tau_0} = \frac{QN(0)\exp(-\xi_0/\Theta)}{1+3s/2+1/2(1+4Dk^2\tau_R)^{1/2}} = \frac{J-J_{th}}{1+3s/2+1/2(1+4Dk^2\tau_R)^{1/2}}, \tag{29}$$

where J is the pumping current, i.e., the number of electrons arriving at the region near $\xi = 0$ per unit time and per unit volume. Formula (29) differs from the usual expression (see, for example,[6]) in that the denominator $1 + 3s/2 + 1/2(1 + 4Dk^2\tau_R)^{1/2}$ exceeds unity, i.e., the fraction of the electrons produced by the source above threshold makes no contribution to the coherent emission.

This difference is due to the following circumstances. The gain of the quanta is proportional to the expression

$$\sum_{\mathbf{p}}\frac{2\gamma}{\xi_1^2+\gamma^2}(2n_0+2n_{2k_1}-1).$$

Its constancy in the stationary regime imposes only certain integral requirements on the distribution function in a region of the order of γ near $\xi_1 = 0$. Yet it is precisely in this region, owing to the presence of the well due to the energy burnout, that the function undergoes significant changes. In fact, substituting the stationary value λ_1^2 in (22), we obtain for the increment of the distribution function at $|\xi| < \Theta$

$$n_0(\xi) \approx \frac{\tau\tau_R}{\tau+\tau_R}\frac{\tau_R Q}{\Theta\tau}\exp\left[-\frac{\xi_0}{\Theta}\right)s\left[\frac{3}{2}+\frac{1}{2s(1+4Dk^2\tau_R)^{1/2}}-\frac{2\gamma^2}{\xi_1^2+\gamma^2}\right]$$
$$\times\left[1+\frac{3s}{2}+\frac{1}{2}(1+4Dk^2\tau_R)^{-1/2}\right]^{-1}.$$

We see that the function $n_0(\xi)$ can even reverse sign under certain conditions ($s(1 + 4Dk^2\tau_R)^{1/2} > 1$). The strong dependence of this function in an appreciable region causes the electron distribution to vary with increasing pumping current when the gain above threshold is constant. At the same time, the total number of electrons is increased, and consequently also the number of recombinations. A similar role is played by the spatially-inhomogeneous parts of the electron distribution ($n_{\pm 2k_1}$). The described effects do not exert a noticeable influence on the single-mode generation regime, in view of the smallness of the parameters s and $(1 + 4Dk^2\tau_R)^{-1/2}$. They do, however, play an important role in a multimode regime.

TWO-MODE REGIME

We proceed to consider the two-mode regime. Since λ_1 and λ_2 are small near threshold, to take the coherent emission into account it suffices to substitute in (3) the density matrix in the zeroth approximation:

$$\rho_{\mathbf{q}}^{(0)}(\mathbf{p}) = \sum_n \frac{\lambda_n \exp[i(\omega_n t+\varphi_n)]}{i\gamma+1/2(\xi_{n,\mathbf{p}}+\xi_{n,\mathbf{p}+\mathbf{q}})}(\delta_{\mathbf{q}+\mathbf{k}_n,0}+\delta_{\mathbf{q}-\mathbf{k}_n,0})$$
$$\times(n^0(\mathbf{p})+n^0(\mathbf{p}+\mathbf{q})-1). \tag{30}$$

In the case of two modes, the following Fourier components of the distribution function differ from zero: $n_0(\mathbf{p})$, $n_{\pm 2k_{1,2}}$, and $n_{k_1\pm k_2}(\mathbf{p})$. The equation for $n_0(\mathbf{p})$ coincides with (20) if one adds to the right-hand side a term that takes into account the action of the field of the second mode,

$$-\lambda_2^2\frac{2\gamma}{\xi_2^2+\gamma^2}2[2n^0(\mathbf{p})-1].$$

The equation for n_{2k} coincides exactly with (24), and the equation for $n_{k_1\pm k_2}$ take the form

$$\left[\frac{\partial}{\partial t}+i\mathbf{v}(\mathbf{k}_1\pm\mathbf{k}_2)+\frac{1}{\tau_R}+\frac{1}{\tau}\right]n_{\mathbf{k}_1\pm\mathbf{k}_2}(\mathbf{p})$$
$$-\frac{1}{\tau}\int K(\xi-\xi')n_{\mathbf{k}_1\pm\mathbf{k}_2}(\mathbf{p}')\,d\xi'\frac{do'}{4\pi} = -\lambda_1\lambda_2[2n^0(\mathbf{p})-1] \tag{31}$$
$$\times\left\{\exp[i(\omega_1-\omega_2)t+i(\varphi_1-\varphi_2)]\left(\frac{1}{\gamma-i\xi_1}+\frac{1}{\gamma+i\xi_2}\right)+\text{c.c.}\right\}.$$

These equations describe the oscillations of the excess

population due to the joint action of the fields of both modes.

It is convenient to seek a solution of (31) in the form

$$n_{k_1 \pm k_2} = \lambda_1 \lambda_2 \{A_\pm(\mathbf{p}, \mathbf{k}) \exp[i(\omega_1 - \omega_2)t + i(\varphi_1 - \varphi_2)] + \text{c.c.}\},$$

and we obtain for the functions $A_\pm$ the equations

$$\left[i(\omega_1 - \omega_2) + iv(\mathbf{k}_1 \pm \mathbf{k}_2) + \frac{1}{\tau_R} + \frac{1}{\tau}\right] A_\pm = \frac{1}{\tau}\int K(\xi - \xi') A_\pm(\mathbf{p}') d\xi' \frac{do'}{4\pi} - (2n^0 - 1)\left(\frac{1}{\gamma - i\xi_1} + \frac{1}{\gamma + i\xi_2}\right). \quad (32)$$

The solution of (32) is obtained by a method analogous to that used in the solution of (20) and (24). As a result we get

$$A_\pm(\xi) = -\tau_\pm \left\{(2n^0 - 1)\left(\frac{1}{\gamma + i\xi_2} + \frac{1}{\gamma - i\xi_1}\right) + \frac{\tau_\pm}{\tau - \tau_\pm}\frac{1}{2\Theta_\pm}\int d\xi' \left(\frac{1}{\gamma + i\xi_2'} + \frac{1}{\gamma - i\xi_1'}\right)(2n^0(\xi') - 1)\exp\left(-\frac{|\xi - \xi'|}{\Theta_\pm}\right)\right\}, \quad (33)$$

where

$$\Theta_\pm^2 = \frac{\Delta^2 \tau}{\tau - \tau_\pm}, \quad \tau_\pm = \tau_\pm' + i\tau_\pm'', \quad \omega_2 - \omega_1 = \Omega,$$
$$\tau_+' = \frac{1}{4kv_0}\left[\operatorname{arctg}\left(\frac{2kv_0 - \Omega}{1/\tau + 1/\tau_R}\right) + \operatorname{arctg}\left(\frac{2kv_0 + \Omega}{1/\tau + 1/\tau_R}\right)\right],$$
$$\tau_+'' = \frac{1}{8kv_0}\ln\frac{(1/\tau + 1/\tau_R)^2 + (2kv_0 + \Omega)^2}{(1/\tau + 1/\tau_R)^2 + (2kv_0 - \Omega)^2}, \quad (34)$$
$$\tau_-' = \frac{1/\tau + 1/\tau_R}{(1/\tau + 1/\tau_R)^2 + \Omega^2}, \quad \tau_-'' = \frac{\Omega}{(1/\tau + 1/\tau_R)^2 + \Omega^2}.$$

The functions $A_\pm(\xi)$ are similar to (26). The main difference is that in addition to the lifetime τ_R and the diffusion time $1/4Dk^2$ we deal with a time Ω^{-1} equal to the period of the oscillations. In particular, the quantity Re Θ_- is the diffusion length of the electron in energy space within the period of oscillations, and Re $\Theta_- \ll \Theta$ by virtue of $\Omega\tau_R \gg 1$. It must also be borne in mind that in the case when the diffusion lengths $\Theta_\pm$ exceed ϵ_0 and ξ_0, appropriate changes must be made in the coefficients of (33).

With the aid of the functions (22), (26), and (33) we obtain from (4) the density matrix in the first-order approximation:

$$\rho_{\pm k_1}^{(1)}(\mathbf{p}) = \frac{2\lambda_1 \exp[i(\omega_1 t + \varphi_1)]}{\xi_1 + i\gamma}[n_0(\mathbf{p}) + n_{2k_1}(\mathbf{p}) + \lambda_2^2(A_+ + A_-)]. \quad (35)$$

Substituting it in the expressions for the current (7), we obtain equations that determine the amplitudes of the field:

$$\partial \mathbf{E}_1 / \partial t = \mathbf{E}_1\{\alpha_1^0 - \beta_1\lambda_1^2 + Q\delta_1 - \Theta_{12}\lambda_2^2\},$$
$$\partial \mathbf{E}_2 / \partial t = \mathbf{E}_2\{\alpha_2^0 - \beta_2\lambda_2^2 + Q\delta_2 - \Theta_{21}\lambda_1^2\}, \quad (36)$$

where α_n, δ_n, and β_n are given by formulas (27′) and (27″), while the coefficient Θ_{12} is equal to

$$\Theta_{12} \approx 4\pi^2 N(0)\frac{e^2 v_{cv}^2}{\hbar\omega_1\gamma}(2n^0 - 1)\left\{\frac{2\tau\tau_R}{\tau + \tau_R}\left(1 + \frac{\Omega^2}{4\gamma^2}\right)^{-1} + \frac{2\tau\tau_R}{\tau + \tau_R}\frac{1}{s} + \operatorname{Re}\sum_\pm\left[\frac{\tau_\pm}{1 - i\Omega/2\gamma} + \frac{1}{s(\tau_R)^{1/2}}\frac{\tau_\pm^2}{(\tau - \tau_\pm)^{1/2}}\right]\right\}. \quad (37)$$

The first two terms in the curly brackets are due to the zeroth Fourier component of the distribution function, the first of them being connected with the well and the second with the smooth part of the function. The expressions under the summation sign stem from the Fourier components $n_{k_1 \pm k_2}$.

The system (36) describes completely the two-mode regime in a state close to stationary. As is well known[7], the stability of the two-mode regime is determined by the sign of the determinant

$$\text{Det} = \beta_1\beta_2 - \Theta_{12}\Theta_{21}. \quad (38)$$

If Det > 0, then a stable two-mode regime is possible (weak coupling); when Det < 0 (strong coupling), the two-mode regime is unstable and is consequently never realized. As equilibrium is established, the system goes over into a state with a single mode.

An analysis of (38) is in the general case quite cumbersome, and we consider therefore the case most frequently encountered in experiment, $1/\tau_R \ll \Omega \ll 1/\tau$. In this region, the values of Θ_{nm} and β_n are close to each other and the determinant can be approximately written in the form[3)]

$$\text{Det} \approx 2\beta(\beta - \Theta) \approx 4\pi^2\beta N(0)\frac{e^2 v_{cv}^2}{\hbar\omega}\frac{\tau(2n^0 - 1)}{\gamma s(4Dk^2\tau_R)^{1/2}}F(x); \quad (39)$$

$$F(x) = \sqrt{2}(1 - u) - \frac{1}{(|x|)^{1/2}} - \left[\frac{1 + (1 + x^2)^{1/2}}{1 + x^2}\right]^{1/2}, \quad (40)$$
$$x = \Omega / 4Dk^2, \quad u = s(4Dk^2\tau_R)^{1/2}.$$

At $u > 1$, the function F(x) is negative for all x, and consequently the two-mode regime is unstable. If u is smaller than unity, then the function F(x) has a root $x_0 = x_0(u)$, and in the region of smaller x it is negative. Thus, $\Omega_0 = x_0 \cdot 4Dk^2$ gives the smallest distance between the modes of the stable two-mode regime. The value of x_0 increases monotonically with increasing parameter u. When this parameter vanishes, x_0 has a minimum value $x_0 \approx 2.3$.

Substituting the expressions s and D, we obtain for the parameter u:

$$u = \frac{\tau\Delta \cdot 2kv_0}{\gamma\sqrt{3}} \quad (41)$$

The parameters in u are not known with sufficient accuracy. If we use approximate values, then u turns out to be of the order of unity. We note that doping the semiconductor leads apparently to a decrease in the parameter u and contributes to the stability of the two-mode regime.

The system (36) enables us also to consider the question of the jumping of the generation from one mode to another. Mode jumping occurs when the generation threshold of the second mode, in which the two-mode regime is unstable, is reached with increasing pump. By determining from the first equation of (36) the stationary value λ_1^2 of the generated first mode (at $\lambda_2 = 0$) and substituting it into the second equation of (36), we obtain

$$\frac{\partial \mathbf{E}_2}{\partial t} = \mathbf{E}_2\left\{-|\alpha_2^0| - \Theta_{21}\frac{\alpha_1^0}{\beta_1} + Q\frac{\beta_1\delta_2 - \Theta_{21}\delta_1}{\beta_1}\right\} \quad (42)$$

(we have taken into account the fact that α_2^0 is negative).

The generation threshold of the second mode can be reached with increasing pump if

$$\det = \beta_1\delta_2 - \Theta_{21}\delta_1 > 0. \quad (43)$$

For smooth distribution functions $n^0(\xi)$ and for a state density $N(\xi)$ we have approximately $\delta_1 \approx \delta_2$ and the sign of this determinant coincides with the sign of Det. Thus, mode jumping is impossible in this approximation.

Allowance for the energy dependences of n_0 and N leads to the existence of a narrow region $\Omega > 0$ in which the condition (43) is satisfied and the determinant (38)

is negative, i.e., a jump to a shorter-wavelength mode is possible.

The model considered in the present paper has certain limitations. A detailed comparison with the experimental results calls for a determination of the parameters and for concrete calculations.

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[1)]The possible distortion of the distribution function was noted in a recent communication [4].

[2)]Spatial diffusion assuming a Fermi distribution function was considered in [5].

[3)]We note that for smooth functions $n^0(\xi)$ the sign of the determinant coincides with the sign of the difference $\beta - \Theta$.

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