Low-temperature electrical conductivity of metals with closed Fermi surfaces

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The low temperature lattice electrical resistance of metals with closed Fermi surfaces is studied, with phonon nonequilibrium (phonon drag) taken into account. It is shown that, in the case when the distances between the electron or hole groups are much smaller than their characteristic dimensions, the temperature T_p below which the electrical resistance ρ should fall in accordance with the Peierls exponential law $\rho \sim e^{-T_0/T}$ is considerably lower than T_0 ($T_0 \approx \Delta ps$, where Δp is the minimum distance between the groups and s is the speed of sound). For the case of one spherical Fermi surface, the behavior of the electrical resistance is studied in detail in the fairly wide range of intermediate temperatures above which a dependence close to the Bloch law, $\rho \sim T^5$ holds, and below which the Peierls exponential law holds. It is shown, in particular, that allowance for phonon drag has a substantial effect on the low-temperature electrical conductivity. The results obtained are in agreement with the experimentally observed temperature dependence of the electrical resistance of Na and K.

The fundamentally important role of Umklapp processes in the electrical conductivity of pure metals became clear as soon as the work of Peierls appeared (1930–1932). At sufficiently low temperatures (roughly speaking, for $T \lesssim \Theta/10$, where T is the temperature and Θ is the Debye temperature), when the probability of Umklapp processes in collisions between phonons is negligibly small, the electrical resistance of a pure metal is due to Umklapp processes in electron-phonon collisions. (Normal collisions do not themselves lead to resistance, since the total quasi-momentum of the electrons and phonons is conserved.) It follows from these considerations that the behavior of the lattice resistance as $T \to 0$ is essentially different for metals with open and closed Fermi surfaces. In the case of an open Fermi surface, electron-phonon U-processes are possible at arbitrarily low temperatures, and the Bloch law for the resistance is valid: $\rho \sim T^5$. But if the Fermi surface is closed, and the number of electrons is not equal to the number of holes, then at temperatures $T \ll T_0 \approx s\Delta p$ (s is the speed of sound and Δp is the minimum distance between isolated electron groups) the probability of an Umklapp process, and with it the electrical resistance, is proportional to $\exp(-T_0/T)$.

We note that the Peierls exponential in the electrical resistance will also occur when the closed Fermi surfaces intersect the Brillouin zone boundaries. Since, in the repeated-zone scheme, connected parts of the Fermi surface do not contain equivalent points, it is always possible to eliminate the intersection by a suitable choice of the unit cell in p-space; the unit cell need not necessarily be bounded by plane boundaries (see Fig. 1).

Up to the present time, there have been no reliable data on the experimental observation of the Peierls exponential, although for certain metals measurements have been performed at temperatures considerably below T_0. It is probable that the most favourable objects for observing the exponential are the alkali metals, in which the distances between the closed electron groups are extremely large: $\Delta p \approx p_F/3$. (For example, for Na the temperature $T_0 \approx 20°K$, and for K $T_0 \approx 10°K$.) However, in the experiments of Woods[1], the following temperature dependence of the resistance of Na was observed: $\rho \sim T^5$ in the interval $15° > T > 9°$ and a steeper curve, $\rho \sim T^6$ approximately, in the interval $9° > T > 5°$. A law close to T^6 was also observed for K for $4.2° > T > 1.3°$ in the work of Tsoĭ and Gantmakher[2], and analogous results were obtained recently by Ekin and Maxfield[3].

Two questions arise in connection with the above account. 1) At what temperatures does the electrical resistance of metals with closed Fermi surfaces and unequal numbers of electrons and holes begin to decrease exponentially as the temperature is lowered? 2) What temperature dependence precedes the exponential?

In our preceding paper[4], a diffusion equation for the electron distribution function, with phonon drag taken into account, was introduced and used to study the low-temperature electrical conductivity of metals with open Fermi surfaces. In the present paper, the electrical conductivity of metals with closed spherical Fermi surfaces is treated using the same diffusion approach. The limitation to spherical Fermi surfaces implies, of course, that the concrete calculations apply directly to the alkali metals, but, as will become clear from the following, certain general conclusions are valid for arbitrary closed Fermi surfaces.

The electrical conductivity of the alkali metals was calculated earlier by Ziman and a number of other authors on the basis of a variational principle, using the drift distribution as the trial function (see[3] for references to these papers). This method leads to an overestimate of the contribution of Umklapp processes (see the note added in proof in[4] and, for more detail,[5]).

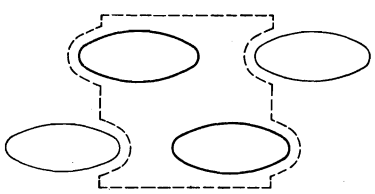

FIG. 1. The dashed line depicts the boundaries of a unit cell that does not intersect the Fermi surface. In the given case, it is impossible to eliminate the intersections by any choice of right-angled cell.

Moreover, phonon drag, which is extremely important at low temperatures[1)], was not taken into account in the papers mentioned (see the end of Sec. 2).

1. IDEALIZED MODEL

First we shall consider a simplified "one-dimensional" model of the band spectrum, in which Fermi spheres positioned on one straight line approach each other closely in the repeated-zone scheme (Fig. 2). The distribution function depends on one coordinate τ, measured along an arc of a great circle from the point of closest approach of the spheres. The electric field $\mathbf{E}$ is directed, naturally, along the reciprocal-lattice vector $\mathbf{g}$. The quantity Δp will be assumed to be extremely small; the method used in the present section is valid if the following strong inequality is fulfilled:

$$\ln (p_F / \Delta p) \gg 1. \tag{1}$$

The dispersion law of the phonons and their interaction with the electrons will be assumed to be isotropic, and the matrix element of the interaction is

$$B_{\mathbf{pkq}} = \sqrt{Mq} \sum_{\mathbf{g}} \delta_{\mathbf{p},\mathbf{k}+\mathbf{q}+\mathbf{g}}$$

($\mathbf{p}$ and $\mathbf{k}$ are the electron momenta and $\mathbf{q}$ is the phonon momentum). Although the model considered is highly idealized, the physical conclusions obtained on the basis of it are valid for real alkali metals (see Sec. 2).

Before proceeding to the calculations, we shall make the following preliminary remarks. The diffusion approximation for normal electron-phonon collisions is applicable if the change q of the electron momentum in a single collision is much smaller than the characteristic dimensions over which the distribution function varies. Far from that region of the Fermi surface from which transitions to the neighboring Fermi sphere are possible (the region of closest approach of the Fermi spheres—we shall call this region the "lune"), the distribution function varies over distances of the order of p_F, and at low temperatures ($T \ll sp_F$) the diffusion approximation is applicable. But within the lune we may expect changes of the distribution function over distances of the order of the dimensions r_0 of the lune.

The quantity r_0 is easily estimated by taking into account that, within the lune, the inequality $q_m - \Delta p \lesssim q_T = T/s$ should be fulfilled ($q_m = |\mathbf{p} - \mathbf{p}'| \approx \Delta p + \tau^2/p_F$ is the distance from a point with coordinate τ to the neighboring Fermi sphere and q_T is the thermal momentum of the phonons), whence $r_0 \approx (q_T p_F)^{1/2}$. Clearly, for $q_T \gtrsim \Delta p$, $\mathbf{q} \approx \mathbf{q}_T$ and so the inequality $q \ll r_0$, which ensures the applicability of the diffusion approximation for normal collisions within the limits of the lune, is fulfilled. But if $q_T \ll \Delta p$, then an electron which has undergone an Umklapp collision acquires an energy of order $\Delta ps \gg T$, i.e., is found in the non-thermal layer of energies. On returning to the Fermi surface in a normal collision with emission of a phonon, such an electron is displaced over a distance $q \approx \Delta p$. Therefore, normal collisions within the limits of the lune can be described as diffusion when the following inequality is fulfilled:

$$\Delta p \ll r_0 \approx (q_T p_F)^{1/2}. \tag{2}$$

The diffusion equation can be derived from the system of kinetic equations for interacting electrons and phonons

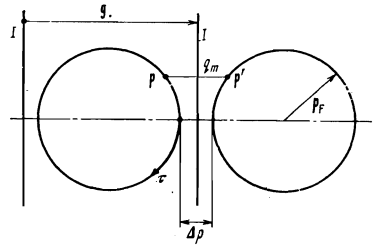

FIG. 2. Repeated-zone sheme. I—Brillouin-zone boundaries.

by the same method that we used previously[4]. The result for the isotropic model has the form

$$\Delta\chi_p - \operatorname{div} \mathbf{a}_p + \Pi_p = -\frac{eE}{D}\cos\theta, \quad \theta = \frac{\tau}{p_F},$$
$$\mathbf{a}_p = \int_{p'} \hat{A}_{pp'} \nabla\chi_{p'} dS_{p'}, \quad D = T^5 \frac{30\zeta(5)}{\pi\hbar^4 v^2} \frac{M}{s^5}, \tag{3}$$
$$\Pi_p = \frac{h^3}{2Dv_F} \int_0^\infty J_p^U d\varepsilon_p.$$

Here Δ and div are the angular parts of the corresponding three-dimensional operators, $-\chi_{\mathbf{p}}\partial n/\partial\epsilon$ is the non-equilibrium correction to the electron distribution function (χ does not depend on ϵ), $n = [e^{(\epsilon-\mu)/T} + 1]^{-1}$, and the explicit form of the kernel $\hat{A}_{\mathbf{pp}'}$ is given in[4]. The first two terms in the left-hand side of (3) describe normal collisions of electrons: the first term describes collisions with equilibrium phonons, and the second integral term takes account of phonon drag; $J_{\mathbf{p}}^U$ is the collision integral for electron-phonon Umklapp collisions[2)]. (The term $\Pi_{\mathbf{p}}$ is absent in the equation obtained previously[4], since in the case of open Fermi surfaces Umklapp processes are taken into account by imposing periodic boundary conditions on the function χ.) The Umklapp-collision integral has the form

$$J_p^U = \sum_{\mathbf{k},\mathbf{q}} (\Gamma^+_{\mathbf{pkq}} - \Gamma^-_{\mathbf{pkq}});$$
$$\Gamma^\pm_{\mathbf{pkq}} = \frac{2\pi}{\hbar} Mq \frac{dN_q}{d\Omega_q} (n_p - n_k)(\chi_p - \chi_k)\delta(\varepsilon_p - \varepsilon_k \pm \Omega_q). \tag{4}$$

Here $N_{\mathbf{q}} = [\exp(\Omega_{\mathbf{q}}/T) - 1]^{-1}$, $\Omega_{\mathbf{q}}$ is the phonon energy, and $\mathbf{k} = \mathbf{p} - \mathbf{q} - \mathbf{g}$ for the right lune and $\mathbf{k} = \mathbf{p} - \mathbf{q} + \mathbf{g}$ for the left lune (see Fig. 2). In (4), we have omitted terms associated with the non-equilibrium of the phonons; it can be shown that taking these into account would lead to a correction proportional to q_T/p_F to the electrical conductivity.

For the calculation of the quantity $\Pi_{\mathbf{p}}$, we note that the characteristic size r_1 of the region on the Fermi surface to which Umklapp processes from some point $\mathbf{p}$ are possible is defined, as is not difficult to understand, by the condition $(q_m^2 + r_1^2)^{1/2} - q_m \approx q_T$, whence $r_1 \approx (2q_m q_T + q_T^2)^{1/2} \ll r_0$. Therefore, a "local" approximation is applicable for U-processes, i.e., in (4) we can put $\chi_{\mathbf{k}} = \chi_{\mathbf{p}'}$, where $\mathbf{p}'$ is the point on the neighboring sphere that is nearest to $\mathbf{p}$ (cf. Fig. 2; with the same accuracy, the point $\mathbf{p}'$ can be assumed to lie on a horizontal line with $\mathbf{p}$). Then, after simple calculations, we obtain

$$\Pi_p = -\frac{f(q_m/q_T)}{30\zeta(5)q_T^3}(\chi_p - \chi_{p'}) \equiv -A(\theta)\chi_p; \tag{5}$$
$$q_m = \Delta p + \frac{\tau^2}{p_F}, \quad f(x) = \int_x^\infty \frac{e^y}{(e^y-1)^2} y^3 dy.$$

Here we have taken into account that, by virtue of the one-dimensionality of the problem and the fact that χ is an odd function of p, $\chi_{\mathbf{p}'} = -\chi_{\mathbf{p}}$. For the function f(x), we can write an interpolation formula which is asymptotically exact for small and large x:

$$f(x) \approx (6\zeta(3) + x^3)e^{-x}.$$

We shall now establish certain relations, useful for the following, which are a consequence of the conservation of quasimomentum in normal collisions:

$$\int_S \mathbf{p}(\Delta\chi - \operatorname{div}\mathbf{a})\,dS = 0. \tag{6}$$

Multiplying Eq. (3) by $\mathbf{p}$ and integrating over the Fermi surface, we obtain, taking (5) and (6) into account,

$$\int A(\theta)\chi\,dS = \frac{eE}{gD}\frac{4}{3}\pi p_F^3. \tag{7}$$

Here we have used the fact that $|\mathbf{p} - \mathbf{p}'| \ll g$; the integration is performed within the limits of the right lune.

We now integrate Eq. (3) over the right half of the Fermi sphere ($0 < \theta < \pi/2$). After simple identity transformations using (7) we obtain

$$2\pi p_F\left[\nabla\chi\left(\frac{\pi}{2}\right) - \mathbf{a}\left(\frac{\pi}{2}\right)\right] = \frac{e\mathbf{E}}{gD}\left(\pi p_F^2 g - \frac{4}{3}\pi p_F^3\right). \tag{8}$$

An analogous relation was obtained earlier in[4].

For the following, it is convenient to write Eq. (3) not in the laboratory frame of reference (in which the lattice is stationary), but in the "co-moving" frame, in which the diffusion equation can be solved by iterations with respect to the integral term $\mathbf{a}_{\mathbf{p}}$. As was shown in[4], the difference $\nabla\chi_{\mathbf{p}} - \mathbf{a}_{\mathbf{p}}$ is invariant under the Galilean transformation $\chi \to \chi + \mathbf{u}\cdot\mathbf{p}$, and therefore Eq. (3) and the relations (7) and (8) have, in the co-moving reference frame and in zeroth order in $\mathbf{a}_{\mathbf{p}}$, the form

$$\Delta\chi - A(\theta)\left(\chi \pm \frac{ug}{2}\right) = -\frac{eE}{D}\cos\theta, \tag{9}$$

$$\int A(\theta)\left(\chi + \frac{ug}{2}\right)dS = \frac{eE}{gD}\frac{4}{3}\pi p_F^3 \tag{10}$$

$$\nabla\chi\left(\frac{\pi}{2}\right) = \frac{e\mathbf{E}p_F}{gD}\left(\frac{2}{3}p_F - \frac{g}{2}\right). \tag{11}$$

The plus sign in (9) corresponds to the right lune, and the the minus sign to the left lune.

It follows from the relation (6) that, in the co-moving reference frame in any order of iteration in $\mathbf{a}_{\mathbf{p}}$, the integral $\int \cos\theta\,\Delta\chi\,dS = 0$. Hence it follows that the electric-current density $j \propto \int \cos\theta\,\chi\,dS = 0$ in the co-moving reference frame, and, therefore, (to within quantities of higher order in q_T/p_F) the electrical conductivity

$$\sigma = \frac{8\pi}{3}\left(\frac{p_F}{h}\right)^3\frac{eu}{E}.$$

To determine the drift velocity u, it is necessary to solve Eq. (9) consistently with the conditions (10) and (11). Far from the lune, the second term in the left-hand side of (9) can be neglected, and in this region of the Fermi surface,

$$\chi = \frac{eEp_F^2}{2D}\cos\theta - b\beta(\theta), \quad \beta(\theta) = \ln\frac{1+\cos\theta}{1-\cos\theta} \tag{12}$$

($b\beta(\theta)$ is the general anti-symmetric solution of the homogeneous equation $\Delta\chi = 0$). We find the value of the coefficient b from (11): $b = eEp_F^3/3gD$.

It is not possible to solve Eq. (9) exactly within the limits of the lune. However, the character of the solution is easily understood from the following considerations. The logarithmic increase of the function χ as the lune is approached is associated with the flow of the diffusion current

$$I_0 \approx 2\pi\tau Db\frac{\partial\beta}{\partial\tau} \approx eE\frac{4}{3}\frac{\pi p_F^3}{g}.$$

Outside the lune this current is constant, but inside the lune the diffusion current is decreased, since part of the total current is carried away by electrons going over on to the neighboring sphere. From a certain portion of the lune, this current is approximately proportional to its area, so that

$$2\pi\tau D\frac{\partial\chi}{\partial\tau} \approx I_0\left(\frac{\tau}{r_0}\right)^2, \quad \tau \leqslant r_0.$$

Thus, inside the lune, χ varies more slowly than $\beta(\theta)$, reaching a finite value at the center; the difference $\chi(r_0/p_F) - \chi(0) \leq b$; this variation can be neglected compared with the quantity $b\beta(r_0/p_F)$, under the condition $\ln(p_F/r_0) \gg 1$. Assuming this condition to be fulfilled, we write

$$\chi \approx \frac{eEp_F^2}{2D} - b\beta\frac{r_0}{p_F}, \quad \theta \leqslant \frac{r_0}{p_F} \approx \left(\frac{q_T}{p_F}\right)^{1/2}. \tag{13}$$

Substituting this expression into (10), we find

$$u = \frac{2eEp_F^3}{3g^2D}\left[\ln\left(\frac{p_F}{q_T}\right) + F^{-1}\right] \equiv u_d + u_U,$$

$$F = \frac{1}{4\pi}\int A(\theta)\,dS = \frac{p_F}{60\zeta(5)\,q_T}\int_{\tau_{\Delta p/q}}^{\infty}\frac{e^y}{(e^y-1)^2}y^3\left(y - \frac{\Delta p}{q_T}\right)dy. \tag{14}$$

The iteration formula for F, asymptotically exact for $q_T \gg \Delta p$ and $q_T \ll \Delta p$, has the form

$$F = \frac{1}{60\zeta(5)}\frac{p_F}{q_T}\left[24\zeta(3) + \left(\frac{\Delta p}{q_T}\right)^3\right]\exp\left(-\frac{\Delta p}{q_T}\right).$$

We shall discuss the physical meaning of the result obtained. The drift velocity u (and with it the electrical conductivity σ) is proportional to the total relaxation time during which an electron describes a closed cycle in p-space, i.e., diffusion through the Fermi surface and a jump between lunes as a result of an Umklapp process. We emphasize that, under conditions of phonon drag, the respective times are summed.

The first term u_d in (14) depends on the temperature like $T^{-5}\ln(p_F s/T)$; it is proportional to the time of diffusion of an electron from central regions of the Fermi surface to a lune. The result $u \sim T^{-5}\ln(p_F/r_0)$ arose in[4] for an open Fermi surface containing a short narrow neck of radius r_0. In the case under consideration, the radius of the effective neck is $r_0 \approx (q_T p_F)^{1/2}$, and this leads to a logarithmic dependence on temperature of the coefficient of T^{-5}.

The second term u_U in (14) is proportional to the time of the Umklapp process and depends on the temperature like $T^{-1}\exp(T_0/T)$, if $T \ll T_0 = s\Delta p$. At sufficiently low temperatures, this term leads to an exponential fall of the electrical resistance. It follows from (14) that the temperature T_P below which the Peierls exponential appears ($u_U \gg u_d$) is considerably less than T_0 for $\Delta p \ll p_F$. This result is connected with two circumstances. 1) Although the probability of a U-process has become exponentially small at $T < T_0$, the transfer of an electron to the neighboring sphere occurs as the result of one collision, whereas the diffusion path through the

whole Fermi surface requires a large number of steps. 2) Each of the three quantities: the square of the matrix element of the interaction, the area of the portion of the Fermi surface in which a transition is possible ($r_1^2 \approx \Delta p q_T$), and the energy layer from which a transition is possible, is $\Delta p/q_T$ times greater for phonons with momenta of order Δp than for thermal phonons. Therefore, the probability of a U-process contains an additional factor $(\Delta p/q_T)^3$, which is large for $T \ll T_0$.

As an illustration, Fig. 3 shows the dependence on the parameter in the problem, $p_F/\Delta p$, of the temperatures T_{p_1} at which $u_U/u_d = 0.25$ and T_{p_2} at which $u_d/u_U = 0.25$. It can be seen that the transitional region of temperatures, from the law $\rho \sim T^5/\ln(p_F s/T)$ to the law $\rho \sim Te^{-s\Delta p/t}$, is fairly broad, and the ratio T_0/T_p increases with increasing $p_F/\Delta p$.

We recall that the expression (14) was obtained in the zeroth approximation in the integral term a_p in Eq. (3). However, as was shown in[4], the corrections to u associated with allowance for the term a_p in the co-moving reference frame (these terms take into account the deviation of the phonon distribution from the drift distribution in the laboratory reference frame) are bounded as $r_0 \to 0$ and are negligibly small for $\ln(p_F/r_0) \gg 1$. We recall also that Eq. (3), in which the diffusion approximation is used for normal collisions, is valid inside the lune, if the condition (2) is fulfilled. It is not difficult to see, however, that when $r_0 \lesssim \Delta p$ and the condition (1) is fulfilled the diffusion time can be neglected compared with the Umklapp time, and so the violation of the diffusion approximation is unimportant.

Although in this section we have considered a highly idealized model, it is not difficult to understand, from physical arguments, which properties of the Fermi surface are actually essential for the results obtained and which conclusions remain valid for arbitrary surfaces. First of all, it is clear that the temperature dependence obtained for the quantity u_U follows entirely from the assumption that the radius of curvature of the Fermi surface in the area of the lune is large compared with Δp. It is also understandable that the conclusion that the temperature T_p is considerably lower than T_0 (roughly in accordance with Fig. 3) is also valid for an arbitrary surface, provided that the characteristic dimensions of the electron or hole groups are substantially greater than the distances between them. As regards the temperature dependence of the quantity u_d, it is extremely sensitive to the geometry of the Fermi surface. If, e.g., the approach of the surfaces arises as a result of the close approach of long narrow protrusions (so that we are concerned with a long broken neck), the quantity u_d can turn out to be proportional to T^{-5} or T^{-4} (cf.[4]). Moreover, the logarithmic dependence obtained above, $u_d \sim T^{-5}\ln(p_F s/T)$, is closely connected with the inequality (1), and may not appear in real cases (cf. the following Section).

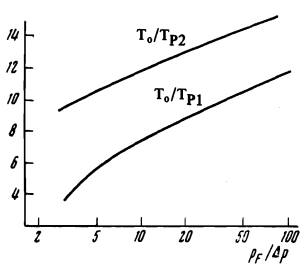

FIG. 3

2. THE ALKALI METALS

Some of the simplifying assumptions used in the preceding Section are certainly invalid for real alkali metals. First of all, even the condition $\Delta p \ll p_F$ is poorly fulfilled (for Na and K we have $\Delta p \approx 0.3p_F$), to say nothing of the fulfillment of the condition (1). Strictly speaking, this implies that, in the region of temperature of interest, in which $\Delta p \gtrsim (q_T p_F)^{1/2} = r_0$, we cannot neglect the variation of the distribution function χ_p within the lune. Moreover, the characteristic scale of variation of the function χ_p is now, clearly, not greater than Δp, and so the diffusion equation (3) itself gives only a qualitative description of the behavior of χ_p within the lune.

Nevertheless, the approach developed in the preceding Section can be used for a quantitative study of the electrical conductivity of the alkali metals. As a detailed analysis shows, the principal error arises in the determination of the effective lune radius appearing in the logarithm (cf. (13)). The point is that condition (10) is not associated with the diffusion approximation, and the drift velocity u obtained from it is relatively insensitive to the behavior of the function χ in the region of the lune. But it is clear from physical considerations that in this region the function χ varies more slowly than logarithmically (cf. Sec. 1).

The purpose of this section is to make the result (14) more exact, for application to real alkali metals. In particular, we shall take into account that, in the repeated-zone scheme, a given Fermi sphere is closely approached by 12 other spheres. The not very large anisotropy of the phonon spectrum is less important, and therefore we shall confine ourselves to the Debye model in this section too.

For the case of a Fermi sphere with several lunes, the diffusion equation analogous to (9) has the following form:

$$\Delta\chi_p - {}^1/_2 A(\theta_k)(\chi_p - \chi_{p'} + u g_k) = -\frac{eE}{D}\cos\theta. \qquad (15)$$

Here the angle θ_k is reckoned from the direction of the reciprocal-lattice vector $\mathbf{g}_k$ corresponding to the lune closest to the point $\mathbf{p}$, and the angle θ is reckoned from the direction of the electric field $\mathbf{E}$, which may conveniently be assumed to be applied along one of the vectors of the reciprocal lattice, $\mathbf{g}_0$ (since the electrical conductivity is isotropic in the model under consideration).

We shall seek the solution of Eq. (15) in the form of a linear combination of solutions of the "one-dimensional" problem corresponding to one pair of lunes (in an analogous way to that followed in[4] for a Fermi sphere open in several directions):

$$\chi_p = \sum_k \tilde{\chi}(\theta_k)\cos\alpha_k + \frac{eEp_F^2}{2D}\cos\theta. \qquad (16)$$

Here, α_k is the angle between the vectors $\mathbf{g}_k$ and $\mathbf{g}_0$, and the function $\tilde{\chi}(\theta)$ satisfies the one-dimensional equation:

$$\tilde{\Delta}\tilde{\chi} - A(\theta)(\tilde{\chi} \pm \tilde{u}g/2) = 0, \qquad (17)$$

where the upper sign refers to the right lune and the lower sign to the left lune.

It is not difficult to verify that a function of the form (16) satisfies Eq. (15) if

$$\tilde{u} = u + \frac{2}{g}\sum_{k\neq 0}\tilde{\chi}(\alpha_k)\cos\alpha_k + \frac{eEp_F^2}{gD}$$

and if the quantity $\tilde{\chi}(\theta_{k\neq k})$ is assumed constant within the k-th lune. It is not difficult to convince oneself that the condition analogous to (10) has the form

$$\int A(\theta)(\tilde{\chi} + \tilde{u}g/2)\,dS = \frac{eE}{gD}\frac{8}{n}\pi p_F^3, \qquad (18)$$

where n is the total number of lunes and the integration is taken over the right lune.

Eq. (17) is equivalent to the variational principle:

$$\delta\int_F \tilde{\chi}[\Delta\tilde{\chi} + A(\theta)(\tilde{\chi} + \mathbf{up})]\,dS = 0. \qquad (19)$$

We seek the solution of Eq. (17) in the following form (cf. (12) and (13)):

$$\tilde{\chi} = -\frac{2eEp_F^3}{ngD}\tilde{\beta}(\theta), \quad \tilde{\beta}(\theta) = \begin{cases} \ln\dfrac{1+\cos\theta}{1-\cos\theta}, & \theta > \theta_0 \\ \ln\dfrac{1+\cos\theta_0}{1-\cos\theta_0}, & \theta < \theta_0. \end{cases}$$

Here, the coefficient of $\tilde{\beta}(\theta)$ is determined from a condition analogous to (11), and θ_0 is a variational parameter, which must be determined jointly with $\tilde{u}$ from (18) and (19); $\theta < \pi/2$. Varying (19) with respect to θ_0, we obtain

$$2\pi p_F^2\int_0^{\theta_0}\theta A(\theta)\left(\tilde{\chi} + \frac{\tilde{u}g}{2}\right)d\theta = \frac{eE}{gD}\frac{4}{n}\pi p_F^3. \qquad (20)$$

The system of equations (18) and (20) can be rewritten in the following form:

$$\tilde{u} = \frac{4eEp_F^3}{ng^2D}\left[\tilde{\beta}(0)\varphi(\theta_0) + \frac{1}{F}\right], \qquad (21)$$

$$\tilde{\beta}(0)[\varphi(\theta_0) - 1] = \frac{1}{F}\left[\frac{1}{2\psi(\theta_0)} - 1\right];$$

$$\varphi(\theta_0) = \frac{p_F^2}{\tilde{\beta}(0)F}\int_0^{\infty}\theta\tilde{\beta}(\theta)A(\theta)\,d\theta, \quad \psi(\theta_0) = \frac{p_F^2}{F}\int_0^{\theta_0}\theta A(\theta)\,d\theta, \qquad (22)$$

where F is defined by the expression (14).

The relation (22) is a complicated transcendental equation for θ_0. The interpolational solution of this equation that is asymptotically exact for large and small F has the form

$$\theta_0 = \left(\frac{q_T}{p_F}\ln[2(FC+1)]\right)^{1/2}, \quad C = \int_{\ln 2}^{\infty} e^{-x}\frac{dx}{x}.$$

Corresponding to this,

$$\tilde{u} = \frac{4eEp_F^3}{ng^2D}\left[\tilde{\beta}(0) - \frac{C\ln 2}{\ln 2 + F} + \frac{1}{F}\right]$$

and the electrical conductivity of the metal is equal to

$$\sigma = \frac{2e}{h^3E}\int_F(\chi_p + up_F\cos\theta)\cos\theta\,dS$$
$$\approx \frac{Ne^2}{2D}p_F\left\{1 - 2\frac{p_F}{g}(1+\cos\theta_0) + \frac{8p_F^2}{ng^2}\left[\sum_k\tilde{\beta}(\alpha_k)\cos\alpha_k + \frac{1}{F} - \frac{C\ln 2}{\ln 2 + F}\right]\right\} + \sigma'. \qquad (23)$$

Here, $N = (8/3)\pi(p_F/h)^3$ is the electron density and σ' is the correction to the electrical conductivity associated with allowance for the integral term $\mathbf{a_p}$ in the co-moving reference frame. (We recall that Eq. (15) corresponds to the zeroth approximation in $\mathbf{a_p}$.)

The analogous correction was calculated previously[4] for a spherical Fermi surface with "opennesses". Using the results of the latter paper, it is not difficult to find

$$\sigma' = \frac{8Ne^2p_F^3}{ng^2D}\sum_{i=3}^{\infty}\frac{2i+1}{i(i+1)}\frac{\lambda_i}{1-\lambda_i}\sum_k[P_i(\alpha_k)]^2\cos\alpha_k, \qquad (24)$$

where for odd i

$$\lambda_i = \frac{2}{i(i+1)}\left[\frac{i!!}{(i-1)!!}\right]^2.$$

$\lambda_i = 0$ for even i, and the $P_i(\alpha)$ are Legendre polynomials.

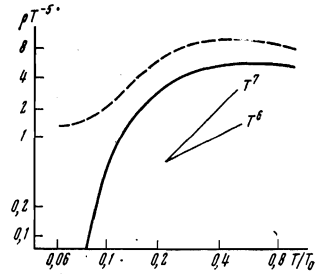

FIG. 4

Figure 4 shows the dependence, calculated from formula (23), of the quantity ρT^{-5} (in arbitrary units) on the relative temperature T/T_0. For comparison, the result obtained in the diffusion approximation used in the present work but without allowance for phonon drag is represented in the same Figure by a dashed line (the distribution function is determined from Eq. (15), in which we must put u = 0; the quasi-momentum balance condition (18) is absent.) It can be seen that phonon drag plays an important role: in the temperature interval $0.15T_0 - 0.6T_0$ the results differ by almost a factor of two, and at lower temperatures the curves behave completely differently.

For potassium, $T_0 \approx 11°K$ (cf.[3]) and the slope of the curve in the region of intermediate temperatures agrees fairly well with the experimental results of[2,3], in which dependences close to $\rho_T \sim T^6$ and $\rho_T \sim T^7$ ($\rho_T = \rho - \rho_0$, where ρ_0 is the residual resistance) were observed at such temperatures. For sodium, $T_0 \approx 20°K$ (cf.[6]), and, as can be seen from Fig. 4, the transition from the law $\rho_T \sim T^5$ to the more rapid dependence occurs at approximately 8°K; this agrees well with the experimental data of Woods[1]. No great significance can be attached to this agreement, however. The point is that formula (23) refers to an ideal impurity-free sample, whereas, in a considerable part of the temperature range considered, $\rho_0 \gg \rho_T$. (For the potassium samples investigated in[2,3], $\rho_0 \approx \rho_T$ at temperature $T \approx 5°K$.)

In the case when $\rho_0 \gg \rho_T$, the kinetic equation for the electrons can be solved by the method of successive approximations, assuming that the electron-impurity collision integral is large compared with the electron-phonon collision integral. As a result, as can easily be shown, the quantity ρ_T is found to depend only on the relative concentration of the different types of impurity. It is possible that the weak dependence, observed by Ekin and Maxfield[3], of the quantity ρ_T on the residual resistance ρ_0 is connected with this fact.

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[1)]We note that phonon drag also plays a significant role in metals with open Fermi surfaces containing narrow necks. It follows from the results of our paper [4], e.g., that for the noble metals allowance for

phonon drag leads to a several-fold reduction of the coefficient in the Bloch law $\rho \sim T^5$.

[2)]Allowance for the effect of Umklapp processes on the phonon distribution function would lead to the appearance of additional terms in the expression for a_p, which, however, are small in the parameter q_T/p_F.

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