# Specific heat and spin paramagnetism of an electron gas in the presence of resonant scattering impurities

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A degenerate gas of electrons interacting with impurities is considered for which resonance of the scattering amplitude is located near the electron Fermi energy. In this case the temperature dependence of the electron-gas specific heat has a peak whose position and width in the one-band model under consideration are defined by the impurity concentration (at not too low concentrations) and density of electron states without impurities. The peak height may be of the same order of magnitude as the free-electron specific heat at the maximum; on application of a magnetic field, the peak splits into two. The magnetic field dependence of spin paramagnetic susceptibility also contains a peak which is strongly temperature dependent.

The effect of impurities on the properties of an electron gas was investigated in a large number of studies. A diagram technique was developed[1] with which to analyze kinetic phenomena in such a system. On the other hand, the thermodynamic characteristics of the system (where they play an important role, e.g., in the analysis of oscillatory effects in a magnetic field[2]), are governed by the fact that scattering by impurities leads to a finite electron mean free path.

We consider in this paper the case when the Fermi energy of the electron gas is close to the resonance of the amplitude for scattering by a single impurity. If the resonance (in the isotropic case) is due to scattering with angular momentum $l \geq 1$, then it is easy to verify that the diagram technique for the calculation of the electron Green's function cannot be applied directly, and it is necessary to use other methods. We derive an expression for the density of states of the electrons, in the form of the first terms of an expansion in powers of the impurity concentration in the case when the impurity concentration is low. Near the pole for the scattering amplitude by a single impurity, of course, it would be necessary to take into account all the terms of the scattering amplitude of the system of impurities in terms of the corresponding amplitudes on the single impurities. This leads to difficulties in the description of the spectrum of such a system at negative energies. For positive energies, on the other hand, the pole of the amplitude goes off from the real energy axis into the unphysical region, as a result of which it becomes possible to describe conveniently the Green's function in the entire vicinity of the resonant energy.

Since the density of states contains in this case a sharp peak there the amplitude resonance, well observable deviations from the usual relations should be observed in the thermodynamic characteristics of the electron gas. In Sec. 2, using the specific heat and the spin paramagnetism as examples, it is hown that even at low impurity concentrations the corrections to these quantities can be of the same order of their values without impurities.

1. Let the potential of an impurity situated at a point $\mathbf{r}_a$ be $v(|\mathbf{r} - \mathbf{r}_a|)$. Then the total potential acting on the electron takes the form

$$V = \sum_a v(|\mathbf{r} - \mathbf{r}_a|).$$

To calculate the density of states of the electron in a field V, we consider the eigenfunctions and the eigenvalues of the operator $VG_0$ (or $G_0V$)[3], where $G_0 = (\omega - \hat{H}_0)^{-1}$ is the Green's function of the electron in the absence of impurities; in the isotropic case, which we shall mainly consider,

$$G_0(\mathbf{rr}'\omega) = -\frac{m^*}{2\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$$

($k = \sqrt{2m^*\omega} > 0$ at $\omega > 0$, $\operatorname{Im}\omega = +0$).

We are thus interested in solutions of the problem

$$\mu_\alpha\psi_\alpha = VG_0\psi_\alpha, \quad \mu_\alpha X_\alpha^+ = X_\alpha^+ VG_0, \qquad (X_\alpha^+\psi_\beta) = \int X_\alpha^+(\mathbf{r})\psi_\beta(\mathbf{r})\,d\mathbf{r} = \delta_{\alpha\beta} \tag{1}$$

(a simple exposition of the theory of non-self-adjoint operators is contained in[4]).

We shall need also the functions

$$\Psi_\alpha = \mu_\alpha^{-1}G_0\psi_\alpha, \quad \chi_\alpha^+ = X_\alpha^+ V, \tag{1a}$$

which obviously satisfy the equations

$$\mu_\alpha\Psi_\alpha = G_0V\Psi_\alpha, \quad \mu_\alpha\chi_\alpha^+ = \chi_\alpha^+G_0V, \quad (\chi_\alpha^+\Psi_\beta) = \delta_{\alpha\beta} \tag{1b}$$

and are connected with $\psi$ and $X^+$ by the relations

$$\psi_\alpha = V\Psi_\alpha, \quad \mu_\alpha X_\alpha^+ = \chi_\alpha^+ G_0. \tag{1c}$$

Weinberg[3] cites the Schrodinger equations and the boundary conditions for the functions $\Psi$ and $X^+$.

If we represent the operator $VG_0$ in the form

$$VG_0 = \sum \mu_\alpha\psi_\alpha X_\alpha^+ = \sum \psi_\alpha\chi_\alpha^+ G_0,$$

then, acting on it from the left with the operator $\omega - \hat{H}_0$, we get

$$V = \sum \psi_\alpha\chi_\alpha^+,$$

from which it is clear that the scattering operator A, defined by

$$A = V + VG_0A,$$

can be expressed in the form

$$A = V + \sum \frac{\mu_\alpha\psi_\alpha\chi_\alpha^+}{1-\mu_\alpha} = \sum \frac{\psi_\alpha\chi_\alpha^+}{1-\mu_\alpha}. \tag{2}$$

The Green's function G of the electron in the impurity field V is expressed in terms of the operator A as follows:

$$G = G_0 + G_0AG_0.$$

This yields for the correction $\rho_1$ to the density of states per unit volume ($\operatorname{Im}\omega = +0$)

$$V_0\rho_1 = -\frac{1}{\pi}\operatorname{Im}\operatorname{Sp} G_0AG_0$$

($V_0$ is the volume of the system), and furthermore

$$\operatorname{Sp} G_0AG_0 = \operatorname{Sp}\sum_\alpha \frac{G_0\psi_\alpha\chi_\alpha{}^+G_0}{1-\mu_\alpha} = \sum_\alpha \frac{\mu_\alpha{}^2(X_\alpha{}^+\Psi_\alpha)}{1-\mu_\alpha}.$$

Differentiating (1) with respect to $\omega$, we obtain

$$\mu_\alpha'\psi_\alpha + \mu_\alpha\psi_\alpha' = -VG_0{}^2\psi_\alpha + VG_0\psi_\alpha',$$

whence, taking the scalar product with $X^+$, we have (see (1))

$$\mu_\alpha' = -(X_\alpha{}^+VG_0{}^2\psi_\alpha) = -\mu_\alpha{}^2(X_\alpha{}^+\Psi_\alpha),$$

which yields for $\rho_1$ the expression

$$V_0\rho_1 = -\frac{1}{\pi}\frac{d}{d\omega}\operatorname{Im}\sum_\alpha \ln(1-\mu_\alpha). \tag{3}$$

We now assume that the impurity concentration $n = N_{\text{imp}}/V_0$ is small, so that $k\bar{r} \gg 1$, where $\bar{r} = n^{-1/3}$ is the mean distance between the impurity centers, and $k = \sqrt{2m^*\omega}$ is the wave vector of a free electron with energy close to resonance.

The eigenvalue $\mu_{nl}$ of the problem (1) in a centrally-symmetrical field v without resonance, at $kr_0 \ll 1$, where $r_0$ is the effective radius of the potential v, is expressed in the form[5] ($l$ is the angular momentum and m is its projection)

$$\mu_{nl} \approx 1 + a_{nl}(\omega-\varepsilon) - i\,2m^*k^{2l+1}c_{nl}{}^+c_{nl}, \tag{4}$$

where

$$a_{nl} = -(2m^*)^{l+1}c_{nl}{}^+c_{nl}b \sim 1/E_0, \quad E_0 = 1/m^*r_0{}^2,$$

b is the coefficient in the principal term of the scattering amplitude near resonance, $\epsilon$ is the energy of the resonance, and $c_{nl}^+c_{nl} \sim r_0^{2l+1}/m^*$ (this follows from (A.5) and from the normalization in (1)).

It is clear from (A.4) that

$$\operatorname{Re}\delta\mu \sim \begin{cases} (r_0/R)^{2l+1}, & kR \lesssim 1,\\ (kr_0)^{2l}\, r_0/R, & kR \gtrsim 1.\end{cases}$$

Since $\operatorname{Im}\mu_{nl} \sim (kr_0)^{2l+1}$ (see (4)), it follows that when $kR \gg 1$ we can disregard the smearing of the levels $\mu_\alpha$ (formula (A.4) was derived for two-particle "collisions" of impurity centers, but $\delta\mu \sim g$ even when many centers are taken into account).

We consider henceforth, by way of example, only two-particle collisions (it will be evident when this approximation is suitable). Since the total number of sublevels into which the given level $\mu_{nm}$ splits is $N_{\text{imp}}$ (and accordingly we have $(2l+1)N_{\text{imp}}$ sublevels for the level $\mu_{nl}$), we can write for any function $f_\nu$ of $\mu_\nu$ (exact solutions in the field V, which go over into $\mu_{nl}$ as $R \to \infty$)

$$\sum f_\nu \approx N_{\text{пр}}p_0f_0 + \frac{N_{\text{пр}}}{2}\int dp_1(R)\,[f_+(R) + f_-(R)].$$

Here $f_0$ denotes $f(\mu_{nl})$, $f_\pm$ denotes $f(\mu_{nm}^\pm)$ (see (A.4)), $p_0 = \exp(-n\bar{v})$, $p_1 = nv\exp(-nv$, $v = \frac{4}{3}\pi R^3$, $\bar{v} = \frac{4}{3}\pi\bar{R}^3$ and the integration is carried out over the region $0 < R < \bar{R}$, where $1/k \lesssim \bar{R} \ll \bar{r}$.

We therefore obtain from the density of states, from (3),

$$\rho_1 = -\frac{n}{\pi}\frac{d}{d\omega}\operatorname{Im}\sum_{n,l}(2l+1)\ln(1-\mu_{nl}) - {}$$
$$-2n^2\frac{d}{d\omega}\operatorname{Im}\sum_{n,m}\int_0 R^2\,dR\left[\ln\frac{1-\mu_{nm}{}^+(R)}{1-\mu_{nm}} + \ln\frac{1-\mu_{nm}{}^-(R)}{1-\mu_{nm}}\right], \tag{5}$$

where $\mu_{nm} = \mu_{nl} = \lim_{R\to\infty}\mu_{nm}(R)$ and $|m| \le l$.

The "tails" of the density of state fall, as will be shown below, in the region $kR \ll 1$, where (see (A.4), the book by Morse and Feshbach[4], and (4))

$$\operatorname{Re}(\mu^\pm - \mu_{nl}) \approx \pm 2m^*c_{nl}{}^+c_{nl}(4l+1)(2l+1)!!\,C_l{}^{lm}/R^{2l+1}.$$

Replacing

$$\operatorname{Im}\ln(1-\mu) = -\operatorname{arcctg}\frac{1-\operatorname{Re}\mu}{\operatorname{Im}\mu} \tag{6}$$

in the second term of (5) by

$$-\pi\vartheta(\omega - \varepsilon \pm B/R^{2l+1}), \tag{6a}$$

where

$$B = \frac{(4l+1)(2l+1)!!}{(2m^*)^l\, b}C_l{}^{lm},$$

$\vartheta(x) = 0$ at $x < 0$ and $\vartheta(x) = 1$ at $x > 0$, we obtain from (5) near a resonance with angular momentum $l$:

$$\rho_1(\omega) \approx \frac{n}{\pi}(1-n\bar{v})(2l+1)\gamma/[(\varepsilon-\omega)^2+\gamma^2] + 2n^2\pi[R(\omega)]^{2l+4}/(2l+1)B, \tag{7}$$

where

$$R(\omega) = (B/|\varepsilon-\omega|)^{1/(2l+1)},$$
$$\gamma = \frac{\operatorname{Im}\mu_{nl}}{a_{nl}} = \frac{\omega^lk}{b} \sim \omega(\omega/E_0)^{l-1/2}.$$

Since $B \sim m^*c^+c/a \sim E_0r_0^{2l+1}$ (see (4)), we can readily show that the second term in (7) is of the order of

$$\frac{n}{\gamma}\frac{n}{k^3}\left(\frac{\gamma}{\sqrt{(\varepsilon-\omega)^2+\gamma^2}}\right)^{(2l+4)/(2l+1)}$$

(it is cut off at $|\omega - \epsilon| \lesssim \gamma$ if it is recognized that it is actually necessary to use (6) rather than (6a), i.e., because $\operatorname{Im}\mu_{nm}$ is finite). In the region of the maximum of $\rho_1(|\epsilon - \omega| \lesssim \gamma)$ the first term is of the order of $n/\gamma$, and the second term is smaller by a factor $n/k^3 \sim (k\bar{r})^{-3} \ll 1$. The second term determines the tails of the density of states at $|\omega - \epsilon| \gg \gamma$ (if $l \ge 2$), but where the two quantities in (7) become comparable (at $|\omega - \epsilon| \sim \gamma(n/k^3)^{-(2l+1)/2(l-1)}$) the correction to the density of states is small (of the order of $(n/\gamma)(n/k^3)^{(2l+1)/(l-1)}$). We recall that without impurities the density of states is $\rho_0 = m^*k/2\pi^2$, and therefore $\rho_0$ is larger than the second term in (7), when it plays an essential role in $\rho_1$, if

$$\gamma/\omega \sim (\omega/E_0)^{l-1/2} \gg (n/k^3)^{3l/(l-1)}. \tag{8}$$

If (8) is satisfied, we can disregard below the higher terms of the expansion in $n/k^3$, i.e., assume that

$$\rho_1 \approx \frac{n}{\pi}(2l+1)\frac{\gamma}{(\varepsilon-\omega)^2+\gamma^2} = \frac{A}{i}\frac{1}{\varepsilon-\omega-i\gamma} + \text{c.c.} \tag{9}$$

where $A = (2\pi)^{-1}n(2l+1)$.

The result means that at low densities, the pertinent expression is the expansion of the Green's functions in the number of "collisions" of the impurity particles. In the diagram technique, the expression for G in terms of the scattering amplitudes is ($\epsilon_{ab} = 1 - \delta_{ab}$):

$$G = G_0 + \sum_a G_0a^aG_0 + \sum_{a,b}G_0a^aG_0a^bG_0\varepsilon_{ab} + \ldots$$

When the two-particle collisions were taken into account, we actually summed all the terms of the type

$$\sum_{a,b}\varepsilon_{ab}G_0(a^a + a^aG_0a^b + a^aG_0a^bG_0a^a + a^aG_0a^bG_0a^aG_0a^b + \ldots)G_0,$$

and, as can be easily seen, the expansion written out above corresponds to a series in which each succeeding term is the product of the preceding term by a quantity of the order of $(\mu - \mu_0)/(1 - \mu_0) \sim g_{ab}/(1 - \mu_0) \gg 1$ at $kR \ll 1$. After summation, all the terms make a contribution on the order of $(1 - \mu_0 - g_{ab})^{-1}$ to the amplitude. At $n/k^3 \ll 1$, the most overwhelmingly probable is $R_{ab} \sim \bar{r}$, when $|g_{ab}| \ll |\mathrm{Im}\,\mu_0|$, so that we can assume in fact that

$$G \approx G_0 + \sum G_0 a^\alpha G_0.$$

2. To determine the specific heat and the spin paramagnetism of the system in question we start, as usual, with a calculation of the thermodynamic potential $\Omega$ of the electron gas[7]. The chemical potential will be dedesignated $\mu$ in this section.

Accurate to $\exp(-\mu/T)$, we can write for $\Omega$

$$\Omega(\mu, T) = -2\int_{-\infty}^{\mu} d\omega\, \rho(\omega)(\mu - \omega) - 2T^2 \int_0^\infty dx \ln(1 + e^{-x})[\rho(\mu + Tx) + \rho(\mu - Tx)], \tag{10}$$

the first term is $\Omega(\mu, 0)$, and the second will be designated $\delta\Omega$. The state density is subdivided, as above, into $\rho_0$ and $\rho_1$, and the same subscripts will be used for all the remaining quantities.

For example,

$$\delta\Omega_0 = -4T^2\rho_0(\mu) \int_0^\infty dx \ln(1 + e^{-x}) = -\frac{\pi^2}{3}\rho_0(\mu) T^2. \tag{11}$$

We express $\delta\Omega_1$ in the form (see (9))

$$\delta\Omega_1 = 4T^2 iA \int_0^\infty \ln(1 + e^{-x}) \frac{(\Delta - i\gamma)\,dx}{(\Delta - i\gamma)^2 - T^2x^2} + \text{c.c.}$$

where $\Delta = \epsilon - \mu$; we have neglected here the dependence of A and $\gamma$ on $\omega$, putting in them $\omega = \mu$ (i.e., the calculations have a relative accuracy $T/\mu$). This integral is calculated with formula 4.319(2) of[8], where, incidentally, the result should be preceded by a minus sign. The calculation yields

$$\delta\Omega_1 = 4A\left\{2\pi T \ln\left[\left|\Gamma\left(\frac{1}{2} + \frac{\gamma + i\Delta}{2\pi T}\right)\right| \Big/ \sqrt{2\pi}\right] + \gamma - \gamma \ln(s/2\pi T) + \Delta \,\mathrm{arctg}(\Delta/\gamma)\right\}, \tag{12}$$

where $\Gamma$ is the Gamma function and $s = \Delta^2 + \gamma^2$.

Since we have a large enough reservoir of states under the peak of the density $\rho_1$ (the total number of states is $\sim n$), it is clear that, say, if $\Delta_0 = \epsilon - \mu_0 > 0$, where $\mu_0$ is the chemical potential at $T = 0$, the value of $\mu$ should decrease rapidly with increasing temperature, until all the states under the peak are filled; at $\mu_0 < \epsilon$ the electrons, to the contrary, will be crowded out from under the peak with increasing temperature, and $\mu$ will therefore increase (a general decrease of $\mu$ sets in at temperatures $T \gtrsim \mu$).

The chemical potential, as usual, is defined by

$$-N = \partial\Omega(\mu_0, 0)/\partial\mu_0 = \partial\Omega(\mu, T)/\partial\mu. \tag{13}$$

Since

$$Z_1(\mu) = 2\int_{-\infty}^{\mu} d\omega\, \rho_1(\omega) = 4A\left(\frac{\pi}{2} - \mathrm{arctg}\left(\frac{\Delta}{\gamma}\right)\right) \tag{14}$$

and

$$\partial\delta\Omega_1/\partial\mu = -\partial\delta\Omega_1/\partial\Delta = 4A(I - \mathrm{arctg}(\Delta/\gamma)),$$

where

$$I = \mathrm{Im}\,\psi\left(\frac{1}{2} + \frac{\gamma + i\Delta}{2\pi T}\right), \quad \psi(z) = \frac{d\ln\Gamma(z)}{dz},$$

condition (13) yields

$$\Delta_0 - \Delta = -\frac{\pi^2}{6}\frac{\rho_0'(\mu)}{\rho_0(\mu)}T^2 + B\left(I - \mathrm{arctg}\frac{\Delta_0}{\gamma}\right), \tag{13a}$$

where $B = 2A/\rho_0(\mu) \sim n\mu/k^3$. The first term on the right is of the order of $T^2/\mu$ and we shall disregard it, neglecting by the same token the decrease of the chemical potential with increasing temperature (we assume, of course, that $|\Delta_0| \ll \mu$).

An investigation of the equation that follows leads, as can be easily seen, to the conclusion that $|\Delta|$ changes little when $T < |\Delta_0|$, after which it increases and reaches a saturation value $\bar{\Delta} = \Delta|_{T=\infty}$ at $T \gtrsim B$ ($B \gtrsim |\Delta_0|$):

$$\bar{\Delta} = \Delta_0 + B\,\mathrm{arctg}(\Delta_0/\gamma). \tag{13b}$$

It remains now to calculate the entropy

$$S = -\frac{\partial\Omega}{\partial T} = \frac{2\pi^2}{3}\rho_0(\mu)T - 4A\left(2\pi \ln\frac{|\Gamma|}{\sqrt{2\pi}} + \frac{\Delta}{T}I - \frac{\gamma}{T}R + \frac{\gamma}{T}\right), \quad R = \mathrm{Re}\,\psi\left(\frac{1}{2} + \frac{\gamma + i\Delta}{2\pi T}\right), \tag{15}$$

and the specific heat

$$C = \frac{2\pi^2}{3}\rho_0(\mu)T + 4A\left(\gamma\frac{dR}{dT} - \Delta\frac{dI}{dT} + \frac{\gamma}{T}\right). \tag{16}$$

Since

$$\frac{dR}{dT} = \frac{I'}{2\pi T}\left(\frac{\Delta}{T} - \Delta'\right) - \frac{\gamma}{2\pi T^2}R', \quad \Delta' = \frac{d\Delta}{dT},$$

where

$$R' = \mathrm{Re}\,\psi'\left(\frac{1}{2} + \frac{\gamma + i\Delta}{2\pi T}\right), \quad I' = \mathrm{Im}\,\psi'\left(\frac{1}{2} + \frac{\gamma + i\Delta}{2\pi T}\right),$$

with $R', I' \lesssim 1$, it follows that in the most probable case $|\Delta_0| \gg \gamma$ in the region of the maximum specific heat we have $\gamma/T + \gamma dR/dT \sim \gamma/\Delta \lesssim \gamma/\Delta_0 \ll 1$, and these terms can be neglected, as will be seen later on.

We next obtain from (13a)

$$\Delta' = -B\,dI/dT, \tag{13c}$$

whence

$$C \approx \frac{2\pi^2}{3}\rho_0(\mu)T + 4A\frac{|\Delta\Delta'|}{B}. \tag{16a}$$

Since the maximum $\Delta' \sim 1$ occurs at $\Delta \sim B$, the correction to the specific heat $C_1 \sim A \sim n$ is of the same order of magnitude (formula wise) as $C_0 \sim m^*kB \sim n$ at $B \gtrsim |\Delta_0| \gg \gamma$. The width of the obtained peak is $\Delta T \sim B$. On the other hand if $B \ll |\Delta_0|$, then the height of the peak decreases.

We now study the effect of the magnetic field on the specific heat and on the Pauli paramagnetism. In the calculation of the classical part of $\Omega$ we can assume that

$$\Omega(\mu, T, H) = \tfrac{1}{2}[\Omega(\mu + \zeta, T) + \Omega(\mu - \zeta, T)],$$

where H is the magnetic field, $\zeta = \mu_0 H$, $\mu_0$ is the effective Bohr magneton, and the quantities $\Omega(\mu, T)$ on the right have already been calculated by us.

As usual, we begin with an examination of the effect of the magnetic field and of the temperature on the chemical potential. Condition (13) now leads to the expression

$$\Delta_0 - \Delta = -\rho_0'(\mu)(\pi^2T^2/3 + \zeta^2)/2\rho_0(\mu) + B[(I_+ + I_-)/2 - \mathrm{arctg}(\Delta_0/\gamma)], \tag{17}$$

where

$$I_\pm = \mathrm{Im}\,\psi\left(\frac{1}{2} + \frac{\gamma + i\Delta_\pm}{2\pi T}\right), \quad \Delta_\pm = \Delta \pm \zeta, \quad \Delta_0 = \Delta|_{T=H=0},$$

and the first term will be neglected, as before.

The correction to the specific heat is now the arithmetic mean of the corresponding quantities taken at $\Delta_+$ and $\Delta_-$ and at $\zeta \ll B$

$$C_1 \approx 2AB^{-1}(\Delta_+\Delta_+' + \Delta_-\Delta_-'),$$

i.e., the initial peak is split in a weak magnetic field. In the general case, of course, it is necessary to solve (17) numerically.

The magnetic moment of the system is

$$M=-\frac{\partial\Omega}{\partial H}=-\frac{\mu_0}{2}\left[\frac{\partial\Omega(\mu+\zeta)}{\partial\mu}-\frac{\partial\Omega(\mu-\zeta)}{\partial\mu}\right] \tag{18}$$
$$=2\mu_0\rho_0(\mu)\zeta+2\mu_0A(I_+-I_-).$$

At $T \ll |\Delta_0|$ we obtain from the asymptotic form of I[8] that Eq. (17) goes over into

$$\Delta_0-\Delta=\frac{B}{2}\left(\operatorname{arctg}\frac{\Delta_+}{\gamma}+\operatorname{arctg}\frac{\Delta_-}{\gamma}-2\operatorname{arctg}\frac{\Delta_0}{\gamma}\right). \tag{17a}$$

We see therefore that $|\Delta|$ changes little at $\zeta < |\Delta_0|$, and then increases and assumes a saturation value $\bar{\Delta} = \Delta_{\zeta=\infty}$ at $\zeta \sim B$ (if $B \gtrsim |\Delta_0|$)

$$\bar{\Delta}=\Delta_0+B\operatorname{arctg}(\Delta_0/\gamma).$$

For the magnetic susceptibility, differentiating (18) at $T \ll |\Delta_0|$, we obtain

$$\chi=\frac{\partial M}{\partial H}=2\mu_0^2\rho_0(\mu)+2\mu_0^2A\left(\frac{\gamma\, d\Delta_+/d\zeta}{\Delta_+^2+\gamma^2}-\frac{\gamma\, d\Delta_-/d\zeta}{\Delta_-^2+\gamma^2}\right), \tag{18a}$$

and if $|\Delta_0| \gg \gamma$ the first term in the parentheses can be neglected in comparison with the second, which gives a peak with a maximum at the point $\zeta_0$, at which $\Delta = \zeta$. From (17a) at $|\Delta_0| \gg \gamma$ we obtain

$$\zeta_0=\Delta_0+B\pi/4$$

and in the vicinity of the maximum ($\delta = \zeta - \zeta_0$)

$$\Delta_-=-\gamma\operatorname{tg}\frac{2(\delta+\Delta_-)}{B}\approx-\gamma\operatorname{tg}\frac{2\delta}{B}$$

(the last equality is valid at $B \gg \gamma$), so that we obtain from (18a) for $\chi_1$ the expression

$$\chi_1\approx\chi_0, \tag{18b}$$

which is suitable when $\pi/2 - 2|\delta|/B \gg (\gamma/B)^{1/2}$. It can be shown that $\partial\chi/\partial H$ has a positive peak near $\zeta = \Delta_0$ and a negative peak near $\zeta = \Delta_0 + B\pi/2$, both of height $\sim\chi_0\mu_0/\sqrt{\beta\gamma}$ and width $\Delta\zeta \sim \sqrt{B\gamma}$.

In conclusion, let us stop to discuss the experimental possibilities of observing the described effects.

The first condition is that the impurity concentration be small, $n \ll k^3$, i.e.,

$$n\ll n_e, \tag{19}$$

where $n_e$ is the electron concentration in the metal (by n is meant only the concentration of the impurity center that lead to resonance near the Fermi energy).

The next condition is determined from the fact that the resonance should be sharp enough, i.e.,

$$(\mu/E_0)^{l+1/2}\ll 1. \tag{20}$$

If it is assumed that $r_0$ is of the same order of magnitude as the Debye radius of the electrons, then $E \sim kp_0/\lambda m_0$, where $m_0$ and $p_0$ are the mass of the free electron and the Fermi momentum in the normal metal, and $\lambda$ is the dielectric constant. Hence

$$\mu/E_0\sim(n_э/n_0)^{1/3}\lambda m_0/m^* \tag{20a}$$

($n_0$ denotes here the concentration of the electrons in the normal metal), i.e., we must have either a low electron concentration or large electron effective masses $m^*$ (narrow bands).

Finally, the mean value of the potential of an impurity with charge Ze

$$\bar{V}\sim Ze^2/\lambda r_0$$

should be larger than or of the order of $E_0$, if a resonance situation is to be possible at all; whence

$$Z\gtrsim[(n_э/n_0)^{1/3}\lambda m_0/m^*]^{1/2},$$

and by virtue of (20) and (20a) the right-hand side here is smaller than unity. The main complication lies therefore in the choice of impurities with resonance $|\mu - \epsilon| \ll \mu$ in a medium with low anisotropy.

Phenomena very similar to those considered by us were observed in graphite[9]. The resonance decay can be attributed to the fact that the spectrum of the graphite has almost rotational symmetry about a threefold axis. An analysis similar to that used in the derivation of (4) shows that in the case of axial symmetry at a moment m we have $\operatorname{Im}\mu_m \propto k^{2m+1}$ (for the eigenfunction $\psi(\rho, z) = \psi(\rho, -z)$, for which $\psi(\rho, p_z) \neq 0$ at $p_z = 0$) or even smaller. Since there is no exact axial symmetry in graphite, decay via the channel with $m = 0$ predominates as $k \to 0$ ($\operatorname{Im}\mu \propto k$). At higher energies, however, however, decay with nonzero m can become predominant.

All the formulas of Sec. 2 are valid in the anisotropic case; it is necessary only to substitute in them the correct values of $\rho_0$ and A (which is now expressed in the form (9)). It should be noted, however, that in the model considered here the positions of the maxima on the specific-heat and paramagnetic-susceptibility curves depend on the value of B, i.e., on the impurity concentration. This dependence ceases to be important at $B \lesssim \Delta_0$, the quantity $(C_1/C_0)_{\max}$ decreases with decreasing B.

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## APPENDIX

We calculate here the eigenvalues $\mu_{nm}$ for a potential made up of two identical spherically-symmetrical potentials located at a distance R from each other. Equation (1) is rewritten in the form

$$\mu\psi=(v_1+v_2)G\psi.$$

If we express $\psi$ as

$$\psi=\psi_1+\psi_2,$$

then

$$\mu\psi_1=v_1G\psi_1+v_1G\psi_2,$$
$$\mu\psi_2=v_2G\psi_1+v_2G\psi_2.$$

The equation for $\psi_1$ is therefore

$$\psi_1=a_1(v_1/\mu)Ga_2(v_2/\mu)G\psi_1, \tag{A.1}$$

where the operator $a(v/\mu)$ in a field $v/\mu$ is given by

$$\mu a=v+vGa.$$

It is clear that

$$a\left(\frac{v}{\mu}\right)=\sum\frac{\psi_\alpha\chi_\alpha^+}{\mu-\mu_\alpha},$$

where $\psi_\alpha$, $\chi_\alpha^+$, and $\mu_\alpha$ are the corresponding eigenfunctions and eigenvalues of the nucleus vG. Expanding $\psi_1$ in the eigenfunctions $\psi_\alpha^{(1)}$ of the kernel $v_1G$ and taking the scalar product of (A.1) and $X_\alpha^{+(1)}$ we obtain ($\psi_1 = \Sigma c_\alpha\psi_\alpha^{(1)}$):

$$c_\alpha=\sum_{\gamma,\beta}\frac{g_{\gamma(2)}^{\alpha(1)}\,g_{\beta(1)}^{\gamma(2)}}{(\mu-\mu_\alpha)(\mu-\mu_\gamma)}c_\beta, \tag{A.2}$$

where

$$g_{\beta(j)}^{\alpha(i)} = (\chi_\alpha^{+(i)} G \psi_\beta^{(j)}),$$

and $\alpha = \{nlm\}$. If the z axis is chosen along **R**, the matrices g are obviously diagonal in the momentum projections m. They are easily calculated if $\exp(R/r_0) \gg 1$:

$$g_{n'l'm(j)}^{nlm(i)} = -2m^* ik\, i^{l-l'} \chi_{nl}^+(k) \psi_{n'l'}(k) \times 4\pi \sum_{\lambda=|l-l'|}^{l+l'} i^\lambda h_\lambda^{(1)}(kR) Y_{\lambda 0}^* \left(\frac{\mathbf{R}_{ij}}{R_{ij}}\right) (Y_{\lambda 0})_{l'm}^{lm}, \tag{A.3}$$

where $\mathbf{R}_{ij} = \mathbf{R}_i - \mathbf{R}_j$, $h_\lambda^{(1)}$ are spherical Hankel functions[4], $(Y_{\lambda 0})_{l'm'}^{lm}$, is a matrix element of $Y_{\lambda 0}$ in terms of the spherical harmonics $Y_{lm}$ and $Y_{l'm_l}$ and is expressed in terms of 3j-symbols[6], and

$$f_l(k) = \int_0^\infty f(r) j_l(kr) r^2\, dr,$$

where $j_l(z)$ are spherical Bessel functions[4].

Therefore, if we seek $\mu(R)$, so that $\mu \to \mu_{nl}$ at $R = |\mathbf{R}_{ij}| \to \infty$, then we obtain for Re $(\mu - \mu_{nl})$, retaining in (A.2) only the diagonal terms:

$$\mathrm{Re}(\mu_{nm}^{\pm} - \mu_{nl}) = \pm 2m^* k \chi_{nl}^+(k) \psi_{nl}(k) (2l+1) \sum_{\lambda=0}^{l} (4\lambda+1) n_{2\lambda}(kR) C_\lambda^{lm}, \tag{A.4}$$

where

$$C_\lambda^{lm} = \begin{pmatrix} l & 2\lambda & l \\ -m & 0 & m \end{pmatrix} \begin{pmatrix} l & 2\lambda & l \\ 0 & 0 & 0 \end{pmatrix}^2,$$

the expressions in the parentheses are 3j-symbols[6], and in $n_l(z)$ is a spherical Neumann function[4],

From the expansion of j(z) at small values of z it follows that if $kr_0 \ll 1$, then

$$\psi_{nl}(k) \approx c_{nl} k^l, \qquad \chi_{nl}^+(k) \approx c_{nl}^+ k^l,$$
$$c_{nl} = [(2l+1)!!]^{-1} \int_0^\infty \psi_{nl}(r) r^{l+2}\, dr. \tag{A.5}$$

To calculate the damping it is necessary in general to take into account also the off-diagonal terms in (A.2), but we do not need the damping; we note only that at $kR \gg 1$ we have $\mathrm{Im}\, \mu(R) \approx \mathrm{Im}\, \mu_{nl}$.

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