Perturbation of a magnetized plasma by a rapidly moving charge

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(Submitted July 25, 1972)
Zh. Eksp. Teor. Fiz. **64**, 146-152 (January 1973)

The flow of a plasma located in a strong external magnetic field $\mathbf{B}_0$ around a weak charge is considered. The velocity of the charge $\mathbf{V}_0$ is much greater than the ion thermal velocity but much smaller than the electron thermal velocity and is inclined to the field. Distributions of the electric potential and perturbations of the electron and ion concentrations δN along the z axis perpendicular to the $(\mathbf{V}_0, \mathbf{B}_0)$ plane and passing through the charge and also distributions for large and small distances from the charge in the $(\mathbf{V}_0, \mathbf{B}_0)$ plane are obtained with quadratic accuracy with respect to a small dimensionless charge. The angular distribution near the symmetry plane $z = 0$ has a petal-like structure characteristic of supersonic streamlining. In contrast to the case of an unmagnetized plasma, now $\delta N_e \neq \delta N_i$. At distances greatly exceeding the ion Larmor radius L_i ($|z| \gg L_i$, $\rho \gg L_i V_0/v_{Ti}$) the results hold for any value of the magnetic field.

1. INTRODUCTION

The problem of the perturbation of tenuous plasma in a magnetic field of arbitrary intensity, either by a rapidly moving large body with dimension $R_0 \gg D$, or by a point-like charge $R_0 \lessdot e|Q|/T \ll D$ ($-e$ is the electron charge, Q is the charge of the body, T is the temperature, and D is the Debye radius of the plasma) was considered by Pitaevskiĭ [1] (see also [2]). He obtained the Fourier components of the electric potential of the plasma (φ_q) and of the perturbations of the particle concentration (δN_q). The inverse Fourier transformation reduces to a calculation of very complicated multiple integrals. Even in the limit of large distances, all that could be obtained were expressions averaged over the spatial coordinate z perpendicular to the $(\mathbf{V}_0, \mathbf{B}_0)$ plane (see[3]).

Perturbation of a plasma with a strong magnetic field, when the Larmor radius of the ion is much less than the characteristic dimension of the problem, $L_i \equiv v_{Ti}/\omega_{Bi} \ll R_0$, was considered by one of the authors[4] for the case of flow around a large body ($R_0 \gg D$). The perturbations of the particle densities and of the electric and magnetic fields in the wake of the body were calculated on the basis of the drift kinetic equations.

The drift equations constitute the zeroth approximation in the expansion of the kinetic equation in the small parameter $\sim 1/B_0$ (see, e.g., [4,5]). When such an expansion is used, the formulas of Pitaevskiĭ's paper[1] become much simpler, and it becomes possible to obtain exact (not averaged) expressions for the asymptotic forms.

In the present article we use the drift equations to solve the problem of stationary flow around a point charge, when $L_i \ll D$. Assuming the charge to be weak

$$|Q'| = \frac{e|Q|}{TD}\frac{v_{Ti}}{V_0} \ll \frac{v_{Ti}}{V_0}$$

we calculate by the perturbation method the Fourier components of the plasma parameters. The Fourier components of the linear approximation can be obtained formally from Pitaevskiĭ's formulas[1] by substituting $B_0 = \infty$, while the quadratic approximation for the case of a strong magnetic field was never calculated before. From the obtained Fourier components we calculate analytically the perturbations of the particle densities and of the field potential for the limiting cases of large and small distances. At very large distances it is necessary to take into account the approximation quadratic in the charge, since the linear expressions fall off more rapidly[1].

2. THE EQUATIONS

Assuming $v_{Te} \gg V_0$, we obtain for the distribution function of the electrons in the coordinate system connected to the charge, with accuracy $\sim V_0/v_{Te}$, the Maxwell-Boltzmann formula

$$f_e(v_\parallel) = N_0\left(\frac{m_e}{2\pi T}\right)^{1/2}\exp\left\{-\frac{m_e}{2T}v_\parallel^2 + \frac{e\varphi}{T}\right\}. \quad (1)$$

The drift kinetic equation for the ion distribution function is

$$(\mathbf{v}_\parallel - \mathbf{V}_{0\perp})\frac{\partial f_i}{\partial \mathbf{r}} - \frac{e}{m_i}\frac{\partial \varphi}{\partial r_\parallel}\frac{\partial f_i}{\partial v_\parallel} = 0. \quad (2)$$

The elliptic potential of the plasma is determined in terms of these distribution functions from the Poisson equation

$$\Delta\varphi = -4\pi e\int_{-\infty}^{\infty}(f_i - f_e)\,dv_\parallel - 4\pi Q\delta^3(r). \quad (3)$$

It is necessary to add to (2) and (3) also the boundary conditions that express the fact that the plasma is not perturbed at infinity:

$$f_i|_\infty \equiv f_i^0 = N_0(m_i/2\pi T)^{1/2} \times \exp\{-m_i(v_\parallel + V_{0\parallel})^2/2T\}, \quad \varphi|_\infty = 0. \quad (4)$$

Here N_0 is the unperturbed particle concentration, m_a is the mass of particles of type a, φ is the electric potential of the plasma, $\mathbf{v}_\parallel$ is the velocity component parallel to the field $\mathbf{B}_0$, and $\mathbf{V}_{0\perp}$ is the component perpendicular to $\mathbf{B}_0$.

Representing the particle distribution in the form $f = f^0 + \delta f$ and changing over to the Fourier transform

$$\delta f_\mathbf{q} = \int d^3r e^{-i\mathbf{q}\mathbf{r}}\delta f(\mathbf{r}), \quad (5)$$

we can write the kinetic equation in the form

$$i(q_\parallel v_\parallel - \mathbf{q}_\perp \mathbf{V}_{0\perp})\delta f_{i,\mathbf{q}} + \frac{e\varphi_\mathbf{q}}{T}iq_\parallel(v_\parallel + V_{0\parallel})f_i^0 = I_\mathbf{q}(v_\parallel),$$
$$I_\mathbf{q}(v_\parallel) = \frac{e}{m}\int d^3r e^{-i\mathbf{q}\mathbf{r}}\frac{\partial\varphi}{\partial r_\parallel}\frac{\partial \delta f_i}{\partial v_\parallel}. \quad (6)$$

Assuming the charge to be sufficiently weak, we seek the solution in the form of an expansion in powers of Q.

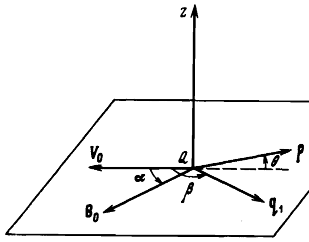

FIG. 1.

3. SOLUTION. LINEAR APPROXIMATION

Elementary calculations yield

$$\varphi_{\mathbf{q}}^{(1)}=\frac{4\pi QD^2}{\Delta_{\mathbf{q}}},$$
$$f_{i,\mathbf{q}}^{(1)}=-\frac{e}{T}\frac{4\pi QD^2}{\Delta_{\mathbf{q}}}\frac{q_\parallel(v_\parallel+V_{0\parallel})}{q_\parallel v_\parallel-\mathbf{q}\mathbf{V}_{0\perp}-i0}f_i^0 \tag{7}$$
$$N_{i,\mathbf{q}}^{(1)}\equiv\int f_{i,\mathbf{q}}^{(1)}dv_\parallel=-N_0\frac{e}{T}\frac{4\pi QD^2}{\Delta_q}K\left(\frac{\mathbf{q}\mathbf{V}_0}{|q_\parallel|v_{Ti}}\right)$$

Here

$$\Delta_{\mathbf{q}}=D^2q^2+\Delta_0,\quad \Delta_0=1+K(\mathbf{q}\mathbf{V}_0/|q_\parallel|v_{Ti}),$$
$$D=(T/4\pi e^2N_0)^{1/2},\quad v_{Ti}=(2T/m_i)^{1/2}, \tag{8}$$
$$K(z)=\frac{\sqrt{\pi}}{2i}w'(z),\ w(z)=\frac{2i}{\sqrt{\pi}}e^{-z^2}\int\limits_{i\infty}^{z}e^{t^2}dt.$$

The Kramp function $w(z)$ has been tabulated (see[6]). The function $K(z)$ has neither zeroes nor singularities in the upper half-plane.

The sought values of the potential and of the densities are given by the inverse Fourier transforms of (7). We consider a cylindrical coordinate system with z axis perpendicular to the $(\mathbf{V}_0, \mathbf{B}_0)$ plane and we measure all the angles in this plane in the positive direction (see Fig. 1). The argument of the function Δ_0 in (7) does not depend on q_z; therefore, integrating with respect to dq_z, we obtain

$$\varphi^{(1)}=\frac{Q}{2\pi D}\int\limits_0^\infty q_1\,dq_1\int\limits_0^{2\pi}\frac{d\beta}{(q_1^2+\Delta_0)^{1/2}}\exp\left\{-\frac{|z|}{D}(q_1^2+\Delta_0)^{1/2}\right.$$
$$\left.-iq_1\frac{\rho}{D}\cos(\beta-\theta)\right\}, \tag{9}$$
$$\Delta_0\equiv\Delta_0\left(\frac{V_0}{v_{Ti}}\frac{\cos\beta}{|\cos(\beta-\alpha)|}\right).$$

Here q_1 and ρ are the projections of $D\mathbf{q}$ and $\mathbf{r}$ on the $(\mathbf{V}_0, \mathbf{B}_0)$ plane, respectively.

Solution on z axis. The integral (9) can be easily calculated at $\rho = 0$ (on the z axis). In fact, replacing the variable β by $-\cos(\beta+\alpha)/\cos\beta = t$ and closing the contour of the integration with respect to t in the upper half-plane, we can evaluate the integral with respect to $d\beta$. The integration with respect to dq_1 is elementary. As a result we get

$$\varphi^{(1)}=\frac{Q}{|z|}\operatorname{Re}\exp\left\{-\frac{|z|}{D}\left[1+K\left(\frac{V_0}{v_{Ti}}e^{i\alpha}\right)\right]^{1/2}\right\},$$
$$N_i^{(1)}=-N_0\frac{eQ}{T|z|}\operatorname{Re}\left\{K\left(\frac{V_0}{v_{Ti}}e^{i\alpha}\right)\exp\left(-\frac{|z|}{D}\left[1+K\left(\frac{V_0}{v_{Ti}}e^{i\alpha}\right)\right]^{1/2}\right)\right\},$$
$$N_e^{(1)}=e\varphi^{(1)}N_0/T. \tag{10}$$

It is seen from (10) that the potential and the density perturbations oscillate, generally speaking, along the z axis. For the case $V_0/V_{Ti} \gg 1$ (as well as for the case $V_0/v_{Ti} \ll 1$), the functions $\varphi^{(1)}$ and $N^{(1)}$ attenuate with increasing z long before these oscillations appear.

Near zone $(e|Q|/T \ll r \ll D)$. In this case we can put $q_1^2 \gg \Delta_0$ under the radical sign in (9), and therefore

$$\varphi^{(1)}\cong\frac{Q}{2\pi}\int\limits_0^{2\pi}d\beta\frac{1}{|z|+i\rho\cos\beta}=\frac{Q}{r}, \tag{11}$$

$$N_i^{(1)}\cong-N_0\frac{eQ}{T}\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{d\beta}{|z|+i\rho\cos(\beta+\alpha-\theta)}K\left(\frac{V_0}{v_{Ti}}\frac{\cos(\beta+\alpha)}{|\cos\beta|}\right). \tag{12}$$

The integral in (12) can be calculated for $V_0/v_{Ti} \gg 1$ and $z \to 0$ (near the symmetry plane $(\mathbf{V}_0, \mathbf{B}_0)$):

$$N_i^{(1)}\cong-2N_0\frac{eQ}{T\rho}h(\sin(\alpha-\theta))\operatorname{Re}K\left(\frac{V_0}{v_{Ti}}\frac{\sin\theta}{\sin(\alpha-\theta)}\right),$$
$$h(x)=\begin{cases}1, & x>0,\\ 0, & x<0.\end{cases} \tag{13}$$

Since $e|Q|/TD \ll 1$, we have the region of applicability of the obtained solutions

$$r\gg e|Q|/T. \tag{14}$$

Far zone $(DV_0/v_{Ti} \ll \rho < DV_0/v_{Ti}|Q^*|\ln|1/Q^*|)$. When the parameter ρ/D in the argument of the exponential in (9) is large enough, we can assume $q_1^2 \ll \Delta_0$ in the expression under the radical sign. We then have for $(|z|D)^{1/2} \gg \rho \gg DV_0/v_{Ti}$ again formulas (10), and for $\rho \gg DV_0/v_{Ti}$, $\rho \gg (|z|D)^{1/2}$ we get

$$\begin{bmatrix}\varphi^{(1)}\\ N_i^{(1)}\end{bmatrix}\cong-Q^*\left(\frac{DV_0}{\rho v_{Ti}}\right)^2\begin{bmatrix}T\Phi_1/e\\ N_0F_1\end{bmatrix}$$
$$\begin{bmatrix}\Phi_1\\ F_1\end{bmatrix}=h(\sin(\alpha-\theta))\frac{\sin\alpha}{\sin^2(\alpha-\theta)}$$
$$\times 2\operatorname{Im}\frac{d}{dt}\left\{\begin{bmatrix}-1\\ K(t)\end{bmatrix}\frac{\exp(-|z|[1+K(t)]^{1/2}/D)}{[1+K(t)]^{1/2}}\right\}, \tag{15}$$
$$t=-\frac{V_0}{v_{Ti}}\frac{\sin\theta}{\sin(\alpha-\theta)}.$$

4. SOLUTION. QUADRATIC APPROXIMATION

The quadratic approximation is significant only in the far zone $\rho > DV_0/v_{Ti}|Q^*|\ln|1/Q^*|$, where it is first comparable with the linear approximation, and then begins to exceed it. Taking (14) into account for

$$V_0/v_{Ti}\gg1,\ q_1\ll(DV_0/v_{Ti})^{-1},\ |q_z|\ll(|Q^*|D)^{-1}$$

we get from (6)

$$\varphi_{\mathbf{q}}^{(2)}=-\frac{T}{e}Q^{*2}\ln\left|\frac{1}{Q^*}\right|\pi D^2\frac{V_0}{v_{Ti}}\sin^2\alpha\,K'\left(\frac{\mathbf{q}\mathbf{V}_0}{|q_\parallel|v_{Ti}}\right)\frac{1}{i|q_\parallel|\Delta_{\mathbf{q}}},$$
$$N_{i,\mathbf{q}}^{(2)}=N_0(D^2q^2+1)e\varphi_{\mathbf{q}}^{(2)}/T. \tag{16}$$

Integrating, in analogy with the procedure described above, we finally obtain for $\rho \gg DV_0/v_{Ti}$, $\rho \gg (|z|D)^{1/2}$, $|z| \gg |Q^*|D$:

$$\begin{bmatrix}\varphi^{(2)}\\ N_i^{(2)}\end{bmatrix}\cong Q^{*2}\ln\left|\frac{1}{Q^*}\right|\frac{DV_0}{\rho v_{Ti}}\begin{bmatrix}T\Phi_2/e\\ N_0F_2\end{bmatrix},$$
$$\begin{bmatrix}\Phi_2\\ F_2\end{bmatrix}=h(\sin(\alpha-\theta))\frac{\sin^2\alpha}{2\sin(\alpha-\theta)}$$
$$\times\operatorname{Im}\left\{\begin{bmatrix}-1\\ K(t)\end{bmatrix}\frac{\exp(-|z|[1+K(t)]^{1/2}/D)}{[1+K(t)]^{1/2}}K'(t)\right\}, \tag{17}$$
$$N_e^{(2)}=N_0\frac{e\varphi^{(2)}}{T},\quad t=-\frac{V_0}{v_{Ti}}\frac{\sin\theta}{\sin(\alpha-\theta)}.$$

Generally speaking, the inverse of the Fourier component $N_{i,\mathbf{q}}^{(2)}$ (16) contains in addition to (7) also a term $\sim\delta(z/D)$. A more rigorous calculation [Eq. (16) does not hold for large values of q_z] leads to a smeared-out δ-function in $N_i^{(2)}$ of the order of $1/Q^*$ for $|z|/D \lesssim |Q^*|$. At such values of z, the assumptions of perturbation theory are violated ($N_i^{(2)}$ turns out to be close to Q^*), and it is necessary to solve a nonlinear problem. This is physically understandable: at small distances r from the charge, the Coulomb potential Q/r exceeds the characteristic plasma potential T/e; the perturbation cannot be regarded as small. Since the plasma particle trajectories far from the charge in a strong magnetic field lie in a plane $z = \text{const}$, the strong perturbation at small ρ and z will change the flow picture at the same values $|z| \lesssim |Q^*| D$ for all ρ.

5. PLOTS AND DISCUSSION OF RESULTS

Figure 2a shows the angular dependence of the linear-approximation quantities in the plane $|z| \sim |Q^*| D$ in the far zone. We consider the case $\alpha = 45°$. The curves are slightly asymmetrical. Near the axis directed along $-V_0$ there is a maximum that is positive for both $\varphi^{(1)}$ and $N_i^{(1)}$. On the two sides of the axis there are

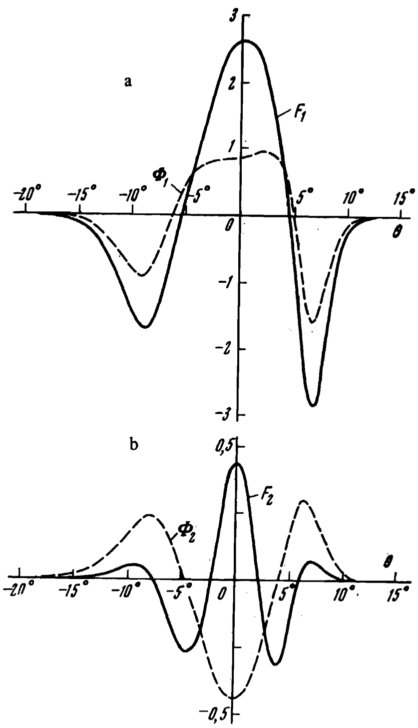

FIG. 2. Angular dependences of the perturbation of the ion density and of the potential in the far zone ($\rho \gg DV_0/v_{Ti}, \sim Q^*D$, $a = 45°$, $V_0/v_{Ti} = 8$).

a) linear approximation

$$F_1 = -N_i^{(1)}/N_0 Q^* (DV_0/\rho v_{Ti})^2, \qquad \Phi_1 = -\varphi^{(1)} \Big/ \frac{T}{e} Q^* \left(\frac{DV_0}{\rho v_{Ti}}\right)^2;$$

b) quadratic approximation

$$F_2 = N_i^{(2)} \Big/ N_0 \frac{DV_0}{\rho v_{Ti}} Q^{*2} \ln \left|\frac{1}{Q^*}\right|, \quad \Phi_2 = \varphi^{(2)} \Big/ \frac{T}{e} \frac{DV_0}{\rho v_{Ti}} Q^{*2} \ln \left|\frac{1}{Q_*}\right|$$

negative minima. The entire perturbation is concentrated in a narrow angle range near the $-V_0$ axis, as is typical of supersonic flow.

In the second approximation (Fig. 2b), the extremal values of the perturbation of the ion density and of the potential (electron density) have opposite signs near the axis. It is interesting that in general the densities become equalized in our problem only in the zeroth approximation. The reason is that the decrease of the perturbations towards the z axis is exponential. The curves of Figs. 2a and 2b for the potential are close to those given by Vas'kov[3]. The point is that the Fourier components at $q_z = 0$ (averaged with respect to z)[3] or $|q_z| \ll L_i^{-1}$ and $q_\perp \ll (L_i V_0/v_{Ti})^{-1}$ (large distances) coincide in the case of arbitrary B_0[1] with the expressions for the strong field ($L_i \ll D$). The results of the averaging, however, must be approached with caution. Thus, Vas'kov[3] obtained an (average) perturbation of the ion density equal to the electron density, whereas the local values of δN should be unequal. Moreover, when averaging over z it turns out that the term with the δ-function (see Sec. 4) is decisive. Consequently, in the case of a strong field the averaged perturbation of the ion density can give a qualitative picture only near the plane $z = 0$ (if it is assumed that the perturbation method is applicable for such z), and ceases to hold true starting with small distances $|z| \gtrsim |Q^*| D$.

In conclusion, the authors are grateful to Ya. L. Al'pert for a discussion and to G. D. Komleva and L. V. Lisina for help with the numerical calculations.

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Translated by J. G. Adashko
14