PROPAGATION OF A RAYLEIGH WAVE ALONG A ROUGH SURFACE

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Submitted June 20, 1972

Zh. Eksp. Teor. Fiz. **63**, 2297–2303 (December, 1972)

Attenuation of a Rayleigh wave due to scattering by disordered surface irregularities is associated with the average characteristics of the surface. Scattering processes involving the production of secondary surface and body waves yield additive contributions to the attenuation. A frequency range of the Rayleigh wave exists in which attenuation involves mainly surface wave production and the surface "traps" sound oscillations because of a spectral gap that separates the Rayleigh wave from body waves.

IT is well-known that a Rayleigh surface wave can propagate in an elastic medium in addition to body sound waves.[1] In the case of an ideally smooth boundary between the material and a vacuum this wave can be characterized by a two-dimensional wave vector **p** that is parallel to the surface of the medium. The wave vector and frequency are related by $\omega = c_t p\zeta$, where the number ζ depends on the velocity ratio $k = c_l/c_t$ of longitudinal and transverse sound and lies within the interval 0.87–0.96 for various bodies.

Since no real surface is ideally smooth, a surface wave will be scattered by individual irregularities. Multiple scattering processes result in attenuation of the initial wave and in the production of secondary surface or body waves having the same frequency ω but a different wave vector.

Steg and Clemens[2] have studied Rayleigh wave scattering by surface defects in connection with American experiments on the Moon. These authors found that surface vibrations continue for a long time without evidence of a transformation into body vibrations. Presenting the hypothesis that a rough surface can be regarded as an aggregate of surface vacancies, they utilized a method that is known from the theory of oscillations of a crystal with local defects.

In the present work the attenuation of a Rayleigh wave scattered by disordered irregularities is calculated for the case when the wavelength is large compared with the average size of an irregularity. In this limit a boundary condition, zero pressure on a rough surface, can apply to an "average" surface that is assumed to be planar. The true surface is regarded as having a random contour, and the attenuation is expressed in terms of its average characteristics. A similar method has previously been used to study the propagation of electromagnetic waves in waveguides,[3] the broadening of electron levels in a film,[4] and the damping of surface magnetic levels in a normal metal[5] and in a superconductor.[6]

1. We now write the equations and a boundary condition that are satisfied by the displacement vector $u_\alpha^\omega(\mathbf{r})$ in a surface wave of frequency ω. In matrix form the vibrational equations are

$$Q_{\alpha\beta}u_\beta\ (\mathbf{r}) = 0, \quad \alpha, \beta = x, y, z,$$
$$Q_{\alpha\beta} = \delta_{\alpha\beta}(\omega^2/c_t^2 + \partial_l\partial_l) + (k^2 - 1)\partial_\alpha\partial_\beta, \tag{1}$$

where $\partial_\alpha \equiv \partial/\partial r_\alpha$ designates differentiation with respect to the coordinates.

Utilizing the known relation between the stress tensor $\sigma_{\alpha\beta}$ and the displacement, the condition of zero surface pressure, $\sigma_{\alpha\beta}n_\beta = 0$, can be written as

$$\Pi_{\alpha\beta}u_\beta^\omega = 0|_{z=\xi(s)}. \tag{2}$$
$$\Pi_{\alpha\beta} = \delta_{\alpha\beta}n_l\partial_l + n_\beta\partial_\alpha + (k^2 - 2)n_\alpha\partial_\beta.$$

This condition can be fulfilled at each point of the surface, which is represented by the equation (see Fig. 1)

$$z = \xi(\mathbf{s}), \quad \mathbf{s} = (x, y). \tag{3}$$

The direction of the normal to the surface is determined from (3):

$$\mathbf{n} \sim (-\partial_x\xi, -\partial_y\xi, 1). \tag{4}$$

We shall assume that $\xi(\mathbf{s})$ is small compared with the characteristic distance λ for the change of u_α^ω in the z direction (the Rayleigh wave is damped below the surface within a distance of the order $\lambda \sim 1/p$), and we expand the left side of (2) in terms of ξ. To second order terms we obtain

$$(\Pi^{(0)} + \Pi^{(1)} + \Pi^{(2)})_{\alpha\beta}u_\beta^\omega = 0|_{z=0};$$
$$\Pi_{\alpha\beta}^{(0)} = \delta_{\alpha\beta}\partial_z + \delta_{\beta z}\partial_\alpha + (k^2 - 2)\delta_{\alpha z}\partial_\beta, \tag{5}$$
$$\Pi_{\alpha\beta}^{(1)} = -\delta_{\alpha\beta}\partial_l\xi\partial_l - \partial_\beta\xi\partial_\alpha - (k^2 - 2)\partial_\alpha\xi\partial_\beta + \xi\Pi_{\alpha\beta}^{(0)}\partial_z,$$
$$\Pi_{\alpha\beta}^{(2)} = \xi\Pi_{\alpha\beta}^{(1)}\partial_z - {}^1/_2\xi^2\Pi_{\alpha\beta}^{(0)}\partial_z\partial_z.$$

The quantities depending on the coordinate **s** are expressed as Fourier transforms:

$$u_\alpha^\omega(\mathbf{r}) = \int\frac{dp}{(2\pi)^2}e^{i\mathbf{ps}}u_\alpha^\omega(\mathbf{p}, z),$$

$$\xi(\mathbf{s}) = \int\frac{dp}{(2\pi)^2}e^{i\mathbf{ps}}\xi(\mathbf{p}).$$

The equations (1) are reduced to a system of ordinary second order differential equations:

$$Q_{\alpha\beta}(\mathbf{p})u_\beta^\omega(\mathbf{p}, z) = 0 \tag{1'}$$

with the boundary condition

$$\Pi_{\alpha\beta}^{(0)}(\mathbf{p})u_\beta^\omega(\mathbf{p}, 0) + \int\frac{dq}{(2\pi)^2}[\Pi^{(1)}(\mathbf{p}, \mathbf{q}) + \Pi^{(2)}(\mathbf{p}, \mathbf{q})]_{\alpha\beta}u_\beta^\omega(\mathbf{q}, 0) = 0. \tag{5'}$$

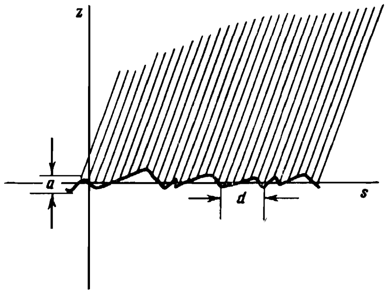

FIG. 1

The matrices Q and Π remain as differential operators with respect to z; the zero argument of the displacement $u^{\omega}_{\beta\gamma}(\mathbf{p}, 0)$ indicates that this quantity is taken at z = 0 following the differentiation. The boundary condition (5′) contains second and third derivatives which can be eliminated by the known procedure using (1′); this operation will not be shown here explicitly.

2. It is now our problem to obtain a solution of (1′) that satisfies the boundary condition (5′) and that approaches zero as $z \to \infty$. This solution depends on the random surface contour, $\xi(\mathbf{s})$, which determines $\Pi^{(1)}$ and $\Pi^{(2)}$ in (5). For this reason the derived solution must be averaged over some distribution of random functions $\xi(\mathbf{s})$. The distribution is not required in an explicit form, because the result obtained below is valid only in second order with respect to ξ and is expressed in terms of the binary correlation function

$$\langle \xi(\mathbf{s})\xi(\mathbf{s}')\rangle = \xi_2(\mathbf{s}-\mathbf{s}'). \tag{5b}$$

This function is a characteristic surface that depends on the difference $\mathbf{s}-\mathbf{s}'$, because average uniformity of the surface is assumed. In our calculations we shall assume that we know the values of $\xi_2(0) \sim a^2$ and the radius d of the region where ξ_2 differs appreciably from zero; the meanings of the parameters a and d are understood from Fig. 1.

Three linearly independent solutions of (1) are given by

$$u^{\omega}_{\beta\gamma}(\mathbf{p}, z) = u^{\omega}_{\beta\gamma}(p, 0)\exp(ip_z^{(\gamma)}z), \qquad p_z^{(\gamma)} = (\omega^2/c_\gamma^2 - p^2)^{1/2}. \tag{6}$$

The index γ that specifies the solutions can be x, y, or z, so that $c_x = c_y = c_t$, $c_z = c_l$.

In (6) we take the value of the root for which $\operatorname{Im} p_z^{(\gamma)} > 0$; consequently $u_{\beta\gamma}(\mathbf{p}, z) \to 0$ as $z \to \infty$. In order to ensure this in the case of $\omega^2/c_l^2 - p^2 > 0$ we assume an infinitesimal addition, with a positive imaginary part, to the frequency ω.

Upon writing out the matrix $Q_{\alpha\beta}$ we find that $u^{\omega}_{\beta\gamma}(\mathbf{p}, 0)$ can be represented by

$$u^{\omega}_{\beta\gamma}(\mathbf{p}, 0) = \begin{vmatrix} p_z^{(t)} & p_y & p_x \\ 0 & -p_x & p_y \\ -p_x & 0 & p_z^{(l)} \end{vmatrix}. \tag{7}$$

We note that

$$p_\beta^{(\gamma)} u^{\omega}_{\beta\gamma}(\mathbf{p}, 0) = 0 \text{ for } \gamma = x, y, \quad u^{\omega}_{\beta z}(\mathbf{p}, 0) = p_\beta^{(l)} \text{ for } \gamma = z. \tag{8}$$

Here and hereinafter the index γ of the solution $p_z^{(\gamma)}$ is not subject to summation when repeated and is to be dropped from the components p_x and p_y.

The general solution of (1) can be written as

$$u_\alpha^{\omega}(\mathbf{p}, z) = \sum_\gamma u^{\omega}_{\alpha\gamma}(\mathbf{p}, z)C_\gamma(\mathbf{p}).$$

The vector **C** is determined from the boundary condition (5′):

$$\Pi^{(0)}(\mathbf{p})u^{\omega}(\mathbf{p}, 0)C(\mathbf{p}) + \int \frac{dq}{(2\pi)^2}[\Pi^{(1)}(\mathbf{p}, \mathbf{q}) + \Pi^{(2)}(\mathbf{p}, \mathbf{q})] \times u^{\omega}(\mathbf{q}, 0)C(q) = 0, \tag{5″}$$

where the matrix indices are omitted if a misunderstanding will not result.

Using (5), (7), and (8), we obtain

$$\Pi^{(0)}_{\alpha\beta}(\mathbf{p})u^{\omega}_{\beta\gamma}(\mathbf{p}, 0) = i[p_z^{(\gamma)}u^{\omega}_{\alpha\gamma}(\mathbf{p}, 0) + p_\alpha^{(\gamma)}u^{\omega}_{z\gamma}(\mathbf{p}, 0) + (k^2-2)\delta_{\alpha z}p_\beta^{(\gamma)}u^{\omega}_{\beta\gamma}(\mathbf{p}, 0)] = i[p_z^{(\gamma)}u^{\omega}_{\alpha\gamma}(\mathbf{p}, 0) + p_\alpha^{(\gamma)}u^{\omega}_{z\gamma}(\mathbf{p}, 0) + \delta_{\alpha z}\delta_{\gamma z}(-2p_z^{(l)2} + p_z^{(t)2} - p^2)]. \tag{9.1}$$

The last equality takes account of the fact that

$$c_t^2(p^2 + p_z^{(t)2}) = \omega^2 = c_l^2(p^2 + p_z^{(l)2}).$$

We obtain, similarly,

$$\Pi^{(1)}_{\alpha\beta}(\mathbf{p}, \mathbf{q})u^{\omega}_{\beta\gamma}(\mathbf{q}, 0) = \xi(\mathbf{p}-\mathbf{q})[u^{\omega}_{\alpha\gamma}(\mathbf{q}, 0)(\mathbf{pq} - q^2 - q_z^{(\gamma)2}) + q_\alpha^{(\gamma)}(p_{s\beta} - q_\beta^{(\gamma)})u^{\omega}_{\beta\gamma}(q, 0) + \delta_{\gamma z}(p_{s\alpha} - q_\alpha^{(l)})(-q^2 - 2q_z^{(l)2} + q_z^{(t)2})], \tag{9.2}$$

$$\Pi^{(2)}_{\alpha\beta}(\mathbf{p}, \mathbf{q})u^{\omega}_{\beta\gamma}(q, 0) = \int \frac{dq'}{(2\pi)^2}\xi(\mathbf{p}-\mathbf{q}-\mathbf{q}')\xi(\mathbf{q}')[(\mathbf{q}'\mathbf{q} - {}^1\!/_2 q_z^{(\gamma)2})u^{\omega}_{\alpha\gamma}(\mathbf{q}, 0) + q_\alpha^{(\gamma)}(q'_{s\beta} - {}^1\!/_2\delta_{\beta z}q_z^{(\gamma)})u^{\omega}_{\beta\gamma}(\mathbf{q}, 0) + \delta_{\gamma z}(-q^2 - 2q_z^{(l)2} + q_z^{(t)2})(q_{s\alpha}' - {}^1\!/_2\delta_{\alpha z}q_z^{(l)})], \tag{9.3}$$

and in all the equations (9) we have $p_{s\alpha} = p_\alpha$ for $\alpha = x,y$ along with $p_{sz} = 0$. The vector **C** will be sought as a series in ξ:

$$C = C^{(0)} + C^{(1)} + C^{(2)} + \ldots,$$

and (5″) will be solved by iterations, with averaging over the irregularities performed at each stage. The following averages must be kept in mind:

$$\langle \xi(\mathbf{p})\rangle = 0, \quad \langle \xi(\mathbf{p})\xi(\mathbf{p}')\rangle = \xi_2(\mathbf{p})(2\pi)^2\delta(\mathbf{p}+\mathbf{p}'). \tag{10}$$

The first of these equalities expresses the fact that the average surface is the z = 0 plane, while the second one follows from the average uniformity of the surface.

It is seen from (10) that the linear in ξ form of (9.2) vanishes:

$$\langle \Pi^{(1)}(\mathbf{p}, \mathbf{q})u^{\omega}(\mathbf{q}, 0)\rangle = 0, \tag{11.1}$$

and the bilinear averages that arise through iterations can be represented as

$$\langle \Pi^{(1)}(\mathbf{p}, \mathbf{q})u^{\omega}(\mathbf{q}, 0)[\Pi^{(0)}(\mathbf{q})u^{\omega}(\mathbf{q}, 0)]^{-1}\Pi^{(1)}(\mathbf{q}, \mathbf{q}')u^{\omega}(\mathbf{q}', 0)\rangle = \langle \Pi^{(1)}(\mathbf{p}, \mathbf{q})u^{\omega}(\mathbf{q}, 0)[\Pi^{(0)}(\mathbf{q})u^{\omega}(\mathbf{q}, 0)]^{-1}\Pi^{(1)}(\mathbf{q}, \mathbf{p})\, u^{\omega}(\mathbf{p}, 0)\rangle(2\pi)^2\delta(\mathbf{p}-\mathbf{q}'), \tag{11.2}$$

$$\langle \Pi^{(2)}(\mathbf{p}, \mathbf{q})u^{\omega}(\mathbf{q}, 0)\rangle = \langle \Pi^{(2)}(\mathbf{p}, \mathbf{p})u^{\omega}(\mathbf{p}, 0)\rangle(2\pi)^2\delta(\mathbf{p}-\mathbf{q}). \tag{11.3}$$

To second order in ξ we obtain for the average value $\langle C\rangle = C^{(0)} + \langle C^{(2)}\rangle$:[1)]

$$\Big\{[\Pi^{(0)}(\mathbf{p}) + \langle \Pi^{(2)}(\mathbf{p}, \mathbf{p})\rangle]u^{\omega}(\mathbf{p}, 0) - \int \frac{dq}{(2\pi)^2}\langle \Pi^{(1)}(\mathbf{p}, \mathbf{q})u^{\omega}(\mathbf{q}, 0) \times [\Pi^{(0)}(\mathbf{q})u^{\omega}(\mathbf{q}, 0)]^{-1}\Pi^{(1)}(\mathbf{q}, \mathbf{p})u^{\omega}(\mathbf{p}, 0)\rangle\Big\}\langle C(p)\rangle = 0. \tag{12}$$

[1)] A matrix equation for a Green's function was solved similarly by Blank and one of the present authors in [6].

The existence condition for a nontrivial solution of the system of algebraic equations (12) is the vanishing of the corresponding determinant, thus determining the spectrum $p = p(\omega)$ of the Rayleigh wave on the rough surface. At $\xi = 0$ we have the operator values $\Pi^{(1)} = \Pi^{(2)} = 0$ and obtain the condition

$$\Delta^{\omega}(p) \equiv \|\Pi^{(0)}(p)u^{\omega}(p,0)\| = ip_z p_z^{(t)}\left[(p_z^{(t)2} - p^2)^2 + 4p^2 p_z^{(t)} p_z^{(l)}\right] = 0. \quad (13)$$

The vanishing of the square-bracketed expression in (13) yields the known spectrum of a Rayleigh wave on an ideally smooth surface, $p = p^0(\omega)$. The treatment given here is valid when the effect of the irregularities is small. Therefore the perturbed spectrum is represented as $p = p^0(\omega) + \delta p(\omega)$, where

$$\delta p(\omega) = \frac{\Delta^{\omega}(p)}{\partial\Delta^{\omega}/\partial p}\,\mathrm{Sp}\left\{\int\frac{dq}{(2\pi)^2}\langle T(p,q)T(q,p)\rangle - [\Pi^{(0)}(p)u^{\omega}(p,0)]^{-1}\langle\Pi^{(2)}(p,p)u^{\omega}(p,0)\rangle\right\}\Bigg|_{p=p^0(\omega)}, \quad (14)$$

with the notation

$$T(p,q) = \Pi^{(1)}(p,q)u^{\omega}(q,0)[\Pi^{(0)}(q)u^{\omega}(q,0)]^{-1}.$$

c We note that $\Delta^{\omega}(p) = 0$ for $p = p^{(0)}(\omega)$; however, the right member of (14) does not vanish, because

$$T(q,p) \sim [\Pi^{(0)}(p)u^{\omega}(p,0)]^{-1} \sim 1/\Delta^{\omega}(p).$$

3. The integral in (14) cannot be calculated explicitly, but its order of magnitude can be estimated. Obtaining the matrices in (9) (for which purpose it is convenient to direct the wave vector **p** of the initial wave along the x axis), we ascertain that the term containing $\Pi^{(2)}$ contributes to only the real part of $\delta p(\omega)$. Wave damping, i.e., $\mathrm{Im}\,\delta p(\omega)$, which alone will interest us, is determined by the integral with $\Pi^{(1)}$.

The q_x, q_y plane is represented in Fig. 2, where the circle of radius $\omega/c_t\zeta$ shows the locations of the poles of the integrand, i.e., the locations of the zeros of $\Delta^{\omega}(\mathbf{q})$. The integration contour about the poles is provided by means of the infinitesimal added imaginary term that was mentioned following Eq. (6). To calculate the contribution of the poles to damping it is sufficient to make the substitution $1/\Delta^{\omega}(\mathbf{q}) \to i\pi\delta(\Delta^{\omega}(\mathbf{q}))$, where $\delta(x)$ is a delta function. The initial point of **p**, which is also determined by the condition $\Delta^{\omega}(\mathbf{p}) = 0$, lies on the same circle.

The region of the plane where the integrand does not vanish is represented in the figure by a circle of radius $1/d$. Outside this circle the Fourier component of the binary function $\xi_2(\mathbf{p}-\mathbf{q})$ under the integral [see Eqs. (9) and (11)] is small; inside the circle $\xi_2(\mathbf{p}-\mathbf{q}) \sim a^2d^2$.

The pole part of the integral determines the damping, $\mathrm{Im}\,\delta p_{ss}$, that results from scattering of the initial surface wave into another surface wave with a different wave vector. The magnitude of the damping depends on the ratio between $1/d$ and p; we obtain, in order of magnitude,

$$\mathrm{Im}\frac{\delta p_{ss}}{p} \sim \begin{cases} a^2dp^3, & pd \gg 1, \quad (15.1)\\ a^2d^2p^4, & pd \ll 1. \quad (15.2)\end{cases}$$

In Fig. 2 the circles of radii ω/c_t and ω/c_l depict the regions where the quantities $q_z^{(t)} = (\omega^2/c_t^2 - q^2)^{1/2}$ and $q_z^{(l)} = (\omega^2/c_l^2 - q^2)^{1/2}$ are real. In these regions transverse and longitudinal body waves, respectively, can propagate. A Rayleigh wave with a given value of **p** can be scattered into a transverse body wave if the circle of radius $1/d$ intersects the circle of radius ω/c_t. In this case the integral in (14) includes an integration region, the region of circle overlap, where the integrand is complex. The corresponding contribution to damping is estimated as follows:

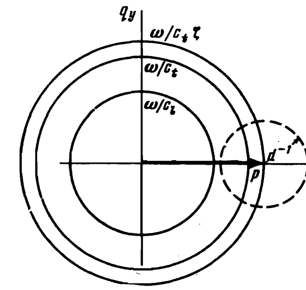

FIG. 2

$$\mathrm{Im}\frac{\delta p_{sv}}{p} \sim \begin{cases} 0, & 1 < pd(1-\zeta), \quad (16.1)\\ (ap)^2/(pd)^{1/2}, & pd(1-\zeta) \ll 1 \ll pd, \quad (16.2)\\ a^2d^2p^4, & pd \ll 1. \quad (16.3)\end{cases}$$

The dependence on the square root of pd in (16.2) results from the fact that the imaginary part of the integrand is proportional to $q_z^{(t)}$ when $q_z^{(t)}$ is small.

In deriving (15) and (16), besides $(ap)^2 \ll 1$ it was assumed that the damping δp is small compared with the gap $p(1-\zeta)$ separating the surface wave from body waves. This last condition permitted the utilization of perturbation theory.

We note that for $pd \ll 1$ the surface wave damping δp_{ss} and the body wave damping δp_{sv} are of the same order of magnitude. Their frequency dependence agrees with the result obtained in[2]: $\mathrm{Im}(\delta p/p) \sim \omega^4$. In the other limiting case, $pd \gg 1$, the surface wave damping is larger on the basis of the parameter pd and depends differently on the frequency.

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Translated by I. Emin
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