NUMBER OPERATOR FOR PHOTONS TRAVERSING A SPATIAL POINT

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A definition is given of a quantum-mechanical operator representing the number of photons traversing a spatial point in an infinite time.

1. In the coordinate representation the wave function of a photon cannot be interpreted as the probability amplitude and, therefore, it is not possible to introduce local space-time characteristics of a photon. Formally this means that the components of the electromagnetic field tensor cannot be used to form a four-vector bilinear in the field and possessing a positive time component (see, for example,[1]).

We shall derive a four-component quantity J_μ which is not a four-vector but has the following properties:

1) $$\partial J_\mu / \partial x_\mu = 0 \quad (\mu = 0, 1, 2, 3); \tag{1}$$

2) J_0 is a positive operator;

3) $$\frac{1}{c}\int J_0\, d^3x = N, \tag{2}$$

where N is the photon number operator;

4) the quantity

$$I_\mu = \int_{-\infty}^{\infty} J_\mu\, dt$$

has the transformation properties of the time integral of a four-vector.

Thus, in spite of the fact that we cannot introduce the density of the flux of photons at a point, we can introduce a time integral of this flux.

2. We shall use the expression for the quantized vector potential of a free electromagnetic field:

$$A_\mu(\mathbf{r}, t) = \frac{\sqrt{\hbar c}}{2\pi}\sum_{\lambda=1,2}\int \frac{d^3k}{k_0} e_\mu^\lambda(\mathbf{k})\left[\exp(ikx)c_\lambda(\mathbf{k}) + \exp(-ikx)c_\lambda^+(\mathbf{k})\right]. \tag{3}$$

Here, $k_0 = |\mathbf{k}|$, $kx = \mathbf{k}\cdot\mathbf{r} - k_0ct$, and the four-vectors $e_\mu^\lambda(\mathbf{k})$ ($\lambda = 1, 2$) satisfy the following conditions in accordance with the selected Coulomb gauge:

$$e_0^\lambda = 0, \quad e_\mu^\lambda e_\mu^{\lambda'} = \delta_{\lambda\lambda'}, \quad k_\mu e_\mu^\lambda(\mathbf{k}) = 0,$$
$$\sum_{\lambda=1,2} e_i^\lambda e_j^\lambda = \delta_{ij} - \frac{k_i k_j}{k_0^2} \quad (i, j = 1, 2, 3),$$

and the creation and annihilation operators c_λ^+ and c_λ^+ obey the commutation relationship

$$[c_\lambda(\mathbf{k}),\, c_{\lambda'}^+(\mathbf{k}')] = k_0\delta_{\lambda\lambda'}\delta(\mathbf{k} - \mathbf{k}'). \tag{4}$$

According to the transformation properties, the quantities c_λ and c_λ^+ are scalar. The photon number operator is of the form

$$N = \sum_\lambda \int \frac{d^3k}{k_0} c_\lambda^+(\mathbf{k})c_\lambda(\mathbf{k}). \tag{5}$$

We shall introduce the operators

$$b_\mu(\mathbf{r}, t) = \sum_\lambda \frac{1}{\sqrt{8\pi^3}}\int e_\mu^\lambda(\mathbf{k})\exp(ikx)\sqrt{k_0}\, c_\lambda(\mathbf{k})\frac{d^3k}{k_0}, \tag{6}$$

$$b_\mu^+(\mathbf{r}, t) = \sum_\lambda \frac{1}{\sqrt{8\pi^3}}\int e_\mu^\lambda(\mathbf{k})\exp(-ikx)\sqrt{k_0}\, c_\lambda^+(\mathbf{k})\frac{d^3k}{k_0}. \tag{7}$$

It is clear from the above expressions that b_μ and b_μ^+ are not four-vectors because of the presence of the factor $\sqrt{k_0}$ in the integrand. We shall also introduce the quantity

$$n(\mathbf{r}, t) = b_\mu^+(\mathbf{r}, t)b_\mu(\mathbf{r}, t) = \sum_{\lambda,\lambda'}\frac{1}{8\pi^3}\iint \frac{d^3k}{k_0}\frac{d^3k'}{k_0'} e_\mu^\lambda(\mathbf{k})e_\mu^{\lambda'}(\mathbf{k}')$$
$$\times \exp[-i(k - k')x]\sqrt{k_0k_0'}\, c_\lambda^+(\mathbf{k})c_{\lambda'}(\mathbf{k}'), \tag{8}$$

which is not a time component of a four-vector because of the presence of the factor $\sqrt{k_0k_0'}$.

Integrating Eq. (8) over the coordinates, we easily obtain the relationship

$$\int d^3r\, n(\mathbf{r}, t) = N. \tag{9}$$

If we differentiate Eq. (8) with respect to t and employ the identity

$$(k_0 - k_0')\exp[-i(k - k')x]$$
$$= i\frac{\mathbf{k} + \mathbf{k}'}{k_0 + k_0'}\exp[i(k_0 - k_0')ct]\nabla\exp[i(\mathbf{k}' - \mathbf{k})\mathbf{r}],$$

we easily find that

$$\dot{n}(\mathbf{r}, t) + \operatorname{div}\mathbf{J}(\mathbf{r}, t) = 0, \tag{10}$$

where

$$\mathbf{J}(\mathbf{r}, t) = \frac{c}{8\pi^3}\sum_{\lambda,\lambda'}\iint \frac{d^3k}{k_0}\frac{d^3k'}{k_0'} e_\mu^\lambda(\mathbf{k})e_\mu^{\lambda'}(\mathbf{k}')\exp[-i(k - k')x]$$
$$\times \frac{\sqrt{k_0k_0'}}{k_0 + k_0'}(\mathbf{k} + \mathbf{k}')c_\lambda^+(\mathbf{k})c_{\lambda'}(\mathbf{k}'). \tag{11}$$

Let n be an arbitrary unit vector. Then, integrating Eq. (11) in the plane $\boldsymbol{\rho}\cdot\mathbf{n} = 0$, we obtain the relationship

$$\iint_{-\infty}^{\infty} d^2\rho \int_{-\infty}^{\infty} dt\, \mathbf{n}\mathbf{J}(\mathbf{r} + \boldsymbol{\rho}, t) = N, \tag{12}$$

which means that the flux of the vector **J** passing in an infinite time through any plane is equal to the number of photons.

We shall introduce the four-component quantities

$$J_\mu = (cn, \mathbf{J}), \quad f_\mu(\mathbf{k}, \mathbf{k}') = \left(\sqrt{k_0k_0'},\ \frac{\sqrt{k_0k_0'}}{k_0 + k_0'}(\mathbf{k} + \mathbf{k}')\right).$$

Then, Eqs. (8) and (11) can be written in the form

$$J_\mu(x) = \sum_{\lambda,\lambda'}\frac{c}{8\pi^3}\iint \frac{d^3k}{k_0}\frac{d^3k'}{k_0'} e_\nu^\lambda(\mathbf{k})e_\nu^{\lambda'}(\mathbf{k}') \tag{13}$$
$$\times \exp[-i(k - k')x]f_\mu(\mathbf{k}, \mathbf{k}')c_\lambda^+(\mathbf{k})c_{\lambda'}(\mathbf{k}'),$$

and Eq. (10) assumes the form (1).

In spite of the relationships (9), (12), and (10), the quantity **J** cannot be regarded as the vector density of the photon flux because f_μ and J_μ do not form four-vectors.

We shall now consider the time integral of the quantity J_μ:

$$I_\mu(\mathbf{r}) = \int_{-\infty}^{\infty} J_\mu(\mathbf{r}, t)\,dt = \sum_{\lambda,\lambda'} \frac{1}{4\pi^2} \iint \frac{d^3k}{k_0}\frac{d^3k'}{k_0'} \exp[-i(k-k')x] \times e_\nu^\lambda(\mathbf{k}) e_\nu^{\lambda'}(\mathbf{k}') \delta(k_0 - k_0') f_\mu(\mathbf{k}, \mathbf{k}') c_\lambda^+(\mathbf{k}) c_{\lambda'}(\mathbf{k}'). \tag{14}$$

The product $\delta(k_0 - k_0') f_\mu(\mathbf{k}, \mathbf{k}')$ can be represented in the form

$$\delta(k_0 - k_0') f_\mu(\mathbf{k}, \mathbf{k}') = \delta(k_0 - k_0')\left(\frac{k_0 + k_0'}{2}, \frac{\mathbf{k}+\mathbf{k}'}{2}\right) = \delta(k_0 - k_0')\frac{k_\mu + k_\mu'}{2}.$$

Therefore, $I_\mu(\mathbf{r})$ can be represented as the time integral of the four-vector

$$I_\mu(\mathbf{r}) = \int_{-\infty}^{\infty} j_\mu(\mathbf{r}, t)\,dt,$$

$$j_\mu(x) = \sum_{\lambda,\lambda'} \frac{c}{8\pi^3} \iint \frac{d^3k}{k_0}\frac{d^3k'}{k_0'} e_\nu^\lambda(\mathbf{k}) e_\nu^{\lambda'}(\mathbf{k}') \times \exp[-i(k-k')x] \frac{k_\mu + k_\mu'}{2} c_\lambda^+(\mathbf{k}) c_{\lambda'}(\mathbf{k}'). \tag{15}$$

The four-vector j_μ also does not represent the density of the photon flux because j_0 is not a positive operator. However, the quantity $I_\mu(\mathbf{r})$, which is defined by Eq. (14) or Eq. (15), has all the necessary properties of the time integral of the photon flux density at a given point.

Thus, avoidance of quantities which are local in time has made it possible to introduce photon characteristics which are local in space.

In this connection we note that

$$\int_V n(\mathbf{r}, t)\,d^3r$$

can be regarded, for a sufficiently large volume V, as the particle number operator in this volume V (see, for example,[2]), in spite of the absence of the necessary transformation properties. This is due to the fact that Eq. (9) is satisfied approximately if the volume V is sufficiently large. A similar situation applies to the vector **J** where the averaging over space is replaced by the averaging over time.

3. Let us consider the operator

$$\nu_{\Sigma T}(\mathbf{r}, t) = \iint_\Sigma d^2\rho \int_{-T/2}^{T/2} d\tau\, \mathbf{nJ}(\mathbf{r}+\rho, t+\tau) \qquad (\mathbf{n}\rho = 0). \tag{16}$$

If $T \to \infty$ (in practice it is sufficient to satisfy the condition $k_0 cT \gg 1$ for those frequencies which actually occur in the expansion), the operator $\nu_{\Sigma T}$ can be regarded as the operator of the number of photons traversing a small area Σ in a time T. No restrictions are imposed on Σ and, in particular, we may consider the case when the dimensions of this area are of the order of the photon wavelength. Although the operator I is not bounded, the operator $\nu_{\Sigma T}$ is bounded for all finite values of Σ and it leads to finite values of the average square of the fluctuations in the number of photons traversing the area Σ.

The operator $\nu_{\Sigma T}$ was introduced by the present author[3] for a medium with a permittivity which varies in space. In that case the formulas are analogous to those given above but instead of expansions in terms of plane waves one uses expansions in terms of solutions of the wave equation for a medium with a specified distribution of the permittivity $\epsilon(\mathbf{r})$. The operator $\nu_{\Sigma T}$ can then be used to consider the diffraction problems in dielectrics from the quantum point of view. It is found that the average values of the number of photons crossing an area Σ in a time T, $\bar{\nu} = \langle \psi | \nu_{\Sigma T} | \psi \rangle$, correspond to the classical solution but they can be used to study the statistical properties of the quantity $\nu_{\Sigma T}$. It is found that the statistics of the fluctuations in the number of photons is different in free space and in the presence of dielectrics causing diffraction. These differences affect the fluctuations

$$\sigma_\nu^2 = \langle \psi | \nu_{\Sigma T}^2 | \psi \rangle - \langle \psi | \nu_{\Sigma T} | \psi \rangle^2.$$

and the correlation function

$$B(\mathbf{r}, \mathbf{r}') = \langle \psi | \nu_{\Sigma T}(\mathbf{r}, t) \nu_{\Sigma T}(\mathbf{r}', t) | \psi \rangle - \langle \psi | \nu_{\Sigma T}(\mathbf{r}, t) | \psi \rangle \langle \psi | \nu_{\Sigma T}(\mathbf{r}', t) | \psi \rangle.$$

In particular, the correlation radius R_0 of the fluctuations in the number of photons in the case of diffraction by an aperture is of the order of the width of a diffraction lobe (in free space it is of the order of the wavelength). The ratio $\sigma_\nu^2/\bar{\nu}$ for coherent light is less than unity, which corresponds to the finite value of the relative fluctuations $\sigma_\nu^2/(\bar{\nu})^2$ if $\Sigma \ll R_0^2$, whereas $\sigma_\nu^2/(\bar{\nu})^2 \approx (\bar{\nu})^{-1} \sim \Sigma^{-1}$ if $\Sigma \gg R_0^2$.

However, it should be pointed out in photon counting experiments one measures not the quantity $\nu_{\Sigma T}$ but the number of photoelectrons m produced as a result of the interaction between light and the detector material. The statistics of m may differ from the statistics of $\nu_{\Sigma T}$. Therefore, the statistical properties of the recorded photon flux do not represent the radiation field itself but the statistics of the photoresponse.

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