SINGULARITIES OF MULTITAIL RING DIAGRAMS FOR FERMI SYSTEMS

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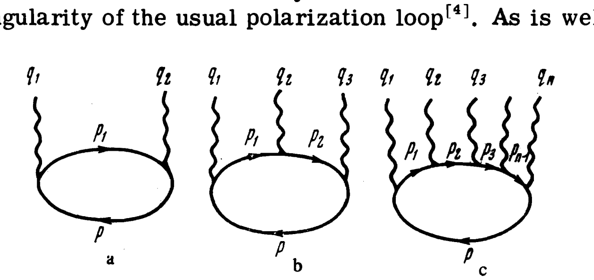
A complete analysis is performed of the possible singularities in the external momenta that arise in multitail ring diagrams for Fermi systems. A method similar to Landau's method is used, making it possible to perform this analysis without explicit integration. A particular case of the hierarchy of singularities found is the Kohn singularity ($n = 2$); all the others are found to be stronger. The asymptotic form of the ring diagrams is found, and the integration of the three-tail diagram is performed explicitly.

1. INTRODUCTION

CHARACTERISTIC in many-body theory of metals is the appearance of closed electron diagrams with an arbitrary number of external-field lines (see the figure)[1,2]. Physically, such diagrams are associated with indirect interaction between the ions through the conduction electrons, and formally they appear in the determination of any quantities that are integrals over the electron spectrum and in the form of a series in the electron-ion interaction. An important point is that each term of such a series factorizes, i.e., breaks down into factors of which one depends on the crystal structure and on the specific character of the electron-ion interaction, and the other depends only on the properties of a uniform system of electrons with a given density. This latter is a connected many-tail diagram with an arbitrary number of electron-electron interaction lines and a fixed number of external static-field "tails" (static by virtue of the validity of the adiabatic approximation—for more detail, see[1]) i.e., an n-tail diagram depends only on the n external three-momenta $q_1, \dots, q_n$.

The purpose of the present paper is to determine the nature and positions of the singularities of these many-tail diagrams as functions of the external momenta. This problem is of substantial interest, inasmuch as such singularities can lead to singular behavior of macroscopic quantities and, in particular, appear explicitly in the dispersion curves of the phonon spectrum. The latter circumstance is connected with the fact that it is a continuous, and not a discrete, change of two of the external momenta of a many-tail diagram that corresponds to a change of the phonon wave-vector in the harmonic problem (cf.[1,2]). The simplest singularity of the type under consideration is the so-called "Kohn anomaly"[3] associated with a singularity of the usual polarization loop[4]. As is well known, the Kohn anomaly has been reliably established in experiments measuring the phonon dispersion law by means of inelastic neutron scattering (cf., e.g.,[5,6]).



We note that the Kohn anomaly, being a singularity of the diagram with two external "tails" ($n = 2$), corresponds to a singularity in the pair indirect interaction between ions. The leading singularities of diagrams with $n \geq 3$ correspond to a many-body interaction, and the observation of these singularities is a direct proof of the existence of non-pair interactions in metals (cf.[1,2]).

For a complete analysis of the problem, a method is developed in the present work which makes it possible to find the singularities without an explicit calculation of the diagram itself. This is extremely important, as only an expression for the "two-tail" diagram or simple polarization loop has been found in analytic form up to the present time (cf.[4]). This method is the analog of the method developed by Landau[7] for determining the singularities of diagrams in quantum field theory. The principal distinctive feature of the problem under consideration is associated with the presence of a background of Fermi particles and of the Fermi surface, and also with the three-dimensionality of the problem. We remark also that, since we are interested in describing only the observable properties, we shall consider only the singularities for real values of the external parameters (on the real axis of the physical sheet).

In the present work, a complete analysis is performed for a definite class of diagrams—for the ring diagrams which do not contain electron-electron interaction lines. To all appearances, diagrams with a fixed number of external-field tails and an arbitrary number of electron-electron interaction lines will not contain stronger singularities than will the ring diagram with the same number of tails. Indeed, as will be seen from the following, a singularity is associated with the presence of a sharp Fermi boundary in momentum space and, by virtue of this, depends on the number of electron propagators. Each Coulomb-interaction line introduces two new electron propagators, but simultaneously introduces an additional fourfold integration.

The Feynman representation used in the work for the diagrams has been found to be highly convenient for analyzing the asymptotic behavior, and also for direct integration. In the last section of the paper, this is demonstrated using the example of the direct calculation of the third-order ring diagram. A detailed discussion of the question of how the singularities found will be manifested in the phonon dispersion curves is given in a separate paper.

2. POSITION OF THE SINGULARITIES

For ring diagrams with static external tails, a distinguishing feature is the fact that the same frequency ω figures in all the electron propagators. For the analysis in this case, it turns out to be very convenient to use the free-electron Green function in the form

$$G_0(\mathbf{p},\omega)=[\omega-\varepsilon_0(\mathbf{p})+i\delta\,\mathrm{sign}\,(\omega-\mu)]^{-1}, \tag{2.1}$$

where μ is the chemical potential. When $\delta\to 0$, this expression coincides with the usual expression[8], but has the advantage that the imaginary correction in the denominator turns out to be the same for all the Green functions of the ring.

An arbitrary diagram (see the figure) can then be written in the form

$$J^{(n)}(\mathbf{q}_1,\ldots,\mathbf{q}_n)=\frac{2}{i}\int\frac{d\mathbf{p}\,d\omega}{(2\pi)^4}\frac{1}{[\omega-\varepsilon_0(\mathbf{p}_1)+i\delta\,\mathrm{sign}(\omega-\mu)]}\cdots$$
$$\cdots\frac{1}{[\omega-\varepsilon_0(\mathbf{p}_n)+i\delta\,\mathrm{sign}(\omega-\mu)]}, \tag{2.2}$$

where all the $\mathbf{p}_i$ are linearly related to $\mathbf{p}$ and the $\mathbf{q}_i$. Using the known Feynman parametrization formula, we can rewrite the expression (2.2) in the form

$$J^{(n)}(\mathbf{q}_1,\ldots,\mathbf{q}_n)=\frac{2}{i}(n-1)!\int\frac{d\mathbf{p}\,d\omega}{(2\pi)^4}\int_0^1\ldots\int_0^1 d\alpha_1\ldots d\alpha_n\frac{1}{f^n}\,\delta\Big(\sum_{i=1}^n\alpha_i-1\Big), \tag{2.3}$$

where

$$f=\omega\Big(\sum_{i=1}^n\alpha_i\Big)-\sum_{i=1}^n\alpha_i\varepsilon_0(\mathbf{p}_i)+i\delta\Big(\sum_{i=1}^n\alpha_i\Big)\,\mathrm{sign}\,(\omega-\mu), \tag{2.4a}$$

or

$$f=\omega-\sum_{i=1}^n\alpha_i\varepsilon_0(\mathbf{p}_i)+i\delta\,\mathrm{sign}\,(\omega-\mu). \tag{2.4b}$$

In (2.4b) we have used in explicit form the condition imposed by the delta-function:

$$\sum_{i=1}^n\alpha_i=1. \tag{2.5}$$

The convenience of the representation obtained lies in the fact that it does not contain complicated integration limits in momentum space, which usually arise because of the Fermi occupation numbers $n(\mathbf{p}_i)$ and make the analysis of the singularities difficult.

According to Hadamard's principle, a singularity of a multiple integral of the type (2.3) appears only for that value of the aggregate of real parameters $\mathbf{q}_i^{(0)}$ to which corresponds the vanishing of f at a point where 1) a second-order zero, or 2) a coincidence with fixed limits of the range of integration ("a limit singularity"), corresponds to each integration variable simultaneously. For a complex region, the first condition can be formulated as the condition for the confluence of two singularities of the integrand as $\mathbf{q}_i\to\mathbf{q}_i^{(0)}$, pinching the corresponding integration contour from two different sides (a "pinch singularity"). For real $\mathbf{q}_i$ these two conditions coincide, and a second-order zero always corresponds automatically to the motion of poles from the two different sides of the contour.

We shall now analyze the expression (2.3) from this point of view.

a) It follows directly from the form of (2.4a) that a singularity in ω can only be of the limit type. In the integration over ω the integration contour for $\omega<\mu$ can be displaced into the lower half-plane, and for $\omega>\mu$ into the upper half-plane; the point $\omega=\mu$ remains fixed, being the limit point of the contour. Hence we obtain the condition

$$\omega=\mu. \tag{2.6}$$

b) A singularity in the variable $\mathbf{p}$, on the other hand, can lie only within the range of integration. Then, for the dispersion law $\epsilon=\mathbf{p}^2/2m$, the condition $\partial f/\partial\mathbf{p}=0$ leads to the relation

$$\sum_{i=1}^n\alpha_i\mathbf{p}_i=0. \tag{2.7}$$

c) A singularity in the variables α_i can be of either the pinch or the limit type. In the first case, from the condition $\partial f/\partial\alpha_i=0$ we find directly for all $\mathbf{p}_i$

$$\varepsilon_0(\mathbf{p}_i)=\omega. \tag{2.8a}$$

Taking account of (2.6), and also of the alternative possibility of the appearance of a limit singularity in any of the variables α_i, we have finally

$$\varepsilon_0(\mathbf{p}_i)=\mu,\quad\text{or}\quad\alpha_i=0. \tag{2.8b}$$

We note, in addition, that it follows from the fact that f (2.4a) is homogeneous in α that from $\partial f/\partial\alpha_i=0$ we automatically obtain $f=0$.

Thus, the necessary condition for the existence of a singularity of the integral (2.3) is the fulfilment of the relations (2.6), (2.7), and (2.8b). In the case $\alpha_i=0$, the propagator corresponding to a given electron line generally drops out, and, as will be seen from the following, the singularity will correspond to the singularity of a diagram of lower order (a "reduced diagram").

The solution of the system of equations (2.7) and (2.8) defines a hypersurface in the space of $\mathbf{q}_1,\ldots,\mathbf{q}_n$, and gives the set of parameters $\alpha_1,\ldots,\alpha_n$ simultaneously at each point on this surface. Then only the part of this hypersurface to which the condition

$$\alpha_i>0,\quad i=1,\ldots,n.$$

corresponds for real $\mathbf{q}_i$ will be singular. The fact that the α_i are positive is an obvious consequence of (2.3), and the condition $\alpha_i<1$ is unimportant because of the homogeneity of f (2.4a).

Most noteworthy is the fact that, for a singularity to appear, it is necessary that all virtual particles lie on the Fermi surface. This reflects the role of the sharp boundary of the Fermi distribution in momentum space. In this connection, it seems that allowance for the interaction between the electrons leads not to a blurring of the singularities, but, on the contrary, to the appearance of stronger singularities. For the first statement the important point is the conservation of the sharp Fermi boundary in momentum space even in the presence of interaction between the particles (cf.[9,8]), and for the second statement the important point is the appearance with each interaction line in the diagram of additional integrations along with the two electron propagators.

3. NATURE OF THE SINGULARITIES

To determine the nature of the singularities arising, we perform the integration over ω and p in (2.3) in explicit form. Keeping in mind the way in which we go around the pole in (2.3), we have

$$\frac{1}{2\pi i}\int_{-\infty}^{\mu} d\omega\left[\omega-\sum_{i=1}^{n}\alpha_i\varepsilon_0(\mathbf{p}_i)-i\delta\right]^{-n}+\frac{1}{2\pi i}\int_{\mu}^{\infty} d\omega\left[\omega-\sum_{i=1}^{n}\alpha_i\varepsilon_0(\mathbf{p}_i)+i\delta\right]^{-n}$$
$$=\frac{1}{2\pi i}\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-2}}{\partial\mu^{(n-2)}}\left\{\left[\mu-\sum_{i=1}^{n}\alpha_i\varepsilon_0(\mathbf{p}_i)-i\delta\right]^{-1}\right. \quad (3.1)$$
$$\left.-\left[\mu-\sum_{i=1}^{n}\alpha_i\varepsilon_0(\mathbf{p}_i)+i\delta\right]^{-1}\right\}=\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-2}}{\partial\mu^{(n-2)}}\delta\left(\mu-\sum_{i=1}^{n}\alpha_i\varepsilon_0(\mathbf{p}_i)\right).$$

For convenience in the following, we introduce the notation

$$\varkappa_s=\mathbf{q}_1+\ldots+\mathbf{q}_s. \quad (3.2)$$

Then

$$\mathbf{p}_i=\mathbf{p}+\varkappa_i, \quad (3.3)$$

and, taking (2.5) into consideration, we obtain

$$\sum_{i=1}^{n}\alpha_i\varepsilon_0(\mathbf{p}_i)=\frac{1}{2m}\left[\left(\mathbf{p}+\sum_{i=1}^{n}\alpha_i\varkappa_i\right)^2-\sum_{i,j=1}^{n}\varkappa_i\varkappa_j\alpha_i\alpha_j+\sum_{i=1}^{n}\alpha_i\varkappa_i^2\right].$$

We shall substitute this result into (3.1) and, shifting the coordinate origin, perform the integration over p in (2.3). We finally obtain

$$J^{(n)}(\mathbf{q}_1,\ldots,\mathbf{q}_n)=\frac{m}{\pi^2}(-1)^{n-1}\frac{\partial^{n-2}}{\partial\mu^{(n-2)}}\int_0^1 d\alpha_1\int_0^{1-\alpha_1}d\alpha_2\ldots \quad (3.4)$$
$$\ldots\int_0^{1-\alpha_1-\ldots-\alpha_{n-2}}d\alpha_{n-1}\left[2m\mu+\sum_{i,j}^{n-1}\varkappa_i\varkappa_j\alpha_i\alpha_j-\sum_{i=1}^{n-1}\alpha_i\varkappa_i^2\right]^{1/2}.$$

Clearly, in this expression the integration is performed only over the region in which the expression under the square root is positive. In deriving (3.4), we have taken into account that

$$\varkappa_n=0, \quad (3.5)$$

and have performed the integration over the α_n by making use of the delta-function in α.

It is interesting that the quadratic form which has appeared under the square root in (3.4) is defined in terms of coefficients that form a Gram determinant, and so is positive-definite. For given values of the external momenta, the expression under the square root in (3.4) has an extremum at the point $\{\bar{\alpha}_i\}$ determined from the solution of the system of equations

$$\sum_{j=1}^{n}\varkappa_i\varkappa_j\bar{\alpha}_j={}^1/_2\,\varkappa_i^2. \quad (3.6)$$

At this point, it becomes equal to

$$\Delta=k_F^2-\varkappa^2,\quad \varkappa^2\equiv\left(\sum_{i=1}^{n-1}\bar{\alpha}_i\varkappa_i\right)^2=\frac{1}{2}\sum_{i=1}^{n-1}\bar{\alpha}_i\varkappa_i^2. \quad (3.7)$$

(It is assumed that the values of κ_j are such that the solution $\{\bar{\alpha}_i\}$ of (3.6) falls in the range of integration of (3.4).)

It is easy to convince oneself that the conditions (2.7) and (2.8b) found above for the existence of a singularity lead to the same system of equations (3.6). In fact, from (2.7) we find the momentum p:

$$\mathbf{p}=-\sum_{i=1}^{n-1}\alpha_i\varkappa_i=\varkappa \quad (3.8)$$

and substitute this expression into the relation (2.8a), having equated $\epsilon_0(\mathbf{p}_i)$ and $\epsilon_0(\mathbf{p}_n)$ using (3.5). We have

$$2\mathbf{p}\varkappa_i+\varkappa_i^2=0. \quad (3.9)$$

Now (3.9) coincides exactly with (3.6), if we substitute (3.8) in place of p. At the actual singular point ($\kappa_i = \kappa_i^{(0)}$), it follows from (2.8b) that $p^2 = k_F^2$ and $\Delta = 0$.

In order to find the character of the singularity, we shall examine the integral (3.4) close to the singular point, i.e., for small Δ. In accordance with what has been said, close to $\{\alpha_i^0\}$ this integral can be represented in the form

$$\int d\alpha_1'\ldots d\alpha_{n-1}'\left(\Delta+\sum_{i=1}^{s}\lambda_i\alpha_i'^2\right)^{1/2}=$$
$$=\frac{1}{(\lambda_1\lambda_2\ldots\lambda_s)^{1/2}}\int dy_1\ldots dy_s\,d\alpha_{s+1}'\ldots d\alpha_{n-1}'\left[\Delta+\sum_{i=1}^{s}y_i^2\right]^{1/2},$$

where s is the rank of a square matrix of order $n - 1$ ($s \le n - 1$)

$$\|\varkappa_i\varkappa_k\|. \quad (3.10)$$

We note that $\lambda_i > 0$ automatically, because the quadratic form is positive-definite.

From a well known property of Gram determinants, s is equal to the number of linearly independent vectors κ_i in the set $\kappa_1, \ldots, \kappa_{n-1}$. Close to a singularity, Δ in the general case (cf. (3.7)) is a linear superposition of deviations of the scalar products $\kappa_i \cdot \kappa_k$ from their values at the singular point (we recall that α_i in the definition (3.7) of κ^2 is the solution of Eq. (3.6)). Then, when (3.6) is fulfilled, $|\kappa_i|$ is the radius of the spherical surface on which lie the tips of all the vectors p_i (cf. (3.8)).

Thus, the general expression determining the character of the singularity of an n-th order multi-point diagram can be written in the following form:

$$J^{(n)}\sim\frac{\partial^{(n-2)}}{\partial\Delta^{(n-2)}}\int_{y_0} dy\,y^{s-1}(\Delta+y^2)^{1/2}, \quad (3.11)$$

where the lower limit $y_0 = 0$ for $\Delta > 0$ and $y_0 = |\Delta|^{1/2}$ for $\Delta < 0$.

We shall make use of the recurrence formula

$$\int dy\,y^{s-1}(\Delta+y^2)^{1/2}=\frac{y^{s-2}(\Delta+y^2)^{3/2}}{s+1}-\frac{s-2}{s+1}\Delta\int dy\,y^{s-3}(\Delta+y^2)^{1/2}. \quad (3.12)$$

The singularity of the multi-tail diagram clearly depends on the behavior of the integral at the lower limit. Because of this, the first term in (3.12) is unimportant. Successively using the recurrence procedure, it is easy to establish that the nature of the singularity is dictated for odd s by the behavior of the integral

$$(-\Delta)^{(s-1)/2}\int_{y_0} dy\,(\Delta+y^2)^{1/2},$$

and for even s by the integral

$$(-\Delta)^{(s-2)/2}\int_{y_0} dy\, y(\Delta+y^2)^{1/2}.$$

Thus, we finally obtain (for odd s)

$$J^{(n)}\sim(-1)^{(2n+s-1)/2}\frac{\partial^{(n-2)}}{\partial\Delta^{(n-2)}}(\Delta^{(s+1)/2}\ln|\Delta|)\sim(-1)^{(2n+s-1)/2}\Delta^{(s-2n+5)/2}\ln|\Delta|, \quad (3.13)$$

where Δ here can have either sign, and (for even s)

$$J^{(n)}\sim(-1)^{(2n+s+2)/2}\frac{\partial^{(n-2)}}{\partial\Delta^{(n-2)}}\Delta^{(s+1)/2}\sim(-1)^{(2n+s+2)/2}\Delta^{(s-2n+5)/2},\quad \Delta>0 \quad (3.14)$$

(in this case the diagram has no singularities for $\Delta < 0$).

We recall that the expressions (3.13) and (3.14) determine the behavior of the diagram with n external tails, close to the leading singularity. Such a diagram, however, will also contain weaker "limit" singularities, corresponding to zero α_j in (2.8). In this case, terms linear in the corresponding α_j appear, and the behavior of $J^{(n)}$ close to a limit singularity with k values $\alpha_j = 0$ will be described by the same expressions (3.13) and (3.14), with the replacement

$$n\to n'=n-k,\quad s\to s'. \quad (3.15)$$

Here s' is the rank of the n'-th order square matrix obtained from the matrix (3.10) by deleting the k rows and columns with indices corresponding to zero α_j.

4. ANALYSIS OF THE SINGULARITIES OF SPECIFIC DIAGRAMS

We proceed now to examine the individual multitail diagrams.

a) The two-tail diagram. In this case (see the figure, case a), the system (3.6) reduces to a single equation $\kappa_1^2\alpha_1 = \frac{1}{2}\kappa_1^2$, the solution of which is $\alpha_1 = \frac{1}{2}$. In this case, from (3.7),

$$\Delta = k_F^2 - \tfrac{1}{4}\varkappa_1^2 = k_F^2 - \tfrac{1}{4}q_1^2. \quad (4.1)$$

Since n = 2 and s = 1, from (3.13) we find for the singular part

$$J^{(n)}(\mathbf{q}_1, -\mathbf{q}_1)\sim(k_F^2-\tfrac{1}{4}q_1^2)\ln|k_F^2-\tfrac{1}{4}q_1^2|. \quad (4.2)$$

We have arrived at the known singularity, characteristic of the polarization loop (cf., e.g.,[4]).

b) The three-tail diagram. For arbitrary values of the vectors $\mathbf{q}_i$ (see the figure, case b), the rank of the matrix (3.10) is s = 2, and the solution of the system of two equations (3.6) gives

$$\alpha_1=\frac{1}{2}\varkappa_2^2\frac{\varkappa_1^2-\varkappa_1\varkappa_2}{\varkappa_1^2\varkappa_2^2-(\varkappa_1\varkappa_2)^2},\quad \alpha_2=\frac{1}{2}\varkappa_1^2\frac{\varkappa_2^2-\varkappa_1\varkappa_2}{\varkappa_1^2\varkappa_2^2-(\varkappa_1\varkappa_2)^2}.$$

The sum of these quantities is equal to

$$\alpha_1+\alpha_2=1-\frac{\varkappa_1\varkappa_2(\varkappa_1-\varkappa_2)^2}{\varkappa_1^2\varkappa_2^2-(\varkappa_1\varkappa_2)^2}.$$

Hence it follows immediately that the fact that α_3 is positive imposes the restriction

$$\sphericalangle(\varkappa_1\varkappa_2)<\pi/2. \quad (4.3)$$

Bearing in mind the definition (3.2) ($\kappa_1 = \mathbf{q}_1$, $\kappa_2 = -\mathbf{q}_3$) and the arbitrariness in the labeling of the external momenta, we can conclude from (4.3) that a singularity exists only under the condition that the vectors $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ form an acute-angled triangle. In this case, $\alpha_1, \alpha_2 > 0$ automatically, so that the requirement found remains the only one.

We shall now determine κ^2 (3.7):

$$\varkappa^2=\frac{1}{4}\frac{(\varkappa_1-\varkappa_2)^2}{1-(\varkappa_1\varkappa_2)^2/\varkappa_1^2\varkappa_2^2}=\frac{1}{4}\frac{q_2^2}{\sin^2\sphericalangle(\mathbf{q}_1\mathbf{q}_3)}\equiv q_R^2. \quad (4.3')$$

Thus, κ^2 is equal to the square of the radius q_R of the circumscribed circle for the triangle formed by the vectors $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$, and

$$\Delta = k_F^2 - q_R^2, \quad (4.4)$$

i.e., a singularity appears when the vectors $\mathbf{q}_i$ form an acute-angled triangle for which the radius of the circumscribed circle is equal to k_F. Of course, the same condition is obtained directly from (2.8b) and (2.7).

Using (3.14) (s even) for the singular part of the three-tail, we obtain

$$J^{(3)}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)\sim(k_F^2-q_R^2)^{1/2},\quad q_R<k_F. \quad (4.5)$$

(For $q_R > k_F$, the singular part is equal to zero.)

We now consider the degenerate case when two vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ are not linearly independent (are parallel), i.e., when between them there exists a relation of the form

$$\mathbf{q}_2 = \beta\mathbf{q}_1.$$

It is easy to check that the system (3.6) is consistent in this case, but only if $\beta = -1$. Then,

$$\mathbf{q}_2=-\mathbf{q}_1,\quad \mathbf{q}_3=0. \quad (4.6)$$

From (3.6), $\alpha_1 = \frac{1}{2}$. Since $\kappa_2 = 0$, from (3.7) we obtain

$$\varkappa^2 = q_1^2/4. \quad (4.7)$$

Taking into account that s = 1 in this case, and using (3.13), we arrive at the stronger singularity:

$$J^{(3)}(\mathbf{q}_1,-\mathbf{q}_1,0)\sim\ln|k_F^2-q_1^2/4|. \quad (4.8)$$

This amplification is a consequence of the distinctive confluence of the Kohn two-tail singularity, which always exists in a three-tail as a singularity of a diagram of lower order, and the intrinsic singularity of the three-tail diagram.

c) The four-tail diagram. In the general case, when the vectors $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ do not lie in one plane, the matrix (3.10) has the maximum rank s = 3, and the system of three equations (3.6) has a unique solution for α_i, which is found by a standard method. The limitations imposed on the external vectors by the conditions $\alpha_i > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 < 1$ can be analyzed directly in this case, and it is not difficult to verify that these restrictions are less stringent than in the preceding case.

Omitting here the tedious analysis of the connection between κ^2 (3.7) and the external vectors with allowance for the corresponding restrictions, we give only the final result for the behavior of the singular part of the four-tail diagram:

$$J^{(4)}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4)\sim\ln|k_F^2-\varkappa^2|. \quad (4.9)$$

Here κ^2 is found from (3.7) and is the radius of the sphere in which is inscribed a tetrahedron with sides $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, $\mathbf{q}_4$, $\mathbf{q}_1 + \mathbf{q}_2$ and $\mathbf{q}_1 + \mathbf{q}_3$. If the vectors $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ lie in one plane, then there exists a linear relation between them, which leads to a reduction of the rank of the matrix (3.10) to s = 2. In this case, we find from (3.14) a stronger singularity (confluence of the singularities of the four-tail and the three-tail diagram occurs):

$$J^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \sim (k_F^2 - \varkappa^2)^{-1/2}, \quad \varkappa < k_F. \tag{4.10}$$

(At $\kappa > k_F$ there is no singularity.)

A still stronger degeneracy occurs if all the vectors are directed along one straight line. Then s = 1, and the condition for the consistency of Eqs. (3.6) reduces to a strict relation between the vectors q_i for the existence of a singularity:

$$J^{(4)}(\mathbf{q}_1, -\mathbf{q}_1, \mathbf{q}_1, -\mathbf{q}_1) \sim J^{(4)}(\mathbf{q}_1, -\mathbf{q}_1, 0, 0) \sim |k_F^2 - q_1^2/4|^{-1}. \tag{4.11}$$

The four-tail diagram naturally also contains the singularities (4.2) and (4.5) of the lower-order diagrams (reduced diagrams), corresponding to a limit singularity (i.e., to the condition $\alpha_i = 0$ in (2.8) and to the absence of the i-th equation in (3.6) in one or two variables α_j respectively).

d) Multi-tail diagrams with n > 4. The distinguishing feature of this case is the existence of a linear dependence between the vectors κ_i. In three-dimensional space, by choosing as the independent vectors any vectors κ_1, κ_2 and κ_3 not lying in one plane, we have

$$\varkappa_t = \sum_{i=1,2,3} \gamma_{ti}\varkappa_i. \tag{4.12}$$

In this case, the rank of the matrix (3.10) will always be bounded by the value s = 3. Therefore, the determinant of the system (3.6) vanishes (we recall that it is a Gram determinant), and as a result the question arises of whether "leading" singularities can exist at all for n > 4.

For the consistency of the system (3.6), it is necessary that the rank of the expanded matrix including a column with the right-hand side of (3.6) be equal to the rank of the matrix $\| \kappa_i \kappa_k \|$. However, it is not difficult to see that this condition is always fulfilled. In fact, it follows from the system (3.6) that, together with (4.12), the condition

$$\varkappa_t^2 = \sum_{i=1,2,3} \gamma_{ti}\varkappa_i^2, \tag{4.13}$$

is also always fulfilled, and the system (3.6) is found to be consistent. In this case, the solution of (3.6) reduces, as in the case n = 4, to the solution of a system of three equations with three unknowns α_i' in place of α_i, where

$$\alpha_i' = \alpha_i + \sum_{t=4}^{n-1} \alpha_t \gamma_{ti}, \quad i = 1, 2, 3. \tag{4.14}$$

Taking into account that s = 3, we obtain for the singular part of a multi-tail diagram

$$J^{(n)} \sim |k_F^2 - \varkappa^2|^{-(n-4)} \ln |k_F^2 - \varkappa^2|. \tag{4.15}$$

Thus, the existence of the geometric relations (4.12) leads to a sharper buildup of singularities with increasing order of the multi-tail diagram.

We note that the fact that, for arbitrary n, a system of third-order equations always arises means also that, for a given set $\{\kappa_i\}$ satisfying the system of equations for the "leading" singularity, the conditions for the existence of the singularities of the lowest diagrams up to n = 4 are also simultaneously satisfied, i.e., there is always a case of superposition of singularities. From the point of view of second-order surfaces, degenerate surfaces, e.g., cylinders, appear, and correspondingly a singular situation is obtained along whole lines or surfaces in α-space.

It is interesting that (3.16) also admits, in particular, the solution $\alpha_t^0 = 0$ (t > 3). Then $\alpha_i^{0\prime} = \alpha_i^0$, and the requirement on the values of κ, κ_2 and κ_3 strictly coincides with the corresponding condition for the appearance of a singularity of the diagram with n = 4. However, in this case, as we have seen, the nature of the singularity of a multi-tail diagram with n > 4 by nono means coincides with that of the leading singularity, at the same point in the three momenta κ_i, of the diagram with four tails; the former is the stronger. This is connected with the fact that, unlike in the case of a reduced diagram, the expression under the square root in (3.4) will not contain linear terms in the α_i close to the singularity.

In this connection it should be noted that in the case of relativistic diagrams with more than five tails (four-dimensional space!), there is an analogous problem of determining the nature of the singularities when there exists a geometric connection (4.13) between the four-vectors[10]. For real values of the scalars $\kappa_i \cdot \kappa_j$, the Landau singularity corresponds to the requirement that the tips of the four-momenta p_i lie on the corresponding mass surfaces. One can easily check through that an analysis completely analogous to that given in this section can be performed in this case. As a result, it is not difficult to convince oneself that the leading singularity continues to get stronger with increasing n for n > s, in contradiction to what has been stated previously[10,11].

ASYMPTOTIC BEHAVIOR OF THE RING-DIAGRAMS

The representation (3.4) for a ring diagram with n external-field tails turns out to be very convenient for determining the asymptotic behavior of multi-tail diagrams. Let $q_{n-1}, q_n \to \infty$ for two neighboring points. (The condition $\Sigma q_i = 0$ always requires that at least two of the external momenta tend to infinity at the same time.) Then in (3.4), only $\kappa_{n-1} \to \infty$. It follows from the fact that the expression under the square root in (3.4) is positive that $\alpha_{n-1} \sim 1/\kappa_{n-1}^2$, and in the quadratic form we can therefore omit terms containing α_{n-1}. After this, the integration over α_{n-1} in (3.4) is performed trivially:

$$J^{(n)}(\varkappa_1, \ldots, \varkappa_{n-1}) \to \frac{2m^2}{\pi^2} \frac{(-1)^{n-1}}{\varkappa_{n-1}^2} \frac{\partial^{(n-3)}}{\partial \mu^{(n-3)}} \int \ldots \int d\alpha_1 \, d\alpha_{n-2}$$
$$\times \left[k_F^2 + \sum_{i,j=1}^{n-2} \varkappa_i \varkappa_j \alpha_i \alpha_j - \sum_{i=1}^{n-2} \alpha_i \varkappa_i^2 \right]^{1/2} = \frac{(-1)}{(n-1)} \frac{2m}{\varkappa_{n-1}^2} J^{(n-1)}(\varkappa_1, \ldots, \varkappa_{n-2})$$
$$\text{for } \varkappa_{n-1} \to \infty. \tag{5.1}$$

Thus, in this case, the multi-point diagram reduces to a ring diagram of lower order. The asymptotic form in a larger number of momenta is found directly from (5.1).

As an example, we may consider the two-tail diagram $J^{(2)}(q, -q)$, which, according to (5.1), must behave asymptotically as $\sim 1/q^2$. It is not difficult to see that this same result is obtained directly if we use the analytic expression for $J^{(2)}$[4].

6. ANALYSIS OF THE THREE-TAIL DIAGRAM

It is of interest to calculate the ring diagram with three tails in explicit form, and to compare its singularities with those predicted in the framework of the analysis performed above. For this, we shall make use of the expression (3.4) and perform the integration over α_2. We first assume that the values of the momenta q_i are such that $q_R < k_F$, and, consequently, in particular, all $q_i < 2k_F$ (cf. (4.3)). Then no restrictions on the range of integration over the α_i appear, and, performing the trivial integration over α_2, we obtain

$$J^{(3)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \frac{2m^2}{\pi^2}\int_0^1 d\alpha_1 \cdot \tag{6.1}$$

$$\times \cdot \ln \frac{2q_3[k_F^2 - q_1^2\alpha_1(1-\alpha_1)]^{1/2} + \alpha_1(q_2^2 - q_1^2 - q_3^2) + q_3^2}{2q_3[k_F^2 - q_2^2\alpha_1(1-\alpha_1)]^{1/2} + \alpha_1(q_2^2 - q_1^2 + q_3^2) - q_3^2}.$$

The remaining integration over α_1 presents no difficulties, although it leads to cumbersome algebraic transformations. As a result, we obtain

$$J^{(3)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \frac{2m^2}{\pi^2}\frac{q_R^2}{q_1q_2q_3}\sum_m \cos\vartheta_m \ln\left(\frac{2k_F + q_m}{2k_F - q_m}\right)$$

$$-2\left[\left(\frac{k_F}{q_R}\right)^2 - 1\right]^{1/2} \operatorname{Arctg} A\left[\left(\frac{k_F}{q_R}\right)^2 - 1\right]^{1/2}; \tag{6.2}$$

$$A = \frac{q_1q_2q_3}{(2k_F)^3}\left[1 - \frac{1}{2}\frac{q_1^2 + q_2^2 + q_3^2}{(2k_F)^2}\right]^{-1} = \frac{\sin\vartheta_1 \sin\vartheta_2 \sin\vartheta_3}{x^3 - x(1 + \cos\vartheta_1\cos\vartheta_2\cos\vartheta_3)}. \tag{6.3}$$

Here we have introduced the notation: $x = k_F/q_R$, $\vartheta_1 = \pi - \measuredangle(\mathbf{q}_2 \cdot \mathbf{q}_3)$ and analogously for ϑ_2 and ϑ_3; q_R, as everywhere above, is the radius of the circle circumscribing the triangle with sides $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$.

We now formally examine the starting expression (3.4) for fixed q_i (and without restrictions on the sign of the expression under the square root), as a function of the complex variable k_F. In particular, this function will also be defined for all real values of k_F to which corresponds, in a certain range of variation of the α_i, a negative value of the expression under the square root. We make use of the relation, valid everywhere on the real axis,

$$\theta(\psi(x))/\sqrt{\psi(x)} = \operatorname{Re}[1/\sqrt{\psi(x)}]$$

($\theta(y)$ is the unit function). Then, to determine $J^{(3)}$ for all values of the parameters, it is sufficient to perform an analytic continuation of (6.2) into the region of smaller real values of k_F and to take the real part of the expression obtained; clearly, the direction in which we go round the singular points $k_F = q_R$ and $k_F = \pm q_m/2$ plays no role in the answer. As a result, we find the following final expression for the three-tail diagram (we shall take the coefficients into account and give immediately the result for $\Lambda^{(3)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$[1], which is proportional to $J^{(3)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$):

$$\Lambda^{(3)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \frac{2m^2}{3\pi^2\hbar^4}\frac{q_R^2}{q_1q_2q_3}\left\{\sum_m \cos\vartheta_m \ln\left|\frac{2k_F + q_m}{2k_F - q_m}\right| - \Delta\left[\begin{matrix}\ln|(1-\Delta A)/(1+\Delta A)| & \text{for } k_F/q_R < 1\\ 2\operatorname{Arctg}\Delta A & \text{for } k_F/q_R > 1\end{matrix}\right]\right\}, \tag{6.4}$$

where

$$\Delta = |(k_F/q_R)^2 - 1|^{1/2}.$$

For the arc tangent, it is necessary to take the branch $0 \le \tan^{-1} x \le \pi$. This is connected with the fact that, in the process of analytic continuation, even before the first singularity $k_F = q_R$ we pass through a point at which the denominator in A goes to zero, and, consequently, A goes from $+\infty$ to $-\infty$. Clearly, this point is not singular, and so we must choose a branch of the arctangent that is continuous at infinity.

We shall now analyze directly the singularities of this expression. We start with the point $k_F = q_R$ ($x = 1$). At this point, according to (6.3), the sign of $(-A)$ is determined by the sign of the product $\cos\vartheta_1\cos\vartheta_2\cos\vartheta_3$, i.e., is positive if the triangle formed by the vectors $\mathbf{q}_i$ is acute-angled, and negative if it is obtuse-angled. Expanding (6.4) in a series in Δ and taking into account the choice of branch of the arc tangent, it is not difficult to obtain that, in the case of an obtuse-angled triangle, $\Lambda^{(3)}$ at the point under consideration is an analytic function, and in the case of an acute-angled triangle,

$$\Lambda^{(3)}_{\text{sing}} \sim \begin{cases} 0, & q_R > k_F \\ 2\pi\Delta, & q_R < k_F \end{cases} \tag{6.5}$$

The result obtained is in complete agreement with (4.5).

We now examine the behavior of (6.4) close to the point $q_m = 2k_F$ or $x = q_m/2q_R = \sin\vartheta_n$ and $\Delta = |\cos\vartheta_m|$. Transforming (6.3), we find that at this point $\Delta A = -\operatorname{sign}(\cos\vartheta_m)$. Hence it follows that at the point $q_m = 2k_F$ the second term in (6.4) is also singular, and it is not difficult to see that the logarithmically divergent terms in the first and second terms cancel exactly, while the remaining terms give a Kohn singularity of the form (4.2), corresponding to the statement that a given diagram must also contain the singularities of diagrams of lower order (reduced diagrams).

Finally, one more characteristic singularity of the expression (6.4) occurs as $q_3 \to 0$ and $q_1 = q_2 \to 2k_F$. In this case, $q_R \to k_F$ and from (6.4) we find directly

$$\Lambda^{(3)}_{\text{sing}}(\mathbf{q}_1, -\mathbf{q}_1, 0) = \frac{m^2}{6\pi^2\hbar^4}\frac{1}{k_F}\ln\left|\frac{q_1 + 2k_F}{q_1 - 2k_F}\right|. \tag{6.6}$$

This result agrees completely with the result (4.8) of the general analysis.

In conclusion, we note that, according to the work of[1], there exists a simple relation between $\Lambda^{(n)}$ and the multi-tail diagrams $\Gamma^{(n)}$ (cf.[1,2]) occurring in the general expression for the energy. In particular,

$$\Gamma^{(3)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \Lambda^{(3)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)/\varepsilon(\mathbf{q}_1)\varepsilon(\mathbf{q}_2)\varepsilon(\mathbf{q}_3). \tag{6.7}$$

The expressions (6.7) and (6.4) have indeed been used by us in concrete calculations of the energy of a metal (cf., e.g.,[12]). In particular, they also result from the direct calculation of the third-order term in the expansion of the energy in powers of the effective electron-ion interaction:

$$\Lambda^{(3)}(\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3) = 2\int\frac{d\mathbf{p}}{(2\pi)^3}n_{\mathbf{p}}\left\{\frac{1}{(\varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{p}+\mathbf{q}_1})(\varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{p}-\mathbf{q}_2})} + \frac{1}{(\varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{p}-\mathbf{q}_1})(\varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{p}+\mathbf{q}_3})} + \frac{1}{(\varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{p}+\mathbf{q}_2})(\varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{p}-\mathbf{q}_3})}\right\},$$

which was given in this form in[2], and also in the paper of Lloyd and Sholl[13]. A calculation of this expression, leading to a result analogous to (6.4), was first published in[13].

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