EXCITATIONS IN THE ONE-DIMENSIONAL HUBBARD MODEL

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The magnetic and electronic properties of a one-dimensional Hubbard chain with an arbitrary electron concentration n are investigated. Analytic expressions for the magnetic susceptibility, and also for the energy spectra, are obtained for the case when $n/c \ll 1$.

1. INTRODUCTION

RECENTLY there has been a great deal of interest in exactly solvable one-dimensional problems. Their investigation is not only helpful in order to understand the physics of phenomena in the usual three-dimensional case, but also because objects exist which can be treated as one-dimensional solids. The complex compounds of tetracyanoquinodimethan (TCNQ), whose crystal structure contains filaments of the TCNQ molecules weakly bound to each other, are an example of such substances. Unpaired electrons can travel along these filaments.

It appears that one can explain the properties of TCNQ compounds by using the one-dimensional Hubbard model.[1] This model is described by the Hamiltonian

$$\mathcal{H} = -\sum_{\langle ij\rangle,\sigma} c_{i\sigma}^{+}c_{j\sigma} + 2c\sum_{i} c_{i\uparrow}^{+}c_{i\uparrow}c_{i\downarrow}^{+}c_{i\downarrow}. \tag{1.1}$$

In Eq. (1.1) $c_{i\sigma}$ ($c_{i\sigma}^{+}$) is the annihilation (creation) operator for an electron of spin component σ at the i-th lattice site. The sum over i and j in the first term runs over nearest-neighbor sites.

If the number of electrons N coincides with the number of lattice sites N_a, then for $c \gg 1$ in second-order perturbation theory with respect to the kinetic energy, the Hamiltonian (1.1) reduces to an effective Hamiltonian of the Heisenberg type:

$$\mathcal{H}_{\text{eff}} = \frac{2}{c}\sum_{\langle ij\rangle} \mathbf{S}_i\mathbf{S}_j, \tag{1.2}$$

where $\mathbf{S}_i$ is the spin operator ($S = \frac{1}{2}$) at the i-th lattice site.

The magnetic properties of a one-dimensional antiferromagnetic chain—the types of excited states, the susceptibility, and the shape of the spectra—have been investigated in a number of articles.[2-5] It should have been anticipated that the magnetic properties of the one-dimensional Hubbard model with $n = N/N_a = 1$ are similar to the same properties of the Heisenberg antiferromagnetic chain. In fact, as was shown by Ovchinnikov,[6] if one disregards the electronic spectrum, which has a gap at n = 1, the magnetic properties of one-dimensional chains described by the Hamiltonians (1.1) and (1.2) are similar. Both in this case and in the other case the long wavelength part of the spectrum of the magnetic excitations is linear with respect to the wave vector q, and the spectrum itself possesses a double periodicity. The magnetic susceptibility for n = 1 was found by Takahashi.[7] It also has an antiferromagnetic nature.

The exact solution for the system described by the Hamiltonian (1.1) was obtained by Lieb and Wu[8] with the aid of a method first used by Yang;[9] this method consists in the following. The quasimomenta k_j ($j = 1, 2, \ldots, N$) of the electrons and the "spin" momenta Λ_α ($\alpha = 1, 2, \ldots, M$) are introduced, where M denotes the number of inverted spins. The electron and spin momenta are related by the following system of equations:

$$N_a k_j = 2\pi I_j - 2\sum_{\beta=1}^{M} \operatorname{arctg}\left(\frac{2\sin k_j - 2\Lambda_\beta}{c}\right), \quad j = 1, 2, \ldots, N, \tag{1.3}$$

$$2\sum_{j=1}^{N} \operatorname{arctg}\left(\frac{2\Lambda_\alpha - 2\sin k_j}{c}\right) = 2\pi J_\alpha + 2\sum_{\beta=1}^{M} \operatorname{arctg}\left(\frac{\Lambda_\alpha - \Lambda_\beta}{c}\right) \tag{1.4}$$
$$\alpha = 1, \ldots, M,$$

where J_α are integers and I_j are integers or half-odd integers. Here the energy of the Hubbard chain is given by

$$E = -2\sum_{j=1}^{N} \cos k_j, \tag{1.5}$$

and the momentum can be expressed in terms of J_α and I_j in the form

$$q = \frac{2\pi}{N_a}\left(\sum_{j=1}^{N} I_j + \sum_{\alpha=1}^{M} J_\alpha\right), \tag{1.6}$$

For the ground state of the system with a fixed spin $S = (\frac{1}{2})(N - 2M) = sN_a$ the numbers J_α and I_j are equidistantly spaced with their differences given by

$$\Delta J_\alpha = J_{\alpha+1} - J_\alpha = 2, \quad \Delta I_j = I_{j+1} - I_j = 1$$

and they are centered around the origin.

After introducing the "electronic" density $\rho(k) = (N_a \Delta k)^{-1}$ and the "spin" density $\sigma(\Lambda) = (N_\alpha \Delta\Lambda)^{-1}$, instead of Eqs. (1.3) and (1.4) we obtain two integral equations for the determination of $\rho(k)$ and $\sigma(\Lambda)$:

$$2\pi\rho(k) = 1 + \cos k \int_{-B}^{B} \frac{4c\sigma(\Lambda)\,d\Lambda}{c^2 + 4(\sin k - \Lambda)^2}, \tag{1.7}$$

$$\int_{-Q}^{Q} \frac{4c\rho(k)\,dk}{c^2 + 4(\sin k - \Lambda)^2} = 2\pi\sigma(\Lambda) + \int_{-B}^{B} \frac{2c\sigma(\Lambda')\,d\Lambda'}{c^2 + (\Lambda - \Lambda')^2}. \tag{1.8}$$

In addition, the electronic and spin densities are normalized, respectively, to the electron density and the density of inverted spins:

$$\int_{-Q}^{Q}\rho(k)\,dk=\frac{N}{N_a}=n,\qquad \int_{-B}^{B}\sigma(\Lambda)\,d\Lambda=\frac{M}{N_a}=m.$$

The energy of the system is expressed in terms of only the function $\rho(k)$ and has the following form:

$$\varepsilon=\frac{E}{N_a}=-2\int_{-Q}^{Q}\rho(k)\cos k\,dk.$$

The limits of integration Q and B in Eqs. (1.7) and (1.8) are monotonically increasing functions of n and m, respectively, and vary within the following limits: $0\le Q\le\pi$ for $0\le n\le 1$ and $0\le B<\infty$ for $0\le m\le n/2$. When $n=1$ the limit $Q=\pi$ and the system of integral equations (1.7)–(1.8) can be reduced to a single equation for the density $\sigma(\Lambda)$.

In the present article we shall be interested in the magnetic susceptibility of the chain, and also in the spectra associated with magnetic and electronic excitations for any arbitrary electron concentration $n<1$. In the case $n>1$, due to the symmetry of the Hamiltonian (1.1) with respect to the interchange particle → hole, the results of the work will remain valid as usual, provided we make the substitution $n\to 2-n$.

2. THE MAGNETIC SUSCEPTIBILITY

In this section we investigate the behavior of the magnetic susceptibility for an arbitrary electron density in a zero field. The case of fields close to the saturation fields has been investigated by Takahashi.[10]

From the well-known theorem about one-dimensional systems by Lieb and Mattis,[11] the ground state of the Hubbard model corresponds to total spin $S=0$. In this case the limit of integration B in the integral equations (1.7) and (1.8) is equal to infinity. In the presence of a small magnetic field h, the total spin of the state of lowest energy is also small. In this connection B is no longer equal to infinity, but remains large. This allows one to expand the energy of the state with small spin in a series in powers of s, as is done in[3,12]. We shall confine our attention to the leading term of the expansion, which is proportional to s^2.

The solution of Eqs. (1.7) and (1.8) for an arbitrary electron concentration can be systematically developed within the framework of perturbation theory in powers of the parameter n/c. In order to do this, it is necessary to expand the kernels of Eqs. (1.7) and (1.8) in even powers of the ratio $(\sin k)/c$. In this connection the energy of the state with spin S has the form

$$\varepsilon(s)-\varepsilon_0=s^2\sum_{l=0}^{\infty}\beta_l\left(\frac{n}{c}\right)^l.$$

We shall assume that the parameter n/c is small and retain only the first term in the expansion of the function $\epsilon(s)$ in powers of n/c. This corresponds to neglecting the presence of $\sin k$ in the kernels of the integral equations (1.7) and (1.8), which can now be written as follows:

$$2\pi\rho(k)=1+\pi\frac{n}{c}\cos k\int_{-A}^{A}\Phi(x)\,g_0(x)\,dx, \tag{2.1}$$

$$\Phi(x)=g_0(x)-\int_{-A}^{A}K(x-x')\Phi(x')\,dx', \tag{2.2}$$

$$x=\frac{2\Lambda}{c},\quad A=\frac{2B}{c},\quad \sigma\left(\frac{2\Lambda}{c}\right)=\frac{n}{c}\Phi(x),\quad K(x)=\frac{1}{2\pi}\frac{1}{x^2+4}$$

$$g_0(x)=\frac{2}{\pi}\frac{1}{x^2+1}.$$

In Eqs. (2.1) and (2.2) it would appear that we must be interested in the function $\Phi(x)$ with $-A\le x\le A$. However, it is more convenient to express all of the physically interesting quantities, for example, the spin s and the energy ϵ with the aid of the tail of the spin density, that is, with $\Phi(x)$ for $|x|>A$. This method was applied by Griffiths[3] in order to determine the susceptibility of a one-dimensional antiferromagnetic chain.

Let us multiply both sides of Eq. (2.2) by the resolvent kernel $J(x-x')$, whose Fourier component is given by

$$J(\omega)=-1+(1+K(\omega))^{-1}=-(1+e^{2|\omega|})^{-1},$$

and let us integrate from $-\infty$ to $+\infty$. In this connection, one obtains the following equation for the tail of the spin density, $\Phi(x>A)$:

$$\Phi(x)=\Phi_0(x)-\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right]J(x-x')\Phi(x')\,dx'$$

$$=\Phi_0(x)-\int_{A}^{\infty}[J(x-x')+J(x+x')]\Phi(x')\,dx', \tag{2.3}$$

$$\Phi_0(x)=e^{-\pi x/2}\sum_{r=0}^{\infty}(-1)^r e^{-\pi r x}. \tag{2.4}$$

(here and in what follows, the subscript 0 indicates that the function is determined from the equations for the ground state).

Since $J(x-x')$ is proportional to $(x-x')^{-2}$ for $|x-x'|\gg 1$ and we are only interested in the values of $\Phi(x)$ for $x>A$, one can neglect in Eq. (2.3) the integral

$$\int_{A}^{\infty}J(x+x')\Phi(x')\,dx',$$

which is of the order of $A^{-2}\Phi(x)$. This neglect corresponds to the fact that in the expression for the energy $\epsilon(s)$ of a state with a given small value of the spin s, terms proportional to $s^2(\ln s)^{-1}$ have been omitted in comparison with the retained term proportional to s^2. In addition, in the summation (2.4) we keep only the single term with $r=0$. This implies that the terms in $\epsilon(s)$ proportional to s^4, s^6, etc. are small in comparison with the term proportional to s^2.

After these simplifying remarks, Eq. (2.3) takes the form

$$\Phi(x)=\Phi_0(x)-\int_{A}^{\infty}J(x-x')\Phi(x')\,dx'. \tag{2.5}$$

We shall seek the solution of Eq. (2.5) in the form

$$\Phi(x)=e^{-\pi A/2}p(y),\quad y=x-A.$$

Then we obtain an equation of the Wiener-Hopf type for p(y):

$$p(y)=e^{-\pi y/2}-\int_{0}^{\infty}J(y-y')p(y')\,dy'.$$

The quantities of interest to us, namely, the spin s and the energy $\epsilon(s)$, can be obtained if we integrate Eq. (2.3), multiplied by unity and by $g_0(x)$, respectively, from $-\infty$ to $+\infty$. Here s and $\epsilon(s)$ are expressed in terms of the tail of the density, $\Phi(x)$ for $x > A$, and consequently in terms of p(y):

$$s = e^{-\pi A/2}\frac{n}{2}\int_0^\infty p(y)\,dy, \tag{2.6}$$

$$\varepsilon(s)-\varepsilon_0 = -4\rho_0(Q_0)\cos Q_0\,\Delta Q + \frac{2n}{c}e^{-\pi A}\int_0^\infty e^{-\pi y/2}p(y)\,dy\int_{-Q_0}^{Q_0}\cos^2 k\,dk. \tag{2.7}$$

The unknown quantity ΔQ appears in Eq. (2.7). However, one can determine it from conservation of the total number of electrons, namely,

$$\int_{-Q_0}^{Q_0}\rho_0(k)\,dk = \int_{-Q_0-\Delta Q}^{Q_0+\Delta Q}\rho(k)\,dk$$

or

$$4\rho_0(Q_0)\cos Q_0\Delta Q = \frac{2n}{c}\int_{-Q_0}^{Q_0}\cos k\cos Q_0\,dk\,e^{-\pi A}\int_0^\infty e^{-\pi y/2}p(y)\,dy.$$

Therefore

$$\varepsilon(s)-\varepsilon_0 = \frac{2n}{c}e^{-\pi A}\int_0^\infty e^{-\pi y/2}p(y)\,dy\int_{-Q_0}^{Q_0}\cos k(\cos k-\cos Q_0)\,dk.$$

Since the spin s is proportional to $e^{-\pi A/2}$ (for $A \gg 1$), and $\epsilon(s) - \epsilon_0 \sim e^{-\pi A}$, then

$$\varepsilon(s)-\varepsilon_0 = Cs^2\frac{8}{nc}\left(Q_0-\frac{1}{2}\sin 2Q_0\right), \tag{2.8}$$

$$C = \int_0^\infty e^{-\pi y/2}p(y)\,dy\Big/\left[\int_0^\infty p(y)\,dy\right]^2. \tag{2.9}$$

The ratio (2.9) is known from the article by Yang and Yang.[12] Its value is equal to $\pi/8$. The limiting momentum is given by $Q = \pi n$, since in our approximation $\rho_0(k) = (2\pi)^{-1}$.

Now it is not difficult to obtain the magnetic susceptibility of the Hubbard chain. In order to determine this quantity, it is necessary to minimize the energy of the system of electrons in a magnetic field:

$$\varepsilon_h = \varepsilon(s) - \mu h s.$$

And finally we obtain

$$\chi = \frac{dM}{dh} = N_a\mu^2\left(\frac{d^2\varepsilon(s)}{ds^2}\Big|_{s=0}\right) = N_a\mu^2\frac{c}{2\pi^2}\left(1-\frac{\sin 2\pi n}{2\pi n}\right)^{-1}. \tag{2.10}$$

In the small-density limit ($n \ll 1$) one finds

$$\chi = N_a\mu^2 3c/4\pi^4 n^2,$$

and in the case $n = 1$ one obtains Takahashi's well-known result[7] from Eq. (2.10):

$$\chi = N_a\mu^2 c/2\pi^2 \quad (c \gg 1).$$

3. THE SPECTRA OF THE ELEMENTARY EXCITATIONS

In this section we shall be interested in magnetic and electronic excitations in the Hubbard model. The elementary magnetic excitations are characterized by the spin (triplet or singlet excitations). The electronic spectrum does not have a gap for $n < 1$ and is similar to the spectrum of a Fermi gas. One can obtain an elementary spin excitation by removing any single number from the set of numbers $\{J_\alpha\}$, which defines the ground, antiferromagnetic state of the system. The new state is a "wave" and corresponds to total spin $S = 1$ (a triplet excitation). It is obvious that upon the withdrawal of the numbers J_1 or $J_{N/2}$ we obtain the ground state for spin $S = 1$. It was shown in the preceding section that $\Delta E(S) = E(S) - E_0 \sim N_a S^2/N_a^2$, and therefore $\Delta E(S) \sim 1/N_a$. Therefore, the spectrum of the triplet excitations does not have a gap. The change in the total momentum of the electron system, upon making the transition from the set $\{J_\alpha\}$ without the number J_1 to the same set without $J_{N/2}$, is equal to πn. Therefore, the period in the triplet-excitation spectrum is also equal to πn.

If we now transpose some of the numbers J_α so that the distance between any pair of neighboring numbers in the resulting set $\{J_\alpha\}'$ becomes smaller than two, then the state which is formed is related to the bound state type and has spin $S = 0$ (a singlet excitation). The spectrum of such elementary excitations is also gapless; however it has a termination point.

Fermion excitations corresponding to spin $S = \frac{1}{2}$ are obtained is one more number I_{N+1} (particle) is added to the set of numbers $\{I_j\}$ or if any I_j (hole) is removed.

Let us proceed to a quantitative description of the spectra. We define the sets of numbers $\{I_j\}_0$ and $\{J_\alpha\}_0$ for the ground, antiferromagnetic state in the following way:

$$\{I_j\}_0 = -\tfrac{1}{2}N+1,\ -\tfrac{1}{2}N+2,\ \ldots,\ \tfrac{1}{2}N;$$
$$\{J_\alpha\}_0 = 1, 3, \ldots, N-1.$$

The elementary excitations are superficially characterized by the changes in the sets $\{I_j\}_0$ and $\{J_\alpha\}_0$:

a) triplet excitations (see, for example,[4-6]):

$$\{I_j\}_t = \{I_j\}_0,$$
$$\{J_\alpha\}_t = 0, 2, \ldots, 2p-2, 2p+1, \ldots, N-1;$$

b) singlet excitations:[5,6]

$$\{I_j\}_s = \{I_j\}_0,$$
$$\{J_\alpha\}_s = 1, 3, \ldots, 2\beta_1-1, 2\beta_1-1, \ldots, 2\beta_2-3, 2\beta_2+1, \ldots, N-1;$$

c) electronic excitations:

$$\{I_j\}_e = -\tfrac{1}{2}N, -\tfrac{1}{2}N+1, \ldots, -\tfrac{1}{2}N+\gamma-1,\ -\tfrac{1}{2}N+\gamma+1, \ldots, N/2,$$
$$\{J_\alpha\}_e = \{J_\alpha\}_0.$$

As has already been indicated above, the triplet and electronic excitations are "waves," that is, the momenta k_j and Λ_α corresponding to these excitations are real. A single pair of complex-conjugate spin momenta exists in the bound state (singlet): $\Lambda_a = \lambda + ic/2$ and $\Lambda_b = \lambda - ic/2$.

The method used to derive the equations for the functions $\rho(k)$ and $\sigma(\Lambda)$ of the excited states is well-known (compare with[4-6]). In all three cases one can write these equations down in the following form:

$$2\pi\rho(k) = 1+\cos k\int_{-\infty}^{+\infty}\frac{4c\sigma(\Lambda)\,d\Lambda}{c^2+4(\Lambda-\sin k)^2} - \frac{1}{N_a}T(k), \tag{3.1}$$

$$\int_{-Q}^{Q}\frac{4c\rho(k)\,dk}{c^2+4(\sin k-\Lambda)^2} = 2\pi\sigma(\Lambda)+\int_{-\infty}^{+\infty}\frac{2c\sigma(\Lambda')\,d\Lambda'}{c^2+(\Lambda-\Lambda')^2}+\frac{D(\Lambda)}{N_a}, \tag{3.2}$$

where for the triplet spectrum one has

$$T_t(k)=0,\quad D_t(\Lambda)=2\pi\sigma(\Lambda-\Lambda_p),$$

for the singlet spectrum one has

$$T_s(k)=\cos k\Big\{2\pi\delta(\sin k-\lambda)-\frac{2c}{c^2+(\sin k-\lambda)^2}+$$
$$+\sum_{m=1}^{2}\Big[\frac{4c}{c^2+4(\overline{\Lambda}_{\beta_m}-\sin k)^2}-2\pi\delta(\sin k-\overline{\Lambda}_{\beta_m})\Big]\Big\},$$
$$D_s(\Lambda)=-4\pi\delta(\Lambda-\lambda)+\frac{4c}{c^2+4(\Lambda-\lambda)^2}+$$
$$+\frac{12c}{9c^2+4(\Lambda-\lambda)^2}+\sum_{m=1}^{2}\Big[2\pi\delta(\Lambda-\overline{\Lambda}_{\beta_m})-\frac{2c}{c^2+(\Lambda-\overline{\Lambda}_{\beta_m})^2}\Big]$$

and for the electronic spectrum one has

$$T_e(k)=2\pi\delta(k-k_\gamma),\quad D_e(\Lambda)=0.$$

The momenta k_γ and Λ_p in the functions $T(k)$ and $D(\Lambda)$ are equal to their unperturbed values, that is, they are found from Eqs. (1.3) and (1.4) by using the sets $\{I_j\}_0$ and $\{J_\alpha\}_0$.

Two additional numbers, $\overline{\Lambda}_{\beta_1}$ and $\overline{\Lambda}_{\beta_2}$, are added to the system of real numbers Λ_α in deriving the equations for the singlet excitations; these additional numbers satisfy the same equations as the numbers Λ_α for $J_{\beta_1}=2\beta_1-1$ and $J_{\beta_2}=2\beta_2-1$. Here the function $\sigma(\Lambda)$ denotes the density of the real numbers Λ_α together with $\overline{\Lambda}_{\beta_1}$ and $\overline{\Lambda}_{\beta_2}$. Just as in[5,6], the constraints

$$|\lambda|>1,\quad |\overline{\Lambda}_{\beta_1}|>1,\quad |\overline{\Lambda}_{\beta_2}|>1. \tag{3.3}$$

are imposed on the spin momenta $\overline{\Lambda}_{\beta_m}$ and λ. The condition for solvability of the system of equations for the numbers Λ_α leads to the relationship $2\beta_1=\beta_2$.

In order for us to determine the energy of the excited states, it is necessary to know the density $\rho(k)$ which is given by the solution of Eqs. (3.1) and (3.2). We shall seek the solution in the form

$$\rho_t(k)=\rho_0(k)+\frac{1}{N_a}\rho_{1t}(k),$$
$$\rho_s(k)=\rho_0(k)+\frac{1}{N_a}\rho_{1s}(k)-\frac{1}{N_a}\Big[\delta(\sin k-\lambda)-\sum_{m=1}^{2}\delta(\sin k-\overline{\Lambda}_{\beta_m})\Big]\cos k,$$
$$\rho_e(k)=\rho_0(k)+\frac{1}{N_a}(\rho_{1e}(k)-\delta(k-k_\gamma)),$$

where $\rho_0(k)$ denotes the electron density in the ground state. Here it is obvious that the integral equations for $\rho_0(k)$ and $\rho_1(k)$ are separable, and we obtain the following equation for $\rho_1(k)$:

$$\rho_1(k)=g(k)+\frac{1}{2\pi}\cos k\int_{-\infty}^{+\infty}d\omega\int_{-Q_0}^{Q_0}\frac{\rho_1(k')\exp[i\omega(\sin k-\sin k')]}{1+e^{|\omega|c}}dk'.$$
$$g_t(k)=-\frac{1}{2\pi}\cos k\int_{-\infty}^{+\infty}\frac{\exp[i\omega(\sin k-\Lambda_p)]}{2\operatorname{ch}(\omega c/2)}d\omega, \tag{3.4}$$
$$g_s(k)=-\frac{1}{2\pi}\cos k\int_{-\infty}^{+\infty}e^{i\omega\sin k}\Big(\sum_{m=1}^{2}\exp(-i\omega\overline{\Lambda}_{\beta_m})-e^{-i\omega\lambda}\Big)\frac{d\omega}{\operatorname{ch}(\omega c/2)},$$
$$g_e(k)=-\frac{1}{2\pi}\cos k\int_{-\infty}^{+\infty}\frac{\exp[i\omega(\sin k-\sin k_\gamma)]}{1+e^{|\omega|c}}d\omega.$$

In the case $n/c\ll 1$ one can neglect the second term on the right-hand side of Eq. (3.4), and then $\rho_1(k)$ will simply coincide with $g(k)$. Now it is easy to write down an expression for the energy of the excited states which depends on the electron and spin momenta k_γ, Λ_p, and $\overline{\Lambda}_{\beta_m}$. The latter, in turn, are related to the total wave vector of the electrons. Such a parametric dependence of the excitation energy on the wave vector has the following form:

a) triplet spectrum:

$$\varepsilon_t=\frac{1}{c}\int_{-Q_0}^{Q_0}\frac{\cos^2 k\,dk}{(\Lambda_p-\sin k)\operatorname{ch}(\pi/c)},$$
$$|q|=2\pi\int_{-\infty}^{\Lambda_p}\sigma_0(\Lambda)\,d\Lambda,\qquad |q|\leqslant\pi n;$$

b) singlet spectrum

$$\varepsilon_s=\frac{2}{c}\int_{-Q_0}^{Q_0}\frac{\cos^2 k\,dk}{(\overline{\Lambda}_{\beta_1}-\sin k)\operatorname{ch}(\pi/c)},$$
$$|q|=4\pi\int_{-\infty}^{\overline{\Lambda}_{\beta_1}}\sigma_0(\Lambda)\,d\Lambda,$$
$$|q|\leqslant q_0=4\pi\int_{-\infty}^{-1}\sigma_0(\Lambda)\,d\Lambda,$$

here q_0 is the termination point of the singlet spectrum (as a consequence of conditions (3.3));

c) electronic spectrum:

$$E(k_\gamma,N-1)-E_0(N)=2\cos k_\gamma+\int_{-Q_0}^{Q_0}\frac{2}{\pi}\cos^2 k\,dk$$
$$\times\int_0^\infty\frac{\cos[\omega(\sin k_\gamma-\sin k)]}{1+e^{\omega c}}d\omega$$
$$|q|=2\pi\int_{-Q_0}^{k_\gamma}\rho_0(k)\,dk.$$

In the case $n/c\ll 1$ which we are investigating, one can change from a parametric representation of the spectra to an explicit expression for the excitation energy as a function of the system's quasimomentum. In order to do this, it is sufficient to keep the following terms in the expansions of $\rho_0(k)$, $\sigma_0(\Lambda)$, and Q_0 in powers of the small parameter:

$$\rho_0(k)=\frac{1}{2\pi}+\frac{n\ln 2}{\pi c}\cos k,$$
$$\sigma_0(\Lambda)=\frac{n}{2c}\Big(\operatorname{ch}\frac{\pi\Lambda}{c}\Big)^{-1},\qquad Q_0=\pi n.$$

Finally the spectra of the magnetic excitations can be written in the following form:

$$\varepsilon_t(q)=\frac{1}{c}\int_{-\pi n}^{\pi n}\cos^2 k\operatorname{ch}^{-1}\Big[\ln\operatorname{tg}\frac{|q|}{2n}-\frac{\pi}{c}\sin k\Big]dk, \tag{3.5}$$
$$\varepsilon_s(q)=\frac{2}{c}\int_{-\pi n}^{\pi n}\cos^2 k\operatorname{ch}^{-1}\Big[\ln\operatorname{tg}\frac{|q|}{4n}-\frac{\pi}{c}\sin k\Big]dk, \tag{3.6}$$

where the singlet spectrum has the termination point

$$q_0=4n\operatorname{arctg}(e^{-\pi/c}). \tag{3.7}$$

Just as one would expect, in the long-wavelength approximation the excitation energy is proportional to $|q|$:

$$\varepsilon_s(q)=\varepsilon_t(q)=|q|\pi\Big(1-\frac{\sin 2\pi n}{2\pi n}\Big)\Big/c. \tag{3.8}$$

And finally let us say a few words about the electronic spectrum. In the same way as the magnetic excitations, the electronic spectrum does not have an energy gap

for $n \neq 1$.[8] This is clear from the fact that

$$E_0(N+1)-E_0(N)=E_0(N)-E_0(N-1)$$
$$=2\cos\pi n+\frac{2}{\pi}\int_{-\pi n}^{\pi n}\cos^2 k\,dk\int_0^\infty\frac{\cos[\omega(\sin\pi n+\sin k)]}{1+e^{\omega c}}d\omega. \quad (3.9)$$

One of the most important characteristics of the electronic spectrum is the Fermi velocity, which is given by

$$v_F=2\sin\pi n+\cos\pi n\int_{-\pi n}^{\pi n}\frac{2}{\pi}\cos^2 k\,dk\int_0^\infty\frac{\omega\sin[\omega(\sin k+\sin\pi n)]}{1+e^{\omega c}}d\omega. \quad (3.10)$$

The Fermi velocity vanishes for $n = 1$, just as it should.

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