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AN INVESTIGATION OF GRAVITATIONAL FIELDS IN AN ANISOTROPIC MODEL WITH MATTER AND NEUTRINOS

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An anisotropic model of the Bianchi type I is considered within the framework of general relativity, assuming that the sources of the gravitational field are matter with the equation of state $p = (\gamma - 1)e$ and free neutrinos. A covariant Dirac equation for four-component neutrinos is used to describe the two fluxes of two-component neutrinos. A qualitative analysis of the Einstein equations is given for the case of axial symmetry and different equations of state. Analytic expressions are obtained for the case of the ultrarelativistic equation of state $e = 3p$, as well as for $e = p$. The influence of free neutrino fluxes on the dynamics is discussed.

1. INTRODUCTION

RELATIVISTIC cosmology in the general theory of relativity (GTR) studies homogeneous anisotropic models, where during the early stages the dynamics of the universe is substantially anisotropic, and which in some cases become asymptotically isotropic. In these models one has to consider as sources of the gravitational field, in addition to matter with isotropic pressure, also weakly interacting particles[1,2]. In this paper we consider gravitational fields in a type I anisotropic model according to the Bianchi classification (with a three-dimensional Euclidean co-moving space) with matter and neutrinos moving freely along one of the axes. During the early stages of evolution of the "hot" model of the Universe, dissipative processes with the participation of neutrinos are important; for the description of such processes it is necessary to take into account in the energy-momentum tensor the contribution related to such processes[1-6]. In the present paper this contribution is not considered and the influence of the expansion of free neutrinos on the dynamics is studied, free neutrinos which appear after the neutrinos are separated from the matter, according to the ideas developed in[1].

The metric in the type I Bianchi model has the form

$$ds^2 = (cd\tau)^2 - R_1^2(\tau)(dx^1)^2 - R_2^2(\tau)(dx^2)^2 - R_3^2(\tau)(dx^3)^2 \quad (1.1)$$

(the notations are those of[7], c is the velocity of light) with functions $R_\alpha(\tau)$, $\alpha = 1, 2, 3$ which have the representation

$$R_\alpha = Re^{\beta_\alpha}, \quad |R| = e^{-\Omega}; \quad \beta_1 = \beta, \quad \beta + \beta_2 + \beta_3 = 0. \quad (1.2)$$

We shall assume that the system (1.1) accompanies the matter, so that the 4-velocity of the matter $u^i = \delta^i_0$, the expansion scalar is (the dot means differentiation with respect to the time τ)

$$\Theta = \dot{R}/R = -\dot{\Omega}, \quad (1.3)$$

and the nonvanishing components of the displacement tensor

$$q_{ik} = 2^{-1}c(u_{i;k} + u_{k;i}) - \Theta(g_{ik} - u_iu_k)$$

have the form

$$q_1{}^1 = \dot{\beta}, \quad q_2{}^2 = \dot{\beta}_2, \quad q_3{}^3 = \dot{\beta}_3, \quad q^2 = q^{ik}q_{ik} = \dot{\beta}^2 + \dot{\beta}_2{}^2 + \dot{\beta}_3{}^2. \quad (1.4)$$

In the Einstein equations of GTR

$$R_i{}^k = (8\pi k/c^4)[T_i{}^k - {}^1/{}_2T\delta_i{}^k] \quad (1.5)$$

the energy-momentum tensor is the sum of the appropriate tensors for matter with isotropic pressure and for the neutrinos $(T_{ik})^N$ (Sec. 2):

$$T_i{}^k = (e+p)u_iu^k - p\delta_i{}^k + (T_i{}^k)^N. \quad (1.6)$$

We shall use an equation of state for matter in the form

$$p = (\gamma - 1)e, \quad (1.7)$$

where p is the pressure and e is the energy density. The constant γ may have values in the interval from $\gamma = 1$ (matter in the form of dust, $p = 0$) to $\gamma = 2$ (matter with an extremely rigid equation of state, with the velocity of sound equal to the velocity of light). For $1 \leq \gamma < 2$ the axially symmetric case is considered qualitatively. This case corresponds to taking in (1.1), (1.2), (1.4) the values

$$R_1 = Re^{\beta}, \quad R_2 = R_3 = Re^{-\beta/2}, \quad q^2 = 3\beta^2/2. \tag{1.1a}$$

For the equation of state $e = 3p$ we have been able to obtain analytic solutions. For $\gamma = 2$ the case of the metric (1.1) is considered in Sec. 3.

2. SPINORS IN THE ANISOTROPIC MODEL

In order to describe simultaneously the two fluxes of two-component neutrinos, we consider the bispinor ψ, formed by the two two-component spinors ξ and η[8]:

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_{\dot 1} \\ \eta_{\dot 2} \end{pmatrix}. \tag{2.1}$$

For ψ we have the Dirac equation in its covariant form[9,10]:

$$\gamma^i \nabla_i \psi = 0; \quad \nabla_i \psi = \frac{\partial \psi}{\partial x^i} - \Gamma_i \psi, \quad \nabla_i \bar\psi = \frac{\partial \bar\psi}{\partial x^i} + \bar\psi \Gamma_i \tag{2.2}$$

In an orthonormal frame with the Pfaffian forms θ^p (p, q vary from 0 to 3; $\tilde g_{pq} = \mathrm{diag}(+1, -1, -1, -1)$) we have[10]

$$ds^2 = g_{ik} dx^i dx^k = \tilde g_{pq} \theta^p \theta^q, \quad dx^k = a_p{}^k \theta^p, \quad \theta^p = b_k{}^p dx^k, \tag{2.3}$$
$$\gamma^k = a_p{}^k \tilde\gamma^p, \quad \tilde\gamma^p \tilde\gamma^q + \tilde\gamma^q \tilde\gamma^p = 2\tilde g^{pq} I, \quad \Gamma_i = {}^1\!/_4 (a_p{}^l \partial b_k{}^p / \partial x^i - \Gamma_{ik}{}^l) \gamma_l \gamma^k.$$

We select for $\tilde\gamma^p$ the spinor representation with the Pauli matrices $\sigma_{1,2,3}$[8]:

$$\tilde\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \tilde\gamma^1 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad \tilde\gamma^2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \tilde\gamma^3 = \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}. \tag{2.4}$$

Here η and ξ satisfy the Weyl equations for the neutrino and antineutrino, respectively (or, more generally, of two types).

For the solution (2.2) it is convenient to represent the metric (1.1) in the form

$$ds^2 = R_1^2(\eta)[d\eta^2 - (dx^1)^2] - R_2^2(\eta)(dx^2)^2 - R_3^2(\eta)(dx^3)^2, \; R_1 d\eta = c d\tau. \tag{2.5}$$

According to (2.3) we have for (2.5)

$$\Gamma_0 = 0, \quad \Gamma_1 = H_1 \tilde\gamma^1 \tilde\gamma^0, \quad \Gamma_2 = H_2 \tilde\gamma^2 \tilde\gamma^0, \quad \Gamma_3 = H_3 \tilde\gamma^3 \tilde\gamma^0, \quad H_\alpha = d \ln R_\alpha / 2R_1 d\eta.$$

We shall assume that ψ does not depend on x^2 and x^3. With the notation

$$\xi^a = \Lambda^a \exp(i\varphi^a), \quad \eta_{\dot a} = M_a \exp(i\psi_a), \quad a = 1, 2, \tag{2.6}$$

we obtain a solution of (2.2) for (2.5), taking (2.4) into account, in the form

$$\begin{aligned} (\Lambda^a)^2 R^3 &= F^a(\eta \mp x^1), \quad \varphi^a = \Phi^a(\eta \mp x^1), \\ M_a^2 R^3 &= \chi_a(\eta \pm x^1), \quad \psi_a = \Psi_a(\eta \pm x^1), \end{aligned} \tag{2.7}$$

where the upper sign corresponds to $a = 1$, and the lower one to $a = 2$.

The energy-momentum tensor of the neutrino field equals ($\bar\psi = \psi^* \tilde\gamma^0$)[10]

$$(T_{ik})^N = {}^1\!/_4 i\hbar c[\bar\psi \gamma_i \nabla_k \psi + \bar\psi \gamma_k \nabla_i \psi - (\nabla_k \bar\psi)\gamma_i \psi - (\nabla_i \bar\psi)\gamma_k \psi]. \tag{2.8}$$

The condition that the components with $i = 0, 1$; $k = 2$ and $i = 0, 1$; $k = 3$ in (2.8) vanish, condition which follows from (1.5) for (2.5), leads to the imposition of the requirement

$$\chi_1 = F^1 = 0 \quad \text{or} \quad \chi_2 = F^2 = 0 \tag{2.9}$$

or $\chi_1 = F^2 = 0$ or $\chi_2 = F^1 = 0$. The current four-vector $j^i = \bar\psi \gamma^i \psi$ for the solution (2.7) and the metric (2.5) has the representation

$$j^i = [(F^1 + \chi_2) l^i + (F^2 + \chi_1) m^i] / R_1^2 R_2 R_3, \tag{2.10}$$

$$j^i j_i = 4(F^1 + \chi_2)(F^2 + \chi_1) / R^6;$$
$$l^i = \delta_0{}^i + \delta_1{}^i, \quad m^i = \delta_0{}^i - \delta_1{}^i, \quad l^i l_i = m^i m_i = 0 \tag{2.11}$$

with the isotropic four-vectors l^i and m^i. In the case (2.9) the current four-vector (2.10) formed as a superposition of isotropic current four-vectors of the two neutrino fluxes with antiparallel momenta along the x^1-direction is timelike. (The case $\chi_1 = F^2 = 0$ or $\chi_2 = F^1 = 0$ would correspond to fluxes with parallel momenta and isotropic total four-current.)

Under these conditions we obtain an expression for (2.8) in (1.6) for the metric (2.5) in the form (with the isotropic vectors (2.11))

$$(T_i{}^k)^N = \hbar c[B(\eta - x^1) l_i l^k + A(\eta + x^1) m_i m^k] / R_1^4 R_2 R_3, \tag{2.12}$$

$$B(\eta - x^1) = -[F^1(\Phi^1)' + \chi_2 \Psi_2'], \; A(\eta + x^1) = -[F^2(\Phi^2)' + \chi_1 \Psi_1'] \tag{2.12a}$$

(the differentiation is with respect to the arguments of the appropriate functions).

For the type I Bianchi model $R_{01} = 0$ in (1.5), (1.6). According to (2.12), (1.6) the condition $\chi_1 = F^2 = 0$ or $\chi_2 = F^1 = 0$ is then impossible. In the model under consideration the case (2.9) is realized, corresponding to the presence of two neutrino fluxes along x^1 with timelike total four-current. We shall consider the case of a type I Bianchi model, when the system with the metric (1.1), (2.5) is accompanying. Then from $R_{01} = 0$ we have according to (2.12), (2.11), (2.12a)

$$(T_{01})^N = 0, \quad B(\eta - x^1) = A(\eta + x^1) = A_0 = \text{const}, \tag{2.13}$$

$$(\Phi^2)' = -\frac{A_0}{F^2}, \quad \Psi_2' = -\frac{A_0}{\chi_2} \quad \text{or} \quad (\Phi^1)' = -\frac{A_0}{F^1}, \quad \Psi_1' = -\frac{A_0}{\chi_1}. \tag{2.13a}$$

The two possibilities in (2.9), (2.13a) correspond to two opposite values of the projections of the neutrino momentum on the x^1 axis. The equation (2.6), (2.7), (2.9), (2.13) give expressions for the spinors ξ, η. According to (2.13), (2.12) for nonvanishing components in (1.6), we have[1)]

$$(T_0{}^0)^N = -(T_1{}^1)^N = 2\hbar c A_0 / R_1^2 R_2 R_3. \tag{2.14}$$

We note that the general properties of the neutrino field in curved space have been considered by a series of authors[10-15].

3. INVESTIGATION OF THE EINSTEIN EQUATIONS IN THE BIANCHI TYPE I MODEL

In the present section we shall use expressions for the metric in the form (1.1)—(1.4). According to the conservation laws[16]

$$u^i (T_i{}^k)_{;k} \equiv \left[\frac{u^k(e+p)}{B(p)}\right]_{;k} = 0, \quad B(p) = \exp\left(\int \frac{dp}{e+p}\right)$$

[1)]The expression (2.14), which determines the gravitational field, is in general valid in the presence of two fluxes, each of which may consist of neutrinos and antineutrinos of both kinds with spins parallel or antiparallel to the x^1 axis, so that for one of the fluxes the spinors depend on $\eta + x^1$, and for the other on $\eta - x^1$.

we have for the equation of state (1.7)

$$e = K/R^{3\gamma}, \quad K = Dc^2/8\pi k = \text{const.} \tag{3.1}$$

From the Einstein equations we use those corresponding to R^α_α, $R^1_1 - (1/3)R^\alpha_\alpha$ and R^0_0, which, with the aid of (1.2), (1.3), (2.14) and (3.1), can be reduced to the form

$$Ne^{-\beta} = -3\Theta R\frac{d(R^3\Theta)}{d\Omega} - \frac{3D(2-\gamma)}{2}R^{4-3\gamma}, \quad N = \frac{8\pi k}{c^2}\cdot 2\hbar c A_0 = \text{const}, \tag{3.2}$$

$$Ne^{-\beta} = \frac{3}{2}R\Theta\frac{d}{d\Omega}\left(R^3\Theta\frac{d\beta}{d\Omega}\right), \tag{3.3}$$

$$\delta\Theta\frac{d\Theta}{d\Omega} - 12\Theta^2 - \Theta^2\left[\left(\frac{d\beta}{d\Omega}\right)^2 + \left(\frac{d\beta_2}{d\Omega}\right)^2 + \left(\frac{d\beta_3}{d\Omega}\right)^2\right] + \frac{D(4-3\gamma)}{R^{3\gamma}} = 0. \tag{3.4}$$

In the case when $R_2 \neq R_3$ it also follows from $R^2_2 - R^3_3$ that

$$\frac{d\beta_2}{d\Omega} = \frac{a_0}{\Theta R^3} - \frac{d\beta}{2d\Omega}, \quad \beta_3 = -(\beta + \beta_2), \quad a_0 = \text{const.} \tag{3.5}$$

For $1 \le \gamma < 2$ in (1.7) we shall consider the axially symmetric case $R_2 = R_3$ with $a_0 = 0$ in (3.5). We introduce the variables[17]

$$\lambda = \frac{D}{\Theta^2 R^{3\gamma}} = \left(\frac{8\pi k}{c^2}\right)\frac{e}{\Theta^2}, \quad \beta' = \frac{d\beta}{d\Omega}. \tag{3.6}$$

According to (1.1a) and (1.4) we have

$$q^2 = 3\beta'^2\Theta^2/2 = 3\beta'^2 D/2\lambda R^{3\gamma},$$

which clarifies the physical meaning of the variable β' as the ratio of the invariant of the displacement tensor q to the dilation scalar Θ.

The system (3.2)–(3.4) yields ($a_0 = 0$, $\beta_2 = \beta_3 = -\beta/2$)

$$Ne^{-\beta} = DR^{4-3\gamma}(-4\lambda + 12 - 3\beta'^2)/4\lambda, \tag{3.7}$$

$$d\lambda/d\Omega = {}^1/_6\lambda[2(4-3\gamma)(\lambda-3) - 3\beta'^2], \tag{3.8a}$$

$$d\beta'/d\Omega = {}^1/_{12}\{-8\lambda + 24 + \beta'[12 + 2(4-3\gamma)\lambda - 3\beta'^2 - 6\beta']\}. \tag{3.8b}$$

The equation for $d\beta'/d\lambda$ can be investigated qualitatively; the fields of integral curves of (3.8a), (3.8b) for $\lambda \ge 0$ are illustrated in Fig. 1 for $1 \le \gamma < 4/3$, and on Fig. 2 for $4/3 < \gamma < 2$. The arrows indicate the direction of increasing Ω. To the singular state $R = 0$ corresponds $\Omega = +\infty$, and to $R = \infty$ corresponds $\Omega = -\infty$. From the meaning of (3.6) we have $\lambda \ge 0$ for $e > 0$.

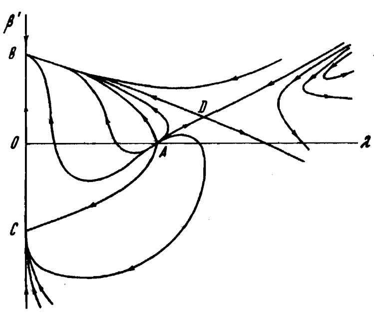

FIG. 1

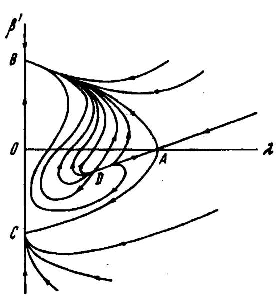

FIG. 2

FIG. 1. The field of integral curves of the equation for $d\beta'/d\lambda$ (3.8a), (3.8b) for $1 \le \gamma < 4/3$.

FIG. 2. The field of integral curves of the equation for $d\beta'/d\lambda$ (3.8a), (3.8b) for $4/3 < \gamma < 2$.

To the case of the absence of neutrinos ($N = 0$) for $e \neq 0$ according to (3.7) corresponds (for any γ) the following solution of (3.8a) and (3.8b):

$$4\lambda = 12 - 3\beta'^2, \quad 3 - \lambda = k_1\exp[-3(2-\gamma)\Omega], \quad k_1 = \text{const.} \tag{3.9}$$

In this case the singular point with $\lambda = 3$, $\beta' = 0$ (the point A in Figs. 1 and 2) corresponds to the isotropic Friedman solution in the flat model: $\Theta^2 = D/3R^{3\gamma}$, $\beta = \text{const}$. The singular points B and C correspond to the two Kasner anisotropic solutions in vacuum ($e = 0$, $N = 0$)[7]. For the solution corresponding to the point B with $\lambda = 0$, $\beta' = 2$, taking into account (3.3), we have

$$\Theta = k_0/R^3, \quad k_0 = \text{const}, \quad \beta = 2(\Omega - \Omega_0). \tag{3.10}$$

In this case in the singular state $R \to 0$ the geometry has the configuration of a "filament" ("cigar"[18,19]). For the solution corresponding to the point C with $\lambda = 0$, $\beta' = -2$, the geometry in the singular state has the configuration of a "pancake" ($\Theta = k_0/R^3$, $\beta = -2(\Omega - \Omega_0)$). The portions of (3.9) with $\beta' > 0$ and $\beta' < 0$ correspond to two types of axially symmetric solutions for $N = 0$, $e > 0$, which near the singularity have different asymptotic behavior, corresponding to the "filament" type (3.10), and the "pancake" type, respectively.

To the presence of a neutrino field with positive energy density correspond values $A_0 > 0$, $N > 0$, according to (2.14). In the (λ, β') plane one should then consider the interior of the region bounded by the integral curves $\lambda = 0$ and the parabola (3.9). The picture of the integral curves is different for $1 \le \gamma < 4/3$ (Fig. 1) and for $4/3 < \gamma < 2$ (Fig. 2).

For $1 \le \gamma < 4/3$ (in particular for dustlike matter) all integral curves (as Ω increases) go out of A and arrive in B. In the case under consideration, for $N > 0$, all solutions have near the singularity an asymptotic behavior of the "filament" type (3.10); in the axially symmetric case, for $N > 0$, a singular state of the "pancake" type is not realized. For all solutions in Fig. 1 for $N > 0$ the function β' vanishes, except at the point A, also in the intermediate state for $\lambda < 3$, corresponding to the vanishing of the invariant q^2 (1.1a), (1.4). Asymptotically, for $R \to \infty$ and for $1 \le \gamma < 4/3$ the solutions tend to the Friedman solutions according to the relations

$$\lambda \approx 3 - \exp[(4-3\gamma)(\Omega - \Omega_0)], \quad R^{3\gamma} \approx {}^3/_4\gamma^2 D\tau^2, \quad \beta \approx \beta_0 R^{3\gamma-4}. \tag{3.11}$$

A comparison of (3.11) with (3.9) shows that for dustlike matter (and in general, for $1 \le \gamma < 4/3$) for fixed λ (i.e., $\Theta^2 R^3$) the quantity β'^2 (i.e., q^2/Θ^2) tends to zero for $N > 0$ and $R \to \infty$ faster than for $N = 0$ (at the same time $e^\beta \to 1$) so that the presence of free neutrinos for $1 \le \gamma < 4/3$ accelerates the isotropization.

For $4/3 < \gamma < 2$ (Fig. 2) the picture of the integral curves changes. For $N > 0$ there is a solution corresponding to the singular point D[2]:

$$\lambda = \lambda_D = {}^9/_2(2-\gamma), \quad \beta' = \beta_D' = 4 - 3\gamma, \\ R^{3\gamma-2} = D(2-3\gamma)^2(\tau - \tau_1)^2/18(2-\gamma). \tag{3.12}$$

[2]An analogous situation for $4/3 < \gamma < 2$ occurs for the homogeneous Bianchi type I model with matter and a magnetic field [18-20]. For the analysis one can also use the variables (3.6).

Near the singularity ($\tau \to \tau_1$) the solution (3.12) does not exhibit Kasner asymptotic behavior: $R_1 \to 0$, $R_2 \to 0$, and we have a point singularity. (3.12) does not become isotropic as $\tau \to \infty$. For $N > 0$ there is also a solution corresponding to the curve D starting from A, having near the singularity (A) the asymptotic behavior (3.11) for $\Omega \to +\infty$, $\tau \to 0$ (a point singularity with Friedman behavior of $R(\tau)$), and for $R \to \infty$ it has the asymptotic behavior (3.12). The other integral curves in the region N start at D and end in B and the corresponding solutions for $R \to 0$ have "filament" singularities (3.10), and for $R \to \infty$ have the asymptotic behavior (3.12) and do not become isotropized.

Let us consider the ultrarelativistic equation of state $e = 3p$, $\gamma = 4/3$, which describes matter near the singularity with $e = \infty$. In this case the singular points D and A coincide, and thus A takes on a more complicated character. For $\gamma = 4/3$ it is possible to obtain analytic solutions. We introduce the new variables ξ and ζ by means of the relations ($N \neq 0$)

$$\lambda = 36N\frac{\xi\zeta^2}{[3(\xi+1)+N\xi\zeta^2]^2}, \quad \beta' = 2\frac{3(\xi+1)-N\xi\zeta^2}{3(\xi+1)+N\xi\zeta^2}. \tag{3.13}$$

Then (3.7), together with (3.1), (2.14) and (3.2) yields

$$e^{-\beta} = D\xi/N, \quad (T_0^0)^N/e = \xi, \tag{3.14a}$$

whereas (3.8a) and (3.8b) with $N \neq 0$ lead to the linear equation

$$d\xi/d\zeta = [\xi(3 - N\zeta^2) + 3]/3\zeta, \tag{3.15}$$

from which we obtain ($\zeta_0 = \text{const}$)

$$\xi = \zeta\exp(-N\zeta^2/6)\int_{\zeta_0}^{\zeta} t^{-2}\exp(Nt^2/6)\,dt. \tag{3.15a}$$

From (3.8a) and (3.8b) it also follows that

$$\begin{gathered} R = e^{-\Omega} = R_0\exp(N\zeta^2/6)\,|\xi|^{1/2}, \quad R_0 = \text{const}, \\ d\tau = \pm R^2 e^{\beta/2}d\zeta = \pm R_1 R_2\, d\zeta. \end{gathered} \tag{3.14b}$$

Equations (3.14a), (3.14b), (3.14c), (3.15a) and (3.1) determine the solution in parametric form for $\gamma = 4/3$.

For $N > 0$, $\xi \geq 0$ in (3.14a) and the variable ζ in (3.15a) varies from $\zeta = \zeta_0 > 0$ ($\xi = 0$) to $\zeta \to +\infty$ or from $\zeta = \zeta_0 < 0$ ($\xi = 0$) to $\zeta \to -\infty$ ($\xi \to 0$). The value $\zeta = \zeta_0$ corresponds to the singular state with the "filament" asymptotic behavior (3.10). Near the singularity the asymptotic formulas have the expression (to an accuracy better than the Kasnerian; $\zeta = \zeta_0(1 + h)$, $h \ll 1$)

$$R \sim h^{1/2}(1 + {}^1\!/_4 N\zeta_0^2 h),$$

$$\tau - \tau_0 \sim h^{3/2}(1 + {}^7\!/_{20} N\zeta_0^2 h), \quad \xi \approx h(1 - {}^1\!/_6 N\zeta_0^2 h).$$

The value $\zeta \to +\infty$ for $\zeta_0 > 0$ or $\zeta \to -\infty$ for $\zeta_0 < 0$ corresponds to $R \to \infty$, $|\tau| \to \infty$. Then according to (3.15), (3.14a) we have $\xi \approx 3/N\zeta^2$, $e^{-\beta} \to 0$, so that the solution under consideration does not become asymptotically isotropic. The quantity β' vanishes, in addition to the asymptotic state, also in the intermediate state for $\lambda < 3$ (with q^2 also vanishing). The ratio of the energy density of the neutrino field and matter tends to zero for $R \to 0$, according to (3.14a) ($(T_0^0)^N \to \infty$) as well as for $R \to \infty$.

The equations (3.13)–(3.15), obtained for $N \neq 0$ lose their applicability for $N = 0$. For $N = 0$ the solution for $\gamma = 4/3$ in the form with the parameter (with asymptotic behavior (3.10)[21]), has, according to (3.14a), the form

$$R^2 = R_0^2\{\exp[2K_0(\zeta - \zeta_0)] - \exp[K_0(\zeta - \zeta_0)]\},$$

$$e^{-\beta} = K_1\{1 - \exp[-K_0(\zeta - \zeta_0)]\}, \quad K_0 = \text{const}.$$

For $N = 0$ the function β' does not change sign and does not vanish for $\lambda < 3$.

We consider the problem of the angular dependence of the temperature of the background radiation in the presence of a directed neutrino flux in the anisotropic model. The instant when the matter becomes transparent for radiation is denoted by the index s. This instant is realized after the neutrinos are separated from matter and corresponds to a state of ultrarelativistic plasma ($e = 3p$) with temperature T_s (constant in the angles) and free neutrinos; for this state the exact solution (3.14), (3.15) is valid. At the instant of observation, denoted by the index 0, the background radiation has a Planck spectrum corresponding to temperature T_0, for which the dependence on the observation angle is given by a general formula derivable from the Bianchi type I model[18,3,22], taking (1.2) into account, in the form

$$T(\theta) = T_s(R_s/R_0)\{\cos^2\theta\exp[2(\beta_0 - \beta_s)] + \sin^2\theta\exp(\beta_s - \beta_0)\}^{-1/2}$$

(θ is the angle formed with the x^1 axis). The ratio between the difference of the longitudinal ($\theta = 0$) and transverse ($\theta = \pi/2$) temperatures to the temperature for $\theta = 0$, taking (3.14a) into account, has the form

$$\Delta T/T = 1 - [\xi_s(e^{\beta_0})D/N]^{3/2}.$$

By appropriate selection of the parameters of the exact solution (3.15) the quantity $\Delta T/T$ can be made arbitrarily small, so that the presence of a directed flux of free neutrinos in the Bianchi type I model does not contradict the observed high degree of isotropy of the background radiation. Numerical estimates have been given by Doroshkevich, Zel'dovich and Novikov[2].

For the equation of state $e = p$ ($\gamma = 2$) one can obtain an exact solution in the general case of the metric (1.1). Making use of (2.5), we obtain from (3.2)–(3.4) (here L_1, L_2, b are scale constants)

$$R_1 = L_1(\eta - \eta_0)^{\alpha_1}e^{Nb\eta}, \quad R_2 = L_2(\eta - \eta_0)^{\alpha_2}, \quad R_3 = (\eta - \eta_0)^{\alpha_3}/L_2c^2b,$$

$$\begin{gathered} \alpha_1 = -{}^1\!/_4 + a^2 + c^2b^2D, \quad \alpha_2 = {}^1\!/_2 - a, \quad \alpha_3 = {}^1\!/_2 + a, \\ a = \text{const} \geqslant 0, \quad b(\eta - \eta_0) \geqslant 0. \end{gathered} \tag{3.16}$$

Near the singularity $\eta = \eta_0$ (3.16) has the asymptotic behavior

$$R_1^3 \sim (\tau - \tau_0)^{1+\gamma_1}, \quad R_2^3 \sim (\tau - \tau_0)^{1+\gamma_2}, \quad R_3^3 \sim (\tau - \tau_0)^{1+\gamma_3},$$

$$\gamma_1 + \gamma_2 + \gamma_3 = 0;$$

$$\gamma_1 = \frac{2\alpha_1 - 1}{\alpha_1 + 1}, \quad \gamma_2 = \frac{3\alpha_2 - \alpha_1 - 1}{\alpha_1 + 1}, \quad \frac{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}{6} = 1 - \frac{3b^2c^2D}{(\alpha_1 + 1)^2},$$

which coincides with the solution for matter with $e = p$ in the absence of neutrinos ($N = 0$)[21], so that the singularity may have a point character with $R_1 \to 0$, $R_2 \to 0$, $R_3 \to 0$ ($\alpha_1 > 0$, $\alpha_2 > 0$), the character of a "filament" ("cigar") ($\alpha_1\alpha_2 < 0$) or the character of a "barrel" ($\alpha_1 = 0$ with $R_1 = \text{const}$, $R_2 \to 0$, $R_3 \to 0$ or $\alpha_2 = 0$)[21,19]. For $b\eta \to +\infty$ ($N > 0$) we have $R \to \infty$, $\tau \to \infty$ in (3.16). For $D = 0$ (3.16) goes over into the solution for neutrinos in vacuo ($e = p = 0$)[1].

For $R_2 = R_3$ and $e = p$, $N \neq 0$, according to (3.8a), (3.8b), $\chi = \text{const}\,(\beta' + 2)^2$ and the points $4\lambda = 12 - 3\beta'^2$ correspond to singularities.

The analysis which we have carried out shows the influence of a directed flux of neutrinos on the dynamics of the anisotropic model, and its dependence on the equation of state of the matter.

[1]A. G. Doroshkevich, Ya. B. Zel'dovich and I. D. Novikov, ZhETF Pis. Red. 5, 119 (1967) [JETP Lett 5, 96 (1967)]; Zh. Eksp. Teor. Fiz. 53, 644 (1967) [Sov. Phys.-JETP 26, 408 (1967)]; ZhETF Pis. Red. 8, 95 (1968) JETP Lett. 8, 58 (1968)]; Astrofizika 5, 539 (1969).

[2]Ya. B. Zel'dovich and I. D. Novikov, Relyativistskaya astrofizika (Relativistic Astrophysics), Nauka, 1967 (Engl. Transl. U. of Chicago Press, 1972).

[3]C. W. Misner, Astrophys. J. 151, 431 (1968).

[4]J. M. Stewart, Monthly Notices Roy. Astron. Soc. 145, 347 (1969) C. B. Collins, and J. M. Stewart, ibidem 153, 419 (1971).

[5]M. A. H. MacCallum, J. M. Stewart, and G. Schmidt, Commun. Math. Phys. 17, 343 (1970).

[6]R. A. Matzner and C. W. Misner, Astrophys. J. 171, 415 (1972). R. A. Matzner, ibid, 171, 433 (1972).

[7]L. D. Landau and E. M. Lifshitz, Teoriya polya (Classical theory of fields), Nauka, 1967; Engl. Translation, Pergamon-Addison Wesley, 1968.

[8]V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, Relyativistskaya kvantovaya teoriya, ch. 1 (Relativistic quantum theory, part 1) Nauka, 1968; Engl. Transl. Addison Wesley 1971.

[9]V. A. Fock und D. Iwanenko, Phys. Zs. 30, 648 (1929); Comptes Rendus (Paris) 188, 1470 (1929); V. A. Fock, Z. Physik 57, 261 (1929).

[10]D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. 29, 465 (1957) (Russian Transl., 1961). J. A. Wheeler, Gravitation, Neutrinos and Cosmology, Varenna Lectures, Acad. Press (Russian Transl. 1962).

[11]D. R. Brill and J. M. Cohen, J. Math. Phys. 7, 238 (1966).

[12]J. B. Grifiths and R. A. Newing, J. Phys. A3, 136 (1970); A4, 208 (1971).

[13]J. Wainwright, J. Math. Phys. 12, 828 (1971).

[14]J. Audretsch and W. Graf, Commun. Math. Phys. 19, 315 (1970).

[15]J. Audretsch, Commun. Math, Phys. 21, 303 (1971).

[16]I. S. Shikin, Ann. Inst. Henri Poincaré A11, 343 (1969).

[17]C. B. Collins, Commun. Math. Phys. 23, 137 (1971).

[18]K. S. Thorne, Astrophys. J. 148, 51 (1967).

[19]K. C. Jacobs, Astrophys. J. 155, 379 (1969); 153, 661 (1968).

[20]I. S. Shikin, Dokl. Akad. Nauk SSSR 176, 1048 (1967) [Sov. Phys.-Doklady 12, 950 (1968)].

[21]I. S. Shikin, Dokl. Akad. Nauk SSSR 179, 817 (1968) [Sov. Phys.-Doklady 13, 320 (1968)].

[22]High Energy Astrophysics, III, Gordon and Breach, N. Y. 1967, p. 432.

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