SOME RELATIONS FROM THE THEORY OF GALVANOMAGNETIC PHENOMENA

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By using natural assumptions regarding the collision integral (Hermitian character and positiveness), a number of inequalities are proven which relate the galvanomagnetic coefficients. It is shown in particular that in metals with equal amounts of electrons and holes $(d\rho/dH^2)_{H=0} \geq (d\rho/dH^2)_{H\to\infty}$, where ρ is the resistance and H the magnetic field strength.

THE possibility of using galvanomagnetic phenomena for an analysis of the electronic energy spectrum (see[1], Appendix III) is based on the independence of many of the theoretical results of the electron-scattering mechanism. Indeed, the dependence of different components of the resistivity tensor $\rho_{ik}(H)$ on a strong magnetic field[1)], as is well known, ([1], Secs. 26–28), is determined only by the geometry of the Fermi surface of the metal. In the case of closed Fermi surfaces, when the number of electrons n_1 is not equal to the number of holes n_2, the resistance transverse to the magnetic field tends to saturate as $H \to \infty$ ($\rho_\perp(H \to \infty) = \rho_\perp^\infty$), and increases quadratically when $n_1 = n_2$ ($\rho_\perp(H \to \infty) \approx \alpha_\infty H^2$). The longitudinal resistance tends in all cases to a constant value ($\rho_\parallel(H \to \infty) = \rho_\parallel^\infty$). In weak fields, the resistance increases quadratically ($\rho = \rho_0 + \alpha_0 H^2$).

The quantities ρ_0, $\rho_\perp^\infty$, $\rho_\parallel^\infty$, α_∞, and α_0 depend, of course, on the scattering mechanisms and cannot be calculated without special model assumptions. It was shown earlier[2], however, that natural very general properties of the collision operator lead to very simple inequalities satisfied by the quantities

$$\rho_\perp^\infty,\ \rho_\parallel^\infty \geqslant \rho_0, \tag{1}$$

and in the present communication we shall show that

$$\alpha_\infty \leqslant \alpha_0 \tag{2}$$

and that $d\rho_\parallel/dH \geq 0$ for an arbitrary value of the magnetic field.

Relation (2) was observed by E. S. Borovik in Bi[3]. He has advanced the hypothesis that this relation is universal. A calculation based on a very simple model of a compensated metal (two isotropic bands, τ-approximation) leads to a quality of the coefficients ($\alpha_\infty = \alpha_0$), and calculation based on a three-band model[4] confirms the existence of the inequality (2).

Since the proof of the inequalities (1) is contained in a collection that is not easily accessible[2], we present here their derivation.

The components of the conductivity tensor can be represented in the form of scalar products of two functions: The components of the velocity $v = \partial\epsilon/\partial p$ of an electron with quasimomentum p and energy ϵ, and the components of the vector ψ:

$$\sigma_{ik} = \langle v_i, \psi_k \rangle. \tag{3}$$

We need not determine the scalar product, which includes integration over p-space. The vector ψ is a solution of the linearized Boltzmann kinetic equation (see[1], Sec. 27):

$$\partial\psi_i / \partial t + \hat{W}_p\{\psi_i\} = v_i, \tag{4}$$

$\hat{W}_p$ is a linearized collision operator, and the variable t has the meaning of motion over the trajectory in the magnetic field.

The asymptotic form of ψ_α (α = x, y; the magnetic field is parallel to the Z axis) in strong fields is given by (we have in mind from now on only closed surfaces):

$$\psi_x \sim -\frac{c}{eH}(p_y - \overline{\hat{W}_p^{-1}}\,\overline{\hat{W}_p p_y}), \quad \psi_y \sim \frac{c}{eH}(p_x - \overline{\hat{W}_p^{-1}}\,\overline{\hat{W}_p p_x}). \tag{5}$$

The bar denotes averaging over the trajectory of the electron motion in the magnetic field.

The operator $\hat{W}_p$ is Hermitian and positive[5,6], i.e.,

$$\langle \varphi, \hat{W}_p\chi \rangle = \langle \chi, \hat{W}_p\varphi \rangle, \quad \langle \varphi, \hat{W}_p\varphi \rangle \geqslant 0. \tag{6}$$

These properties of the operator $\hat{W}_p$ allow us to introduce a new definition of the scalar product:

$$(\varphi, \chi) \equiv \langle \varphi, W_p\chi \rangle, \tag{7}$$

which we shall find convenient in what follows.

In weak fields

$$\psi_\alpha = \hat{W}_p^{-1}v_\alpha - \hat{W}_p^{-1}\frac{\partial}{\partial t}\hat{W}_p^{-1}v_\alpha + \hat{W}_p^{-1}\frac{\partial}{\partial t}\hat{W}_p^{-1}\frac{\partial}{\partial t}\hat{W}_p^{-1}v_\alpha - \ldots \tag{8}$$

The series (8), the structure of which is obvious, corresponds to expansion of ψ_α in powers of the magnetic field. The operator $\partial/\partial t$ is anti-Hermitian:

$$\left\langle \varphi, \frac{\partial}{\partial t}\chi \right\rangle = -\left\langle \frac{\partial}{\partial t}\varphi, \chi \right\rangle. \tag{9}$$

We introduce, furthermore, the operator $\hat{u}$, which has the dimension of mobility (sec/g):

$$\hat{u} = \frac{c}{eH}\hat{W}_p^{-1}\frac{\partial}{\partial t}, \tag{10}$$

and which can be easily shown to be anti-Hermitian in the sense of the new definition of the scalar product (7):

$$(\varphi, \hat{u}\chi) = -(\hat{u}\varphi, \chi). \tag{11}$$

We shall henceforth denote the quantities $p_\alpha - \overline{\hat{W}_p^{-1}}\,\overline{\hat{W}_p p_\alpha}$ simply by p_α ($\overline{W_p^{-1}W_p p_\alpha}$ is the α-th coordi-

[1)]We have in mind magnetic fields such that $l \gg r_H$, where l is the mean free path and r_H is the radius of the Larmor orbit. We do not consider quantum phenomena (such as the Shubnikov–de Haas effect, see [1], Sec. 31).

nate of the center of the orbit in the plane p_z = const).

We shall assume that the tensor σ_{ik} is of the form[2)]

$$\sigma_{ik} = \begin{pmatrix} \sigma_{\alpha\beta} & 0 \\ 0 & \sigma_{zz} \end{pmatrix}. \tag{12}$$

The resistivity tensor $\rho_{ik} = \sigma_{ik}^{-1}$ then takes a similar form, with

$$\rho_{\alpha\beta} = \sigma_{\alpha\beta}^{-1}, \quad \rho_{zz} \equiv \rho_{\parallel} = 1/\sigma_{zz}. \tag{13}$$

The inequality $\rho_{\parallel}^{\infty} > \rho_0$ follows directly from the natural fact $\sigma_{xx,yy,zz}(H) < \sigma_{xx,yy,zz}(0)$, which was proved in[2]:

We shall prove below a much stronger inequality, namely, $d\sigma_i/dH \le 0$, where σ_i is the i-th principal value of the specific electric conductivity (i = x, y, z), from which it follows according to (13) that $d\rho_{\parallel}/dH \ge 0$.

We choose the axes X and Y such as to diagonalize the tensor $\sigma_{\alpha\beta}(0)$ (α, β = x, y). Using the notation introduced above, we can express the components of the tensor $\sigma_{\alpha\beta}(H)$ in the following manner:

a) In weak fields, accurate to terms quadratic in H

$$\sigma_{xx} \approx (\hat{u}p_y, \hat{u}p_y) - \left(\frac{eH}{c}\right)^2 (\hat{u}^2 p_y, \hat{u}^2 p_y) \quad (x \leftrightarrow y),$$

$$\sigma_{xy} \approx \frac{eH}{c}(\hat{u}^2 p_x, \hat{u}p_y) + \left(\frac{eH}{c}\right)^2 (\hat{u}^2 p_x, \hat{u}^2 p_y), \tag{14}$$

The symbol $(x \longleftrightarrow y)$ below denotes an expression obtained from the preceding one by interchanging subscripts x and y;

b) In strong fields, accurate to terms $\sim 1/H^2$

$$\sigma_{xx} \approx \left(\frac{c}{eH}\right)^2 (p_y, p_y) \quad (x \leftrightarrow y), \tag{15}$$

$$\sigma_{xy} \approx \frac{c}{eH}(p_y, \hat{u}p_x) - \left(\frac{c}{eH}\right)^2 (p_x, p_y).$$

The condition that the tensor $\sigma_{\alpha\beta}(0)$ be diagonal becomes

$$(\hat{u}p_x, \hat{u}p_y) = 0, \tag{16}$$

and when dealing with compensated metals, the condition that the numbers of electrons and holes be equal ($n_1 = n_2$) takes the form

$$(p_x, \hat{u}p_y) = 0. \tag{17}$$

In addition to inequality (1), it is proved in[2] that $d\rho/dH^2|_{H=0} \ge 0$. In our notation (according to (13) and (14)):

$$\left.\frac{d\rho_{xx}}{dH^2}\right|_{H=0} = \frac{e^2}{c^2}\frac{1}{(\hat{u}p_y \hat{u}p_y)^2}\left\{(\hat{u}^2 p_y, \hat{u}^2 p_y) - \frac{(\hat{u}^2 p_y, \hat{u}p_x)^2}{(\hat{u}p_x, \hat{u}p_x)}\right\}, \quad (x \leftrightarrow y). \tag{18}$$

Non-negativity of $d\rho_{xx}/dH^2|_{H=0}$ is ensured by the fact that the product of the squares of the norms of the two vectors is not less than the square of their scalar product (the vectors $\hat{u}^2 p_y$ and $\hat{u}p_x$, and accordingly, $(x \longleftrightarrow y)$).

Since $\rho_{xx}(0) = 1/(\hat{u}p_y, \hat{u}p_y)$, $(x \longleftrightarrow y)$, and $\rho_{xx}(\infty) = (p_y, p_y)/(p_x, \hat{u}p_y)^2$ $(x \longleftrightarrow y)$ when $n_1 \ne n_2$ (see (15) and (13)), the condition $\rho_{xx}(\infty) \ge \rho_{xx}(0)$ $(x \longleftrightarrow y)$ follows from the inequalities

$$(p_x, p_x)(\hat{u}p_y, \hat{u}p_y) \geqslant (p_x, \hat{u}p_y)^2 \quad (x \leftrightarrow y). \tag{19}$$

Let us prove the inequality (2) formulated above (we recall that we are dealing with a metal with $n_1 = n_2$).

According to (18)

$$\alpha_0 = \frac{e^2}{c^2}\frac{1}{(\hat{u}p_y, \hat{u}p_y)^2}\left\{(\hat{u}^2 p_y, \hat{u}^2 p_y) - \frac{(\hat{u}^2 p_y, \hat{u}p_x)^2}{(\hat{u}p_x, \hat{u}p_x)}\right\}, \tag{20}$$

and from (15) and (13) it follows under the condition (17) that[3)]

$$\alpha_\infty = \frac{e^2}{c^2}\frac{(p_x, p_x)}{(p_y, p_y)(p_x, p_x) - (p_x, p_y)^2}. \tag{21}$$

We consider the vector

$$q = \hat{u}^2 p_y - \frac{(\hat{u}^2 p_y, \hat{u}p_x)}{(\hat{u}p_x, \hat{u}p_x)}\hat{u}p_x. \tag{22}$$

The square of its norm is

$$(q, q) = (\hat{u}^2 p_y, \hat{u}^2 p_y) - \frac{(\hat{u}^2 p_y, \hat{u}p_x)^2}{(\hat{u}p_x, \hat{u}p_x)}. \tag{23}$$

The vector q is orthogonal to the vector p_x. Indeed,

$$(q, p_x) = (p_x, \hat{u}^2 p_y) - \frac{(\hat{u}^2 p_y, \hat{u}p_x)}{(\hat{u}p_x, \hat{u}p_x)}(p_x, \hat{u}p_x) = 0, \tag{24}$$

since the first term is equal to zero in accordance with the condition (16) and the second vanishes because the operator u is anti-Hermitian. In addition, owing to the condition (17),

$$(q, p_y) = -(\hat{u}p_y, \hat{u}p_y). \tag{25}$$

Therefore (20) can be rewritten in the form

$$\alpha_0 = \frac{e^2}{c^2}\frac{(q, q)}{(p_y, q)^2}, \tag{26}$$

and the ratio is

$$\frac{\alpha_0}{\alpha_\infty} = \frac{(q, q)[(p_y, p_y)(p_x, p_x) - (p_x, p_y)^2]}{(p_y, q)^2 (p_x, p_x)}.$$

The inequality (2) follows from the inequality

$$(p_y, p_y) \geqslant \frac{(p_x, p_y)^2}{(p_x, p_x)} + \frac{(q, p_y)^2}{(q, q)}, \tag{27}$$

which is valid, since the square of the norm of the vector p_y does not exceed the sum of the squares of its projections on two unit vectors (the unit vectors $p_x/\sqrt{(p_x, p_x)}$ and $q/\sqrt{(q, q)}$).

We now prove that $d\sigma_i/dH < 0$. The operator u(10) does not depend on the value of the magnetic field. Therefore, if the kinetic equation (4) is rewritten by introducing the operator u, then the resultant equation

$$\frac{eH}{c}\hat{u}\psi_i + \psi_i = \hat{W}_p^{-1} v_i \tag{28}$$

contains the magnetic field only explicitly. We put

$$\hat{W}_p^{-1} v_i \equiv w, \qquad \psi_i \equiv \psi. \tag{29}$$

It then follows from (3) and (7) that

$$\sigma_i = (w, \psi). \tag{30}$$

With the aid of (28) we can calculate $\psi(H + \Delta H)$ in the form of an expansion in powers of $\Delta H/H$:

$$\psi(H + \Delta H) = \psi(H) + \frac{\Delta H}{H}\psi^{(1)}(H) + \ldots, \tag{31}$$

and to determine $\psi^{(1)}$ we have the equation

$$\frac{eH}{c}\hat{u}\psi^{(1)} + \psi^{(1)} = -\frac{eH}{c}\hat{u}\psi. \tag{32}$$

According to (30) and (31),

[2)]It can be verified that this is certainly so in those cases when a magnetic field is directed along an axis of twofold or higher symmetry.

[3)]We use the value ρ_{xx}, and the expression for ρ_{yy} is obtained by the substitution $y \leftrightarrow x$.

$$\sigma_i(H+\Delta H)=\sigma_i(H)+\frac{\Delta H}{H}(w,\psi^{(1)})+\ldots \tag{33}$$

Replacing w by the left-hand side of (28), we get

$$(w,\psi^{(1)})=\frac{eH}{c}(\hat{u}\psi,\psi^{(1)})+(\psi,\psi^{(1)}),$$

and we transform the second term with the aid of Eq. (32):

$$(w,\psi^{(1)})=\frac{eH}{c}(\hat{u}\psi,\psi^{(1)})-\frac{eH}{c}(\psi,\hat{u}\psi)-\frac{eH}{c}(\psi,\hat{u}\psi^{(1)})$$

$$=-2(-eH/c)(\hat{u}\psi,\psi^{(1)}).$$

We have used the fact that the operator $\hat{u}$ (11) is anti-Hermitian. In the last expression we replace $-eHc^{-1}\hat{u}\psi$ by the left-hand side of (32), and obtain

$$(w,\psi^{(1)})=-2(\psi^{(1)},\psi^{(1)})\leqslant 0. \tag{34}$$

It follows directly from (34) and (33) that

$$d\sigma_i/dH\leqslant 0. \tag{35}$$

The relations proved here apparently account for all the statements that can be made concerning the components ρ_{ik} in metals with closed Fermi surfaces, without special assumptions concerning the electron scattering mechanism. It must be emphasized that all the results are based on the assumption that the energy spectrum of the electrons remains unchanged in the magnetic field, and in particular that there is no magnetic breakdown ([1], Sec. 10). In addition, it is assumed that the magnetic field does not influence the scattering mechanisms, i.e., W_p does not depend on the magnetic field (e.g. in ferro- and antiferromagnets, in which scattering by spin waves is significant, this condition is not satisfied and, e.g., a situation is possible wherein $\rho_\infty < \rho_0$ as a result of the decrease in the number of spin waves in a strong magnetic field).

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