MOBILITY OF A NEGATIVE ION IN He^3

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A kinetic equation for an ion in a degenerate Fermi gas is presented. It is shown that for an ion with a radius a which is so large that $p_F a \gg 1$ (p_F is the Fermi momentum) the kinetic equation is valid at both low and high temperatures. In the intermediate temperature range for which the kinetic equation is not valid, the ion mobility varies by a factor $(p_F a)^6$. A diagram technique is proposed for calculating the mobility of an ion interacting with a fermi gas. This technique is applicable if one of the parameters, either the ion mass or radius, is sufficiently large. The calculations are performed under the assumptions that the ion is a large-radius rigid sphere ($p_F a \gg 1$). It is shown that the corrections to the mobility in the high temperature range qualitatively change on transition from the region $T_\tau \gg 1$ to the region $T_\tau \ll 1$ (τ is the transport time). In the region $T_\tau \ll 1$ the growth of the corrections with decreasing temperature is slower than in the region $T_\tau \gg 1$. Consequently, the high temperature result for mobility is valid for a much broader temperature region than was previously assumed[1] when the estimates were based on the kinetic equation.

A remarkable property of the negative ion in He^3 is its large radius, a ~ 15–20 Å, which is much larger than the dimension of the atom. For such a large particle one can expect a qualitative change in the temperature dependence of the mobility compared with particles of atomic dimensions. This question is investigated here from two points of view. The first part of the paper discusses the kinetic equation and the ensuing results. In the second part we develop a diagram technique that makes it possible to find the corrections to the high-temperature value of the mobility. The analysis is carried out for an ion in a Fermi gas, since Gould and Ma[1] have shown that the corrections to the ion mobility for the interaction between the gas particles are small and decrease with decreasing temperature.

KINETIC EQUATION. LIMITS OF ITS APPLICABILITY

We assume that the degeneracy condition $\epsilon_F \gg T$ is satisfied. The ion has an energy on the order of T, as a result of which the only particles that take part in the scattering are those in the region of thermal smearing of the Fermi function. The momentum of the scattered particle remains on the Fermi surface—it can change direction, but its magnitude changes little, to the extent that T/ϵ_F is small. The amplitude of the scattering of a Fermi particle by an ion, as will be shown below, can depend strongly on the particle scattering angle φ, but depends little on the initial or final particle momentum. We assume therefore that the amplitude depends only on the angle φ, which in turn is fully specified by the momentum transfer in accordance with the relation

$$\sin\frac{\varphi}{2}=\frac{|\mathbf{P}-\mathbf{P}'|}{2p_F}, \tag{1}$$

where $\mathbf{P}$ and $\mathbf{P}'$ are the initial and final momenta of the ion, and p_F is the Fermi momentum. We introduce the symbol $F(\mathbf{P}-\mathbf{P}')$ for the amplitude. The probability of scattering of an ion from the state $\mathbf{P}$ to the state $\mathbf{P}'$ is given by the usual formula

$$w(\mathbf{P},\mathbf{P}')=4\pi\left|\frac{2\pi F(\mathbf{P}-\mathbf{P}')}{m}\right|^2\int\frac{d^3\mathbf{p}}{(2\pi)^3}n(\mathbf{p})[1-n(\mathbf{p}+\mathbf{P}-\mathbf{P}')]\times\delta[\varepsilon(\mathbf{p})+\mathscr{E}(\mathbf{P})-\varepsilon(\mathbf{p}+\mathbf{P}-\mathbf{P}')-\mathscr{E}(\mathbf{P}')], \tag{2}$$

where the spin factor is taken into account; m is the Fermi-particle mass, which we assume to be much smaller than the ion mass M, $\epsilon(\mathbf{p})$ and $n(\mathbf{p})$ are the energy and distribution function of the Fermi particles, and $\mathscr{E}(\mathbf{P})$ is the ion energy. Since $\mathscr{E} \sim T$ and $\epsilon \sim \epsilon_F$, the integral in (2) can be calculated by neglecting the terms $\sim \mathscr{E}/\epsilon \sim T/\epsilon_F \ll 1$, and we thus obtain for the probability

$$w(\mathbf{P},\mathbf{P}')=4\pi\frac{|F(\mathbf{P}-\mathbf{P}')|^2}{|\mathbf{P}-\mathbf{P}'|}\frac{\mathscr{E}-\mathscr{E}'}{\{1-\exp[\beta(\mathscr{E}'-\mathscr{E})]\}}\theta(2p_F-|\mathbf{P}-\mathbf{P}'|). \tag{3}$$

Here $\beta = 1/T$, and the step function θ limits the momentum transfer to the value $2p_F$.

The kinetic equation for the ion momentum distribution function $f(\mathbf{P})$ is written, using (3), in the form

$$\mathbf{E}\frac{\partial f}{\partial\mathbf{P}}=4\pi\int\frac{d\mathbf{P}'}{(2\pi)^3}\frac{|F(\mathbf{P}-\mathbf{P}')|^2}{|\mathbf{P}-\mathbf{P}'|}\frac{\mathscr{E}-\mathscr{E}'}{e^{\beta\mathscr{E}}-e^{\beta\mathscr{E}'}}[f(\mathbf{P}')e^{\beta\mathscr{E}'}-f(\mathbf{P})e^{\beta\mathscr{E}}]. \tag{4}$$

The integral in (4) is over $|\mathbf{P}-\mathbf{P}'| < 2p_F$. Here $\mathbf{E}$ is the external force acting on the ion. We see that when $\mathbf{E} = 0$ the solution of the kinetic equation is the Maxwellian function $f \sim e^{-\beta\mathscr{E}'}$. The limitation of the integration with respect to $\mathbf{P}'$ becomes significant when $\mathbf{P}^2 \gtrsim p_F^2$, i.e., for temperatures $T > m\epsilon_F/M$. At such temperatures, the ion loses in one collision a small fraction of its momentum, and Eq. (4) can be transformed into a Fokker-Planck equation by expanding in powers of the parameter p_F^2/MT. The mobility in this limit was calculated by Davis and Dagonnier[2].

In the case of low temperatures $T \ll m\epsilon_F/M$, the limitation of the integration with respect to $\mathbf{P}'$ is immaterial, since the scattering is only through small angles. Denoting by F_0 the zero-angle scattering amplitude, linearizing (4) with respect to $\mathbf{E}$, and making the resultant equation one-dimensional, we obtain for the mobility at low temperatures

$$\mu = \frac{2\pi^{1/2}}{3M^2T^2F_0^2}\int_0^\infty \varphi(x)\, e^{-x^2}x^3\, dx, \tag{5}$$

where the function $\varphi(x)$ is defined by the equation

$$-xe^{-x^2} = \left\{\int_0^x \frac{y^2}{x^2}\left[\frac{y\varphi(y)}{3} - x\varphi(x)\right] + \int_x^\infty \left[\frac{x\varphi(y)}{3} - y\varphi(x)\right]\right\}\frac{x^2-y^2}{e^{x^2}-e^{y^2}}\, dy. \tag{6}$$

The relation $\mu \sim T^{-2}$ at low temperatures was obtained by a number of workers[3-5]. Recently Kramer[6] expressed doubts concerning the validity of this result, and presented a calculation according to which $\mu \to$ const as $T \to 0$. Kramer did not solve the kinetic equation, but used the momentum-balance relation. This relation is identically satisfied for the exact solution of the kinetic equation, but is not an equation by itself. Reasonable results can be obtained from the balance relation by approximating correctly the distribution function contained in it. At high temperatures, practically any approximation gives the correct order of the mobility. At low temperatures, a very important factor hindering the scattering is the smallness of the momentum-space volume (ΔP^3) to which the ion can go after the scattering, $(\Delta P)^3 \sim T^{3/2}$. Taking into account the factor $T^{1/2}$ that results from $(\mathcal{E} - \mathcal{E}')/|\mathbf{P} - \mathbf{P}'|$ in (4), we obtain precisely the factor T^2 in the scattering probability. All these factors drop out if we consider the momentum balance for an ion moving with constant velocity, i.e., neglecting recoil. This is precisely the reason for Kramer's incorrect result.

We indicate also a result that follows from (4) in the limit as $T \to 0$. This condition means in fact that the ion momentum in the field exceeds its thermal momentum. Putting $T = 0$ in (4) we obtain for the ion velocity the nonlinear relation

$$\bar{v} = \text{const}\,\frac{1}{M}\left(\frac{EM}{|F_0|^2}\right)^{1/5}. \tag{7}$$

Let us discuss now the results obtained with the aid of the kinetic equation. We begin with the simplest case, when the scattering amplitude F of a Fermi particle by an ion is small, so that $p_F F \ll 1$. We then have for the mobility the expressions

$$\mu \sim (MT)^{-2}F^{-2}, \quad MT \ll p_F^2, \tag{8}$$

$$\mu \sim p_F^{-4}F^{-2}, \quad MT \gg p_F^2, \tag{9}$$

which become of the same order of magnitude at $MT \sim p_F^2$. If we take the criterion for the validity of the kinetic equation to be

$$T\tau \equiv TM\mu \gg 1, \tag{10}$$

then we see that at $p_F F \ll 1$ the kinetic equation is valid at all temperatures. With increasing ion dimension, the position changes radically. If $p_F a \gg 1$, then at low temperatures $(MT \ll a^{-2})$ the Fermi particles become scattered through the angles $\varphi < (p_F a)^{-1}$, and the amplitude F_0 is determined by the diffraction scattering, so that

$$F_0 \sim a(p_F a), \quad \mu \sim (MT)^{-2}a^{-2}(p_F a)^{-2}, \quad MT \ll a^{-2}. \tag{11}$$

The criterion for the applicability of the kinetic equation $MT \ll a^{-2}(p_F a)^{-2}$ is violated while the diffraction approximation for the amplitude is still valid. The mobility at the limit of applicability of the kinetic equation is

$$\mu_{\lim} \sim a^2(p_F a)^2, \quad T_{\lim} \sim p_F^2(p_F a)^{-4}/M. \tag{12}$$

It is interesting that $\mu_{\lim}$ does not depend on the ion mass. At high temperatures the mobility is given by the same formula (9) with $F \sim a$ (see formula (39) below). The mobility ratio at the limits of applicability of the high- and low-temperature results is of the order of $(p_F a)^6 \gg 1$. The temperature dependence of the mobility in the intermediate region is unknown.

DIAGRAM TECHNIQUE. HIGH TEMPERATURE CORRECTIONS

We write down the Hamiltonian of an ion interacting with a Fermi gas:

$$H = H_0 + \frac{\mathbf{P}^2}{2M} + \sum_{\mathbf{kp}} U(\mathbf{k})\, e^{i\mathbf{kR}}\, a_{\mathbf{p}}^+ a_{\mathbf{p}+\mathbf{k}}, \tag{13}$$

where H_0 is the Hamiltonian of the Fermi gas, $\mathbf{P}$, $\mathbf{R}$, and M are respectively the momentum, the coordinate, and the mass of the ion, a^+ and a are the Fermi-particle creation and annihilation operators. The ion mobility, determined by the Kubo formula, is best calculated for imaginary frequencies

$$\mu(\omega_n) = \frac{i}{\omega_n}\int_0^\beta \exp\{\omega_n\tau_1\}d\tau_1\, \mathrm{Sp}_{\text{ion}}\mathrm{Sp}_{\text{gas}}\{e^{-\beta H}v_z(\tau_1)v_z(0)\}, \tag{14}$$
$$\mathbf{v}(\tau) = e^{H\tau}\mathbf{P}e^{-H\tau}/M, \quad \omega_n = 2\pi i n T.$$

The mobility at real frequencies is obtained by continuing $\mu(\omega_n)$ from the points of the upper half-plane.

The expression for μ can be significantly transformed by averaging in the general form over the states of the ion. To this end, we replace Sp_{ion} in the gas by a continuous integral along the ion trajectories[7]:

$$\mathrm{Sp}_{\text{ion}}\mathrm{Sp}_{\text{gas}}\{e^{-\beta H}v_z(\tau_1)v_z(0)\} = \int D\{\mathbf{R}(\tau)\}\exp\left\{-\int_0^\beta \frac{M\mathbf{v}^2(\tau)}{2}d\tau\right\} \tag{15}$$
$$\times v_z(\tau_1)v_z(0)\,\mathrm{Sp}_{\text{gas}}\left[T_\tau \exp\left\{-\int_0^\beta H(\tau)\,d\tau\right\}\right],$$

where

$$H(\tau) = H_0 + \sum_{\mathbf{kp}} U(\mathbf{k})\, e^{i\mathbf{kR}(\tau)}\, a_{\mathbf{p}}^+ a_{\mathbf{p}+\mathbf{k}}, \quad \mathbf{v}(\tau) = \frac{d\mathbf{R}(\tau)}{d\tau}. \tag{16}$$

If we write

$$H(\tau) \equiv H(\tau) + i\pi\mathbf{v}(\tau) - i\pi\mathbf{v}(\tau), \tag{17}$$

where

$$\pi = \sum_{\mathbf{p}} \mathbf{p}\, a_{\mathbf{p}}^+ a_{\mathbf{p}}$$

is the total momentum of the gas, and change over to the interaction representation in terms of $i\pi \cdot \mathbf{v}(\tau)$, then

$$\mathrm{Sp}_{\text{gas}}\left\{T_\tau \exp\left(-\int_0^\beta H(\tau)\,d\tau\right)\right\} \tag{18}$$
$$= \mathrm{Sp}_{\text{gas}}\left\{T_\tau \exp\left(-\int_0^\beta \left[H_0 + \sum_{\mathbf{kp}} U(\mathbf{k})\, a_{\mathbf{p}}^+ a_{\mathbf{p}+\mathbf{k}} - i\pi\mathbf{v}(\tau)\right]d\tau\right)\right\}$$
$$= \left\langle T_\tau \exp\left(i\int_0^\beta \pi(\tau)\mathbf{v}(\tau)\,d\tau\right)\right\rangle'.$$

It is recognized here that

$$\exp\left\{-i\int_0^\beta \pi\mathbf{v}(\tau)\,d\tau\right\} = 1,$$

since $\mathbf{R}(0) = \mathbf{R}(\beta)$ in accord with the definition of the trace, and we have put $\mathbf{R}(0) = 0$. The angle brackets denote the Gibbs average over the states of the gas with Hamiltonian

$$H_0 + \sum_{\mathbf{kp}} U(\mathbf{k}) a_{\mathbf{p}}^{+} a_{\mathbf{p}+\mathbf{k}},$$

$\boldsymbol{\pi}(\tau)$ is the operator π in the Heisenberg representation with this Hamiltonian. We note that

$$H_0 + \sum_{\mathbf{kp}} U(\mathbf{k}) a_{\mathbf{p}}^{+} a_{\mathbf{p}+\mathbf{k}}$$

is the Hamiltonian of a gas interacting with a static potential U fixed at the origin. We assume that the solutions of the single-particle problem for such a potential are known.

We see now that after we substitute (18) in (15) the integral becomes Gaussian. Calculating the auxiliary integral

$$\int D\{\mathbf{R}(\tau)\} \exp\left[-\int_0^\beta \frac{M\mathbf{v}^2(\tau)}{2} d\tau\right] \left\langle T_\tau \exp\left(i\int_0^\beta \pi(\tau)\mathbf{v}(\tau)d\tau\right)\right\rangle \quad (19)$$

$$= \left\langle T_\tau \exp\left\{-\int_0^\beta \frac{\pi^2(\tau)d\tau}{2M} + \frac{1}{2M\beta}\left(\int_0^\beta \pi(\tau)d\tau\right)^2\right\}\right\rangle ,$$

(the second term in the argument of the exponential appears if it is recognized that $\int_0^\beta \mathbf{v}(\tau)d\tau = 0$), and taking the variational derivatives $\delta^2/\delta\pi_z(\tau_1)\delta\pi_z(0)$ of both halves of (19), we obtain for the mobility

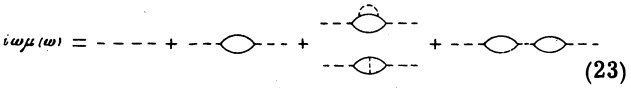
$$\mu(\omega) = \frac{i}{\omega}\left\{\frac{1}{M} - \frac{1}{M^2}\langle T_\tau \pi_z(\tau_1)\pi_z(0)S(\beta)\rangle_\omega\right\}, \quad \omega \neq 0. \quad (20)$$

$$S(\beta) = T_\tau \exp\left\{-\int_0^\beta \frac{\pi^2(\tau)}{2M}d\tau + \frac{1}{2M\beta}\left(\int_0^\beta \pi(\tau)d\tau\right)^2\right\}. \quad (21)$$

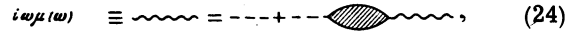
It is convenient to represent (20) graphically. The Green's functions of the gas particles are represented by solid lines, and we introduce the new graphical relations

$$-\frac{1}{M} = \text{------}; \qquad \pi(\tau) = \text{——|——} . \quad (22)$$

The vectors $\boldsymbol{\pi}$ joined by a dashed line form a scalar product, and a dashed line carries any frequency except zero. The first terms of the expansion of $\mu(\omega)$ in terms of M^{-1} are represented by the diagrams

$i\omega\mu(\omega) =$  (23)

The sum of such a series will be represented by a wavy line. We express it in terms of the sum of diagrams, for a loop, which are not reducible with respect to this line. To this end we write down a diagram equation for summing the chain

$i\omega\mu(\omega) \equiv$  (24)

The solution of which is

$i\omega\mu(\omega) = -\left[M +$  $\right]^{-1} \equiv -\left[M + \mathcal{M}(\omega)\right]^{-1}$. (25)

Let us indicate several diagrams for $\mathcal{M}$ in the first orders of perturbation theory

 (26)

We note that diagrams with two gas loops appear only in third order perturbation theory (we take the order of perturbation theory to be equal to the number of wavy lines). This is so because the vertices contain the vectors $\boldsymbol{\pi}$, and each loop should have an even number of vertices, meaning that an even number of wavy lines should emerge from it, including the external vertex of the loop.

We make the concrete calculations in the representation of functions with definite angular momentum $\psi_{\mathbf{k}lm}$, as defined in Sec. 33 of[8]. The matrix element of the momentum for these functions consists of a term corresponding to the free motion and a term determined by the matrix element of the force in accord with the equation $\dot{\mathbf{p}} = -\nabla U$:

$$\langle k'l'm'|\mathbf{p}|klm\rangle = (\mathbf{n})_{lm}^{l'm'}\left[k\delta(k-k') + \frac{\operatorname{sign}(l'-l)}{\varepsilon'-\varepsilon}\left(\frac{dU}{dr}\right)_{kl}^{k'l'}\right]. \quad (27)$$

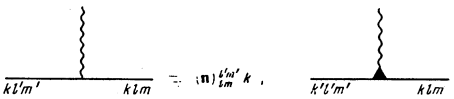
Here ϵ' and ϵ denote the energies $\epsilon(\mathbf{k}')$ and $\epsilon(\mathbf{k})$,

$$(n_z)_{lm}^{l-1\,m} = \left[\frac{l^2-m^2}{(2l+1)(2l-1)}\right]^{1/2}; \quad \left(\frac{dU}{dr}\right)_{kl}^{k'l'} = \int_0^\infty R_{k'l'}(r)R_{kl}(r)\frac{dU(r)}{dr}r^2\,dr, \quad (28)$$

$R_{kl}(r)$ is the solution of the radial Schrödinger equation in the field U(r). The matrix elements n_x and n_y are obtained by comparison with Sec. 29 of[8]. The integral of an expression of the type $(\epsilon - \epsilon')$ is taken in the sense of principal value. The product of such expressions is transformed by using the identity

$$\frac{1}{(x-y)(x-z)} = \frac{1}{y-z}\left(\frac{1}{x-y} - \frac{1}{x-z}\right) + \frac{\pi^2}{2}\delta(x-y)\delta(x-z). \quad (29)$$

We shall henceforth use the notation:

 (30)

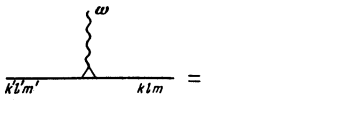
$$= (\mathbf{n})_{lm}^{l'm'}\frac{\operatorname{sign}(l'-l)}{\varepsilon'-\varepsilon}\left(\frac{dU}{dr}\right)_{kl}^{k'l'}.$$

We recall that it is necessary to integrate with respect to all the k in the diagram from 0 to ∞.

It is possible to decrease appreciably the number of diagrams in each order of perturbation theory by changing from the momentum correlator to the force correlator in accordance with the formula

$$\langle \pi\pi\rangle_\omega = -\omega^{-2}\langle \nabla U \nabla U\rangle_\omega. \quad (31)$$

Accordingly we shall henceforth designate the outer vertex by a light triangle

 (32)

$$= (\mathbf{n})_{lm}^{l'm'}\frac{\operatorname{sign}(l'-l)}{\omega}\left(\frac{dU}{dr}\right)_{kl}^{k'l'}.$$

The most important characteristic of the ion in He^3, in our opinion, is its large dimension a, and the condi-

tion $p_F a \gg 1$ is well satisfied. This condition means that the gas approximation does not hold, since $(p_F a)^2 \gg 1$ gas particles interact simultaneously with the ion. It will be shown that the procedure developed above yields an expansion in inverse powers of the gas parameter. Accordingly, we choose for the concrete calculations the simplest type of potential U(r):

$$U(r) = \infty, \quad r < a, \quad U(r) = 0, \quad r > a. \tag{33}$$

It is convenient to calculate the force matrix element for a finite value U(r < a) and then take the limit as U(r < a) → ∞. We present the final result:

$$\left(\frac{dU}{dr}\right)_{kl}^{k'l'} = -\frac{1}{\pi}\frac{(kk')^{1/2}}{m}\left[k^2-\left(\frac{l}{a}\right)^2\right]^{1/4}\left[k'^2-\left(\frac{l'}{a}\right)^2\right]^{1/4}; \tag{34}$$

$$k,\ k' \gg a^{-1}; \quad l,\ l' \gg 1.$$

The matrix element vanishes if $ka < l$ or $k'a < l'$.

Let us calculate the contribution made to $\mathcal{M}(\omega)$ by the loop without inserts. After summing over the internal frequency of the loop

$$\mathcal{M}_0(\omega) = \text{} = \tag{35}$$

$$= -\frac{4}{\omega^2}\sum_{ml}(n_z)_{lm}^{l-1\,m}(n_z)_{l-1\,m}^{lm}\int_0^\infty dk\int_0^\infty dk'\left(\frac{dU}{dr}\right)_{kl}^{k'l-1}\left(\frac{dU}{dr}\right)_{k'l-1}^{kl}\frac{n(\varepsilon)-n(\varepsilon')}{\omega+\varepsilon-\varepsilon'}$$

The factor 4 takes into account the presence of spin and the fact that the indices $l - 1$ and l, which pertain to the upper and lower lines, can be interchanged without changing the value of the diagram. Summation over m yields the factor $l/3$. Substituting (34) in (35) and changing from summation over l to integration, we get

$$\mathcal{M}_0(\omega) = -\frac{8a^2m^2}{3\pi^2\omega^2}\int_0^\infty d\varepsilon\int_0^\infty d\varepsilon'\varphi(\varepsilon,\varepsilon')\frac{n(\varepsilon)-n(\varepsilon')}{\omega+\varepsilon-\varepsilon'}, \tag{36}$$

where $\varphi(\epsilon, \epsilon')$ is the function produced after integration with respect to l:

$$\varphi(\varepsilon,\varepsilon') = \int_0^{la(\varepsilon,\varepsilon')}[(\varepsilon-t)(\varepsilon'-t)]^{1/2}dt = \frac{(\varepsilon+\varepsilon')(\varepsilon\varepsilon')^{1/2}}{4} \tag{37}$$

$$-\frac{(\varepsilon-\varepsilon')^2}{8}\ln\left|\frac{\varepsilon^{1/2}+\varepsilon'^{1/2}}{\varepsilon^{1/2}-\varepsilon'^{1/2}}\right|.$$

It is possible to perform one more integration in (36) by using (37), so that $\mathcal{M}_0(\omega)$ is expressed in terms of an integral with respect to the variable $\epsilon' - \epsilon = \nu$, which we write down by separating the real and imaginary parts after replacing ω by $\omega + i\delta$,

$$\mathcal{M}_0(\omega+i\delta) = \frac{i}{\omega\mu_0}\Phi\left(\frac{\omega}{\varepsilon_F}\right) - \frac{2}{\mu_0}\int_0^\infty\frac{d\nu}{\omega^2-\nu^2}\Phi\left(\frac{\nu}{\varepsilon_F}\right). \tag{38}$$

Here ω is already considered real, μ_0 denotes mobility at zero frequency,

$$\mu_0 = 3\pi/p_F^2(p_Fa)^2, \tag{39}$$

and we have introduced the notation

$$\Phi(x) = \varphi_1(x) + \theta(1-x)\varphi_1(-x); \tag{40}$$

$$\varphi_1(x) = (1+x)^{1/2}\left[\frac{1}{x}+\frac{1}{3}+\frac{x}{4}\right] - x\left(\frac{x}{8}+\frac{1}{4}\right)\ln\left|1+\frac{2}{x}+\frac{2(1+x)^{1/2}}{x}\right|.$$

The result (39) was obtained by Gould and Ma[1], who calculated a quantity that is given exactly by (35), but confined themselves to finding the low-frequency limit of its imaginary part. The real part (38) contains a frequency-dependent correction to the ion mass, due to the interaction with the Fermi gas. It is convenient to replace the function $\Phi(\nu/\epsilon_F)$ in the integrand of (38) by $\Phi(\nu/\epsilon_F) - 1$. This does not change the value of the integral at $\omega \neq 0$, but eliminates the singularity of the integrand at $\omega \to 0$, since $\Phi(0) = 1$ and, according to (40), $\Phi(x)$ can be expanded near zero in even powers of x. Recognizing that

$$\int_0^\infty\frac{1-\Phi(x)}{x^2}dx = \frac{\pi^2}{8}-1, \tag{41}$$

we obtain for the mobility at low frequencies

$$\mu(\omega) = \mu_0\left\{1+\frac{i\omega}{\varepsilon_F}\left[\frac{3\pi}{2}\frac{M}{m(p_Fa)^2}-\left(\frac{\pi^2}{4}-2\right)\right]\right\}. \tag{42}$$

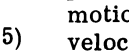
We see that the increment to the ion mass is $\approx -0.0993m(p_Fa)^2$. It will be shown that the corrections to (42) at high temperature are small regardless of the value of M.

Formula (38) was obtained in the first approximation of the formal calculation scheme. Let us indicate the possibility of a simpler derivation. If we specify the ion motion in the form $\mathbf{v}(t) \sim e^{-i\omega t}$, where $\mathbf{v}$ is the ion v velocity and t is the present time, the coefficient of the proportionality of the ion velocity to the momentum t transfer per unit time from the ion to the gas turns out to be $i\omega\mathcal{M}_0(\omega + i\delta)$. The result (25) follows then from the equation for the ion motion under the influence of the force $\mathbf{E}(\omega)$:

$$-i\omega M\mathbf{v}(\omega) = i\omega\mathcal{M}_0(\omega+i\delta)\mathbf{v}(\omega) + \mathbf{E}(\omega). \tag{43}$$

The value of $\mathcal{M}_0$ is thus calculated neglecting the recoil of the ion, and is thus independent of the ion mass. The recoil is taken into account in the next stage—in Eq. (43). In a situation that admits of the use of the kinetic equation, the analogous approximation is the Fokker-Planck approximation.

Let us estimate the corrections to the result (38) and (39). We begin with a diagram containing one wavy line:

$$\mathcal{M}_1(\omega) = \text{  } = \tag{44}$$

$$= A(p_Fa)^2\frac{p_F^4}{\omega^2}\int_{-\infty}^\infty\frac{\nu\,d\nu}{\omega-\nu}T\sum_{\omega_1}\frac{i\mu(\omega_1)}{\omega_1+\omega-\nu}.$$

We have summed here over the internal frequency of the loop and integrated with respect to the energy of one of the gas lines at a fixed energy difference ν. A smooth function similar to $\Phi(\nu/\epsilon_F)$ in (38) has been taken outside the integral sign and included in the numerical factor A. Representing $\mu(\omega_1)$ in the form

$$\mu(\omega_1) = \frac{\omega_1}{\pi i}\int_{-\infty}^\infty\frac{\mu(x)\,dx}{x^2-\omega_1^2}, \tag{45}$$

where $\mu(x)$ is the mobility at real frequencies, we can calculate the sum over ω_1. Since the smooth function ν was taken outside the integral sign in (44), this expression can be used only to calculate the imaginary part of $\mathcal{M}_1(\omega + i\delta)$. Taking the integral with respect to ν and going over to the limit $\omega \ll T$, we obtain

$$\operatorname{Im}\mathcal{M}_1(\omega+i\delta) = -\frac{A(p_Fa)^2p_F^4}{\omega}\int_{-\infty}^\infty\frac{\mu(x)\,dx}{4T}\left(\frac{2T}{x}\operatorname{cth}\frac{x}{2T}-\operatorname{sh}^{-2}\frac{x}{2T}\right). \tag{46}$$

The integral is determined by the relation between

the frequency that is characteristic of μ and the temperature. For the relation

$$\mu(x)=\frac{\mu_0}{(1-i\tau x)}, \qquad \tau=M\mu_0, \tag{47}$$

(we assume for simplicity that $M \gg m(p_F a)^2$) we obtain in the limiting cases

$$\operatorname{Im}\mathcal{M}_1(\omega+i\delta)=\begin{cases}-\omega^{-1}A(p_F a)^2 p_F^2\pi p_F^2/6MT, & T\tau\gg 1. \quad (48a)\\ \omega^{-1}A(p_F a)^2 p_F^2 p_F^2\mu_0\ln(2\pi T\tau), & T\tau\ll 1. \quad (48b)\end{cases}$$

Formula (48a) is in fact valid for all $\mu(x)$, while formula (48b) is valid for all $\mu(x)$ with logarithmic accuracy.

Comparing (48a) with (38) and (39), we see that the corrections to the mobility are of relative order of p_F^2/MT at high temperatures. The corrections to formula (9) for the mobility of a particle of small dimensions ($p_F F \ll 1$) are of the same order. The criterion for the suitability of such a correction in the latter case is $MT \gg p_F^2$. For a large-radius particle, the criterion $T\tau \gg 1$, which if satisfied causes the correction to take the form (48a), is much more stringent: $MT \gg p_F^2(p_F a)^2$. With decreasing temperature, the growth of the correction slows down, and at $T\tau \ll 1$ it depends, according to (48b), on the temperature logarithmically and its relative value is $p_F^2\mu_0$. The result (48b) is meaningful if $p_F^2\mu_0 \ll 1$. Writing this criterion in the form

$$p_F^2\mu_0=\frac{p_F^2}{MT}T\tau\ll 1, \qquad \frac{p_F^2}{MT}\ll\frac{1}{T\tau}, \tag{49}$$

we see that its satisfaction means smallness of the increment to the Fokker-Planck value of the mobility at $T\tau \sim 1$, meaning also at lower temperatures. Satisfaction of a criterion that is the reverse of (49) means that $p_F^2/MT \sim 1$ at $T\tau \sim 1$, meaning also at lower temperatures. Satisfaction of a criterion that is the reverse of (49) means that $p_F^2/MT \sim 1$ at $T\tau \gg 1$, i.e., that the Fokker-Planck approximation is not valid if the kinetic equation is valid. We shall therefore consider henceforth the case $p_F^2\mu_0 \ll 1$, i.e., the case of low mobility. It is seen from (39) that a large value of the parameter $p_F a$ ensures satisfaction of this inequality. For the ion in He^3, according to[9], $\mu_0 \sim 0.01$ cm^2/V-sec and $p_F \approx 10^8$ cm^{-1}, so that

$$(p_F^2\mu_0)_{\lim}\approx 0{,}06. \tag{50}$$

We shall try to explain the cause of the transition from (48a) to (48b). We write down the mean-squared momentum of the ion in the form

$$\overline{P^2}=M^2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega\mu(\omega)\operatorname{cth}\frac{\omega}{2T} \tag{51}$$

$$=2M^2\left(\int_0^{\infty}\omega\mu'(\omega)\frac{d\omega}{2\pi}+2\int_0^{\infty}\omega\mu'(\omega)(e^{\beta\omega}-1)^{-1}\frac{d\omega}{2\pi}\right)$$

(μ' is the real part of the mobility). The first integral can be regarded as convergent at frequencies $\sim\tau^{-1} \sim (M\mu_0)^{-1}$ and is of the order $(4\pi\tau M)^{-1}$, regardless of the relation between τ and T. The second integral depends significantly on this relation:

$$2\int_0^{\infty}\omega\mu'(\omega)(e^{\beta\omega}-1)^{-1}\frac{d\omega}{2\pi}=\begin{cases}T/2M, & T\tau\gg 1\\ \pi T^2\tau/4M, & T\tau\ll 1\end{cases}. \tag{52}$$

Comparing these formulas with the aforementioned of the first integral in (51), we see that $\overline{P^2} \sim MT$ if $T\tau \gg 1$ and $\overline{P^2} \sim (2\pi\mu_0)^{-1}$ if $T\tau \ll 1$. We can therefore assume that the correction to (39) is of the order of p_F^2/P^2, just as in the case when the kinetic equation is applicable, and the transition from (48a) to (48b) is due to the change in the temperature dependence of $\overline{P^2}$ on going from $T\tau \gg 1$ to $T\tau \ll 1$. The logarithmic temperature dependence of the correction, and also the numerical factors in (48), cannot be explained in such a rough manner. Josephson and Lekner[10] obtained a correction similar to (48b) for the mobility by a less rigorous method.

We consider now diagrams containing two wavy lines. Without presenting the calculations, we indicate that

$$\operatorname{Im}\ \text{}\ \sim \tag{53}$$

$$\sim\frac{1}{\omega}B(p_F a)^2\frac{p_F^6}{T}\int_0^{\infty}\int_0^{\infty}\frac{\mu'(x)\mu'(y)\,dx\,dy}{x+y}$$

for $\omega \ll T$. If it is assumed that the integral is of the order of μ_0/M, then the diagram yields a correction of the order of $(p_F^2/MT)p_F^2\mu_0$ to the main result. It can be shown that a diagram with two intersecting wavy lines contains a contribution that cancels (53), so that their sum does not contain terms proportional to T^{-1}. This cancellation is possible because interchange of, say, the lower ends of the wavy lines in (53) does not change the factor made up of the matrix elements of the unit vector. This is easiest to see in the plane-wave representation, where a point vertex is a free-particle momentum matrix element diagonal in the plane-wave momentum. Therefore interchange of point vertices between which there are no triangular ones does not change the angular dependence of the diagram. By calculating a diagram with one wavy line with triangular vertices we can verify that in the case $T\tau \gg 1$ it is of the order of p_F^2/MT, the same order as the diagram with the point vertices. No calculations were made for more complicated diagrams. Nonetheless, it is likely that diagrams with triangular vertices will be of the same order of magnitude as the diagrams with point vertices which were estimated above, but will not cancel each other. If this is so, then the corrections to formula (39) are of the order of $(p_F^2/MT)p_F^2\mu_0$, and since $p_F^2\mu_0 \ll 1$, this formula is valid also for lower temperatures than previously assumed[1] when the criterion of its applicability was $p_F^2 \ll MT$.

CONCLUSION

The mobility of a negative ion in He^3 was measured in[9] at temperatures 0.03—1°K. When the temperature was lowered to this region, the mobility was decreased about 10%, but the rate of decrease slowed down with decreasing temperature. Such a relation contradicts the results obtained for an ion in a free Fermi gas and was explained by Gould and Ma[11], who have shown that the interaction between fermions increases the mobility by $\sim T^2$ lnT. Knowledge of the Fermi-liquid interaction constants is insufficient to calculate the value of this correction, but the experimental points fit this temperature dependence well. It is almost obvious that the mobility should increase with further decrease of the tem-

perature. The earlier estimate in[1] presupposed that the mobility should increase noticeably at $p_F^2/MT \sim 1$ or at $T \sim 0.01°K$. This estimate was based on the kinetic equation.

The present results are shown schematically in the figure. The applicability of the kinetic equation to a large-ion radius was considered. The criterion of the applicability was assumed to be $T\tau \gg 1$, where τ is the transport time. It turned that for an ion of radius a, such that $p_F a \gg 1$, the kinetic equation holds at low and at high temperatures. The corresponding sections of the temperature axis are cross-hatched in the figure. In the intermediate temperature region, where the kinetic equation does not hold, the mobility changes by a factor $(p_F a)^6$. It is seen from the figure that the relation $p_F^2/MT \sim 1$ is satisfied precisely in the intermediate region, and therefore the estimate given in[1] for the temperature below which the mobility begins to increase is not valid.

The diagram technique developed by us has made it possible to estimate the corrections to the mobility at high temperatures. We have shown that at $T\tau \gg 1$ the first correction to the mobility is of relative order p_F^2/MT, as follows also from the kinetic equation. With decreasing temperature, the relation $T\tau \sim 1$ is still reached at $p_F^2/MT \sim p_F^2\mu_0 \sim (p_F a)^{-2} \ll 1$. With further decrease of temperature, the first correction to the mobility remains unchanged, so that it can be assumed that the high-temperature result holds for the mobility to a temperature lower than indicated in[1]. An examination of the higher-order corrections will allow us to assume that the increase of mobility with increasing temperature should start at $T \lesssim (p_F^2/M)(p_F a)^{-2}$, and it is precisely on this basis that the line μ = const was drawn on the figure.

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