ON THE KINETIC COEFFICIENTS OF A SUPERFLUID FERMI LIQUID

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The transport coefficients appearing in the equations of two-fluid hydrodynamics are calculated for a superfluid Fermi liquid in Gor'kov's model.

IN the study of a number of kinetic phenomena in superfluid Fermi systems, it is necessary to take into account those deviations of the distribution of quasi-particles from equilibrium whose characteristic relaxation frequencies are of the order of the frequency of collisions between the quasi-particles. An important example of kinetic characteristics associated with distributions of this type are the transport coefficients appearing in the hydrodynamic equations of the theory of superfluidity[1]. These equations, which generalize Landau's phenomenological equations of an ideal superfluid liquid[2] and which were derived (also on a phenomenological basis) to be applied to liquid helium, also turn out to be valid for superfluid Fermi systems. In the case of Landau's equations, this was shown in[3,4], where they were obtained from the exact microscopic equations, with, in the case of a Fermi system, the role of the superfluid component being played by the condensate and that of the normal component by the gas of Fermi quasi-particle excitations. The dissipative corrections to Landau's equations can be found when the residual interaction (after elimination of the self-consistent field) of the quasi-particles, which ensures the relaxation to equilibrium, is taken into account. In this paper, these corrections, together with the kinetic coefficients occurring in them, are calculated for a neutral Fermi liquid with direct effective particle-particle interaction, described by Gorkov's Hamiltonian[5]. External fields are assumed absent. We succeed in obtaining expressions for the kinetic coefficients of this system in explicit form, and at the same time the specific features of the calculation of, e.g., the coefficients of second viscosity can be clearly seen.

We shall start from the kinetic equation for superconductors derived in[6], which is formulated for the matrix distribution function $\gamma(\mathbf{r}, \mathbf{r}')$ and has the following form:

$$i\partial\gamma(\mathbf{r}, \mathbf{r}')/\partial t = \hat{\varepsilon}(\mathbf{r})\gamma(\mathbf{r}, \mathbf{r}') - \gamma(\mathbf{r}, \mathbf{r}')\hat{\varepsilon}(\mathbf{r}') + iL^{(2)}\{\gamma\} \tag{1}$$

(here and below, we use the notation of[6]). This equation is supplemented by two self-consistency equations:

$$\Delta(\mathbf{r}) = {}^1/_2|g|\operatorname{Tr}(\sigma_x\gamma(\mathbf{r}, \mathbf{r})), \tag{2}$$

$$0 = \operatorname{Tr}(\sigma_y\gamma(\mathbf{r}, \mathbf{r})). \tag{3}$$

The matrix $\hat{\epsilon}(\mathbf{r})$ is given by the formula

$$\hat{\varepsilon}(\mathbf{r}) = \sigma_z\frac{(\hat{\mathbf{p}} + \sigma_z\mathbf{p}_s(\mathbf{r}))^2}{2m} + \sigma_z\frac{\dot{\chi}(\mathbf{r}) - |g|n(\mathbf{r})}{2} + \sigma_x\Delta(\mathbf{r}),$$

where

$$\mathbf{p}_s(\mathbf{r}) = m\mathbf{v}_s(\mathbf{r}) = {}^1/_2 m\nabla\chi(\mathbf{r}).$$

To find the solution of the kinetic equation, we shall make use of the Chapman-Enskog scheme[7] and shall seek the solution in the form of a series in gradients of the macroscopic quantities. As is well known, the starting point of this expansion is the assumption of local equilibrium in the zeroth approximation, which corresponds to small microscopic relaxation times compared with the "hydrodynamic" times under consideration. To construct the expansion, we must take into account that Eq. (1) contains two "kinetic" relaxation times, and the form of γ depends essentially on the relation between these times[6]. These are the time τ_1 of formation of the quasi-particle condensate and the relaxation time τ_2 of the quasi-particles with respect to the momenta. Near zero temperature $\tau_1 \sim T_c^{-1}$, and τ_2 is exponentially large; as the critical temperature is approached, τ_1 increases like $(T_c - T)^{-1}$, whereas τ_2, while remaining finite, tends to its value in the normal state. We shall confine ourselves to studying the region in which the condition $\tau_1 \ll \tau_2$ is fulfilled, i.e., almost the whole range of temperature except a small region about T_c; its upper bound is determined by the condition

$$\Delta/T_c \gg |g|mp_F(T_c/\varepsilon_F)^{1/2}. \tag{4}$$

With this assumption, times long compared with τ_1 are characterized in the zeroth approximation by a diagonal γ in the representation of the locally uniform matrix $\hat{\epsilon}$. In mixed-representation variables, it has the form

$$\hat{\varepsilon}(\mathbf{p}, \mathbf{R}) = \sigma_z\frac{(\mathbf{p} + \sigma_z\mathbf{p}_s(\mathbf{R}))^2}{2m} + \sigma_z\frac{\dot{\chi}(\mathbf{R}) - |g|n(\mathbf{R})}{2} + \sigma_x\Delta(\mathbf{R}).$$

By introducing the projectors $\hat{E}_\sigma(\mathbf{p}, \mathbf{R})$ on to the proper subspaces of the matrix $\hat{\epsilon}$, corresponding to the eigenvalues

$$\varepsilon_\sigma = \sigma\varepsilon + \mathbf{p}\mathbf{v}_s(\mathbf{R}), \quad \varepsilon = \sqrt{\xi^2 + \Delta^2}, \quad \xi = p^2/2m - \mu,$$
$$\mu(\mathbf{R}) = -{}^1/_2\dot{\chi}(\mathbf{R}) - {}^1/_2 m v_s^2(\mathbf{R}) + {}^1/_2|g|n(\mathbf{R}), \quad \sigma = \pm 1,$$

we can represent γ in the form

$$\gamma = \sum_\sigma \hat{E}_\sigma f_\sigma. \tag{5}$$

The two-component vector $\mathbf{f}_\sigma(\mathbf{R}, \mathbf{p})$ has the meaning of the quasi-particle distribution function and is determined from the solubility conditions for the equation for the correction to the distribution function (5) that is non-diagonal in the $\hat{\epsilon}$-representation. These conditions, in which terms linear in the gradients are retained, have the form of a pair of standard kinetic equations:

$$\frac{\partial f_\sigma}{\partial t}+\frac{\partial \tilde{\varepsilon}_\sigma}{\partial \mathbf{p}}\frac{\partial f_\sigma}{\partial \mathbf{R}}-\frac{\partial \tilde{\varepsilon}_\sigma}{\partial \mathbf{R}}\frac{\partial f_\sigma}{\partial \mathbf{p}}=I_\sigma\{f\}, \tag{6}$$

where

$$I_\sigma\{f\}=\mathrm{Tr}\left(\hat{E}_\sigma L^{(2)}\left\{\sum_{\sigma'}\hat{E}_{\sigma'}f_{\sigma'}\right\}\right).$$

Complete local equilibrium is defined by the condition

$$I_\sigma\{f\}=0,$$

and the corresponding distribution function is of the form

$$f_\sigma^{(0)}=1-n_0((\tilde{\varepsilon}_\sigma-\mathbf{p}\mathbf{v}_n)/T), \tag{7}$$

where $n_0(x) = (1 + e^x)^{-1}$, and $\mathbf{v}_n(\mathbf{R})$ and $T(\mathbf{R})$ are the average velocity and temperature of the quasi-particles.

We shall seek the nonequilibrium correction to $f_\sigma^{(0)}$ in the form

$$f_\sigma^{(1)}=f_\sigma^{(0)}(1-f_\sigma^{(0)})\varphi_\sigma(\mathbf{p}). \tag{8}$$

The equations determining $\varphi_\sigma(\mathbf{p})$ must again satisfy the solubility conditions, which, as is well known, are the hydrodynamic equations of the zeroth approximation and have the form of conservation laws. Corresponding with the dependence of the distribution (7) on four parameters, the solubility conditions form four conservation laws: for the flux density $\mathbf{j}$ and the energy E[6].

When the condition (5) is fulfilled to second order in the gradients, these quantities can be written in the form

$$\mathbf{j}=n\mathbf{v}_s+\mathbf{j}_0,\quad \mathbf{j}_0(\mathbf{R})=\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{\mathbf{p}}{m}(1-\mathrm{Tr}\,\gamma(\mathbf{R},\mathbf{p})),$$

$$E=E_0+\mathbf{p}_s\mathbf{j}_0+nmv_s^2/2,$$

$$E_0(\mathbf{R})=-\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{p^2}{2m}\mathrm{Tr}(\sigma_z\gamma(\mathbf{R},\mathbf{p}))-\frac{\Delta^2(\mathbf{R})}{|g|}-\frac{|g|n^2(\mathbf{R})}{4}.$$

Since, to calculate the kinetic coefficients, we must know the fluxes occurring in the first-approximation hydrodynamic equations to second order in the gradients, we immediately write down the equations exact to this order. Without dwelling on the calculation, which can be carried out conveniently in matrix form in the coordinate representation, we remark only that the equations obtained preserve the form of the zeroth-approximation equations derived in[3,4]:

$$\frac{\partial j_i}{\partial t}+\frac{\partial}{\partial R_k}\Pi_{ik}=0,\quad \Pi_{ik}=nv_{si}v_{sk}+j_{0i}v_{sk}+j_{0k}v_{si}+\Pi_{0ik}, \tag{9}$$

$$\Pi_{0ik}=-\int\frac{d\mathbf{p}}{(2\pi)^3}\sum_\sigma\frac{p_ip_k}{m^2}\frac{\xi}{\varepsilon}\sigma f_\sigma-\delta_{ik}\left(\frac{\Delta^2}{m|g|}+\frac{|g|n^2}{4m}\right), \tag{10}$$

$$\frac{\partial E}{\partial t}+\mathrm{div}\,\mathbf{Q}=0,\quad \mathbf{Q}=\mathbf{Q}_0+E\mathbf{v}_s+\mathbf{j}_0\left(\frac{mv_s^2}{2}-\frac{|g|n}{2}\right)+(\Pi_0\mathbf{p}_s), \tag{11}$$

$$\mathbf{Q}_0=-\int\frac{d\mathbf{p}}{(2\pi)^3}\sum\frac{p^2\mathbf{p}}{2m^2}f_\sigma. \tag{12}$$

It is easily seen that corrections to the conservation laws associated with the non-diagonal nature of γ are of order τ_1/τ_2 compared with the corrections associated with the nonequilibrium nature of the quasi-particle distribution function, and, because of the condition (4), are not taken into account. In the approximation (5), the self-consistency condition (2) has the form

$$1=\frac{|g|}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}\sum_\sigma\frac{\sigma}{\varepsilon}f_\sigma, \tag{13}$$

as regards condition (3), in this approximation it is satisfied identically, and this leads to an indeterminacy in the problem (we need an equation determining the phase $\chi(\mathbf{R})$ of the order parameter). This difficulty is connected with the well known fact that the approximation (5) is not sufficiently accurate for the continuity equation

$$\partial n/\partial t+\mathrm{div}\,\mathbf{j}=0 \tag{14}$$

to be fulfilled for the particle density $n(\mathbf{R})$ defined by the relation

$$n(\mathbf{R})=\int\frac{d\mathbf{p}}{(2\pi)^3}[1-\mathrm{Tr}(\sigma_z\gamma(\mathbf{R},\mathbf{p}))]. \tag{15}$$

Indeed, it is not difficult to see that Eq. (14) does not follow from the kinetic equation (6), although in general form it is a consequence of Eqs. (1)–(3). This makes it possible to treat the continuity equation, which, in the approximation (5), is independent, as the missing equation for the phase of the order parameter. In local equilibrium, the fulfilment of (14) means that the quantity μ, occurring in the formula for $\tilde{\epsilon}_\sigma$ and connected with the phase by the relation

$$\mu=-{}^1\!/_2\dot{\chi}-{}^1\!/_2mv_s^2+{}^1\!/_2|g|n, \tag{16}$$

is the chemical potential of the true particles. Equations (9)–(16), with f_σ given by the expression (7), are a complete system of zeroth-order hydrodynamic equations.

To perform the subsequent calculations, we shall introduce a limitation on the velocities of the normal and superfluid components: namely, we shall assume that their difference $\mathbf{v} = \mathbf{v}_n - \mathbf{v}_s$ is small compared with the critical velocity $v_{cr} \sim \Delta/p_F$. In this approximation, the problem has spherical symmetry, and as a result the structure of the dissipative terms is simplified, leading to agreement with the equations of[1].

The nonequilibrium correction to Eq. (16) is found from the following arguments. We fix the four arbitrary constants on which the solution of the kinetic equation depends by the conditions $\mathbf{j}_0^{(1)} = 0$ and $E_0^{(1)} = 0$ where $\mathbf{j}_0^{(1)}$ and $E_0^{(1)}$ are the non-equilibrium corrections to $\mathbf{j}_0$ and E_0 respectively. Then, by virtue of Eq. (14), the correction $n^{(1)}$ to the particle density must also be equal to zero. This equation, together with the self-consistency equation (13), determines the correction $\delta\Delta$ to the equilibrium value Δ and the correction to Eq. (16), which we denote by $\delta\mu$.

We write this system in explicit form:

$$\frac{\partial n}{\partial\mu}\delta\mu+\frac{\partial n}{\partial\Delta}\delta\Delta=-n_1,\quad \frac{\partial F}{\partial\mu}\delta\mu+\frac{\partial F}{\partial\Delta}\delta\Delta=-F_1, \tag{17}$$

where F denotes the right-hand side of Eq. (13) with f_σ from the equality (7), and

$$n_1=-\int\frac{d\mathbf{p}}{(2\pi)^3}\sum_\sigma\frac{\xi}{\varepsilon}\sigma n_0\left(\frac{\sigma\varepsilon}{T}\right)n_0\left(-\frac{\sigma\varepsilon}{T}\right)\varphi_\sigma(\mathbf{p}), \tag{18a}$$

$$F_1=\frac{|g|}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}\sum_\sigma\frac{\sigma}{\varepsilon}n_0\left(\frac{\sigma\varepsilon}{T}\right)n_0\left(-\frac{\sigma\varepsilon}{T}\right)\varphi_\sigma(\mathbf{p}). \tag{18b}$$

The linearized kinetic equation for the nonequilibrium correction to the distribution function, after certain transformations of the left-hand side using Eqs. (9)–(16) and also the thermodynamic relations in local equilibrium[1,3], is brought to the form

$$n_0\left(\frac{\sigma\varepsilon}{T}\right)n_0\left(-\frac{\sigma\varepsilon}{T}\right)\left\{\left(\frac{sv}{Tj_0}-\frac{\xi}{mT^2}\right)\mathbf{p}\nabla T-\right. \tag{19}$$

$$-\sigma\frac{\xi}{\varepsilon}\frac{p_ip_k}{2mT}\left(\frac{\partial v_{ni}}{\partial R_k}+\frac{\partial v_{nk}}{\partial R_i}-\frac{2}{3}\delta_{ik}\operatorname{div}\mathbf{v}_n\right)$$
$$+\sigma\frac{\xi}{\varepsilon}\frac{1}{T}\left(\frac{\partial n}{\partial T}\right)\operatorname{div}\mathbf{i}+\sigma\frac{\xi}{\varepsilon}\left[\frac{s}{n}\left(\frac{\partial\mu}{\partial s}\right)_n\right.$$
$$\left.-\frac{p^2}{3mT}\right]\operatorname{div}\mathbf{v}_n+\left[\sigma\varepsilon\frac{s}{T^2}\left(\frac{\partial T}{\partial s}\right)_n-\sigma\frac{\Delta}{\varepsilon}\frac{s}{T}\left(\left(\frac{\partial\Delta}{\partial T}\right)_\mu\left(\frac{\partial T}{\partial s}\right)_n\right.\right.$$
$$\left.\left.+\left(\frac{\partial\Delta}{\partial\mu}\right)_T\left(\frac{\partial\mu}{\partial s}\right)_n\right)\right]\operatorname{div}\mathbf{v}_n+\left[\sigma\varepsilon\frac{1}{T^2}\left(\frac{\partial T}{\partial n}\right)_s-\sigma\frac{\Delta}{\varepsilon}\frac{1}{T}\left(\left(\frac{\partial\Delta}{\partial T}\right)_\mu\left(\frac{\partial T}{\partial n}\right)_s\right.\right.$$
$$\left.\left.+\left(\frac{\partial\Delta}{\partial\mu}\right)_T\left(\frac{\partial\mu}{\partial n}\right)_s\right)\right]\operatorname{div}\mathbf{j}\Big\}=\hat{I}_\sigma\varphi$$

where s is the entropy density; the linearized collision integral $\hat{I}_\sigma$ has the form[6]:

$$\hat{I}_\sigma\varphi=-\frac{g^2}{(2\pi)^5}\sum_{\sigma_1\sigma_2\sigma_3}\iiint d\mathbf{p}_1\,d\mathbf{p}_2\,d\mathbf{p}_3\,\delta(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_1-\mathbf{p}_3)$$
$$\times\delta(\sigma\varepsilon+\sigma_2\varepsilon_2-\sigma_1\varepsilon_1-\sigma_3\varepsilon_3)\cdot\frac{1}{8}\left(1-\sigma\sigma_2\frac{\xi\xi_2\pm\Delta^2}{\varepsilon\varepsilon_2}\right)$$
$$\times\left(1-\sigma_1\sigma_3\frac{\xi_1\xi_3+\Delta^2}{\varepsilon_1\varepsilon_3}\right)n_0\left(-\frac{\sigma\varepsilon}{T}\right)n_0\left(-\frac{\sigma_2\varepsilon_2}{T}\right)$$
$$\times n_0\left(\frac{\sigma_1\varepsilon_1}{T}\right)n_0\left(\frac{\sigma_3\varepsilon_3}{T}\right)[\varphi_\sigma(\mathbf{p})+\varphi_{\sigma_2}(\mathbf{p}_2)-\varphi_{\sigma_1}(\mathbf{p}_1)-\varphi_{\sigma_3}(\mathbf{p}_3)]. \tag{20}$$

To transform the collision integral, we note that its kernel is symmetric with respect to interchange of the indices 1 and 3. This makes it possible to integrate both the δ-functions occurring in the kernel:

$$\sum_{\sigma_3}\iiint d\mathbf{p}_1\,d\mathbf{p}_2\,d\mathbf{p}_3\,\delta(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_1-\mathbf{p}_3)\,\delta(\sigma\varepsilon+\sigma_2\varepsilon_2-\sigma_1\varepsilon_1-\sigma_3\varepsilon_3)$$
$$=m^3\int\frac{d\varphi_2\,d\varphi\sin\theta\,d\theta}{\cos\theta/2}\cdot\int' d\xi_1\,d\xi_2\frac{|\sigma\varepsilon+\sigma_2\varepsilon_2-\sigma_1\varepsilon_1|}{[(\sigma\varepsilon+\sigma_2\varepsilon_2-\sigma_1\varepsilon_1)^2-\Delta^2]^{1/2}},$$

where θ and φ_2 are the angles defining the direction of the vector $\mathbf{p}_2$ in the spherical coordinate system associated with the vector $\mathbf{p}$; φ is the angle between the planes formed by the vectors $\mathbf{p}$, $\mathbf{p}_2$ and $\mathbf{p}_1$, $\mathbf{p}_3$[8]; the prime on the integral over the energies limits the range of integration by the condition

$$|\sigma\varepsilon+\sigma_2\varepsilon_2-\sigma_1\varepsilon_1|\geqslant\Delta.$$

Equation (19) is a system of two integral equations determining the two-component function $\varphi_\sigma(\mathbf{p})$. Certain simplifications of this system are associated with the symmetry properties of the solution. From the expressions (10), (12) and (18) determining the fluxes, it can be seen that in them a part of the solution that is either even or odd in the index σ appears. For such functions, as follows from (20), Eq. (19) is decoupled into two independent equations, which pass over into each other on replacing $\sigma\to-\sigma$. Below, in all the energy integrals encountered, we shall go over to dimensionless variables, and then the lower limit, equal to $-\mu/T_c$ in order of magnitude, can be extended to $-\infty$ because of the rapid convergence of these integrals, thereby establishing a symmetric range of integration. Taking this into account, it is not difficult to see that the fluxes are determined by solutions of definite parity in the variable ξ, which makes it possible to simplify the kernel of the operator (20) and, confining ourselves to the region $\xi>0$, to go over to integration over the more convenient variable ϵ. We note also that the integral operator (20) carries over to the solution, without change, the symmetry of the inhomogeneous term in the variables σ and ξ, thereby realizing a selection rule which supplements the selection rule in the tensor dimensionalities. Even with these simplifications, Eq. (19) remains so complicated that it is impossible to find its solution in the general case. We shall therefore seek the solution in two limiting cases: at temperatures close to zero, and at temperatures close to T_c. Correspondingly, the kinetic coefficients will also be calculated in these two regions.

1. COEFFICIENTS OF FIRST VISCOSITY AND THERMAL CONDUCTIVITY

The coefficient of first viscosity η is determined by the tensor part of the general solution of Eq. (19); we represent this part in the form:

$$\varphi_\sigma(\mathbf{p})=p_ip_k\left(\frac{\partial v_{ni}}{\partial R_k}+\frac{\partial v_{nk}}{\partial R_i}-\frac{2}{3}\delta_{ik}\operatorname{div}\mathbf{v}_n\right)\varphi_\sigma(\xi).$$

In view of the fact that the nonequilibrium correction to the momentum-flux tensor $\Pi^{(1)}_{0ik}$ is connected with η by the relation

$$\Pi^{(1)}_{0ik}=-\eta\left(\frac{\partial v_{ni}}{\partial R_k}+\frac{\partial v_{nk}}{\partial R_i}-\frac{2}{3}\delta_{ik}\operatorname{div}\mathbf{v}_n\right),$$

the first viscosity coefficient is expressed in terms of $\varphi_\sigma(\xi)$ as follows:

$$\eta=\frac{p_F^5}{15\pi^2m}\int d\xi\sum_\sigma n_0\left(\frac{\sigma\varepsilon}{T}\right)n_0\left(-\frac{\sigma\varepsilon}{T}\right)\frac{\xi}{\varepsilon}\sigma\varphi_\sigma(\xi). \tag{21}$$

First we shall consider temperatures close to zero, when the parameter $u=\Delta/T$ is great: $u\gg1$.

We go over to the dimensionless variable $\epsilon/\Delta\to\epsilon$ and introduce the function $\psi_a(\epsilon)$, connected with $\varphi_\sigma(\xi)$ for $\xi>0$ by the relation

$$\varphi_\sigma(\xi)=\frac{4\pi^4p_F^2}{\lambda^2m^2\Delta^3}\sigma\psi_a(\varepsilon),\qquad\lambda=|g|mp_F.$$

Then the equation for $\psi_a(\epsilon)$ takes the form

$$u\frac{\sqrt{\varepsilon^2-1}}{\varepsilon}n_0(u\varepsilon)=\sum_{\sigma_1\sigma_2}\int_1^\infty d\varepsilon_2\,K_{\sigma_1\sigma_2}(\varepsilon,\varepsilon_2)$$
$$\times\left[\left(1-\frac{\sigma_2}{\varepsilon\varepsilon_2}\right)\frac{\varepsilon_2}{\sqrt{\varepsilon_2^2-1}}\psi_a(\varepsilon)-\frac{1}{5}\frac{\sqrt{\varepsilon^2-1}}{\varepsilon}\psi_a(\varepsilon_2)\right], \tag{22}$$

where
$$K_{\sigma_1\sigma_2}(\varepsilon,\varepsilon_2)=\int_1^{\infty\prime}d\varepsilon_1\left(1-\frac{\sigma_1}{\varepsilon_1(\varepsilon+\sigma_2\varepsilon_2-\sigma_1\varepsilon_1)}\right)\frac{\varepsilon_1}{\sqrt{\varepsilon_1^2-1}}$$
$$\times\frac{|\varepsilon+\sigma_2\varepsilon_2-\sigma_1\varepsilon_1|}{\sqrt{(\varepsilon+\sigma_2\varepsilon_2-\sigma_1\varepsilon_1)^2-1}}n_0(-u\sigma_2\varepsilon_2)\,n_0(u\sigma_1\varepsilon_1)\,n_0(u\varepsilon+u\sigma_2\varepsilon_2-u\sigma_1\varepsilon_1).$$

For large u, the integrand of the matrix $K_{\sigma_1\sigma_2}$ contains, in the principal approximation, exponentials which decrease rapidly on moving away from the boundaries of the integration. The number of these exponentials, which is different for different matrix elements, determines their order of smallness in the parameter e^{-u}. It is easily verified that the element K_{+-} with the range of integration $1\le\epsilon_1\le\epsilon-\epsilon_2-1$ has the maximum (zeroth) order. Since $\epsilon_2\ge1$, it follows from the form of the range of integration that this element gives no contribution to the equation for $\epsilon<3$. At the same time, it can be seen from (21), rewritten for the function $\psi_a(\epsilon)$, that the region close to $\epsilon=1$ gives the important contribution to the integral. Therefore, the principal contribution to the solution is made by elements $K_{\sigma_1\sigma_2}$ of first order, i.e., the kernel of the collision integral is effectively exponentially small. The subsequent transformations reduce to taking the slowly varying factors multiplying the truncating exponentials outside the integrals, after which, by the replacement

$$\varepsilon = 1 + \tau / u, \quad \psi_a(\varepsilon) = ue^u \nu_a(\tau) \tag{23}$$

the integral operator of Eq. (22) is brought to a sum of operators decreasing according to a power law (in the parameter u^{-1}). Retaining the principal term in this sum, we obtain for $\nu_a(\tau)$ the following simple equation:

$$\tau = \nu_a(\tau)\left[\Gamma\left(\frac{3}{4}\right) + \sqrt{\tau}\int_0^\infty d\tau_1 e^{-\tau_1^2}\left(\frac{\pi}{2} - \arcsin\frac{\tau_1^2}{\sqrt{\tau_1^2+\tau}}\right)\right]. \tag{24}$$

Substituting (23) into (21) and putting Δ equal to its value Δ_0 at T = 0, we obtain

$$\eta = \frac{4\pi^2 C_a}{15}\frac{p_F^7}{\lambda^2 m^3 \Delta_0^2}, \quad C_a = \int_0^\infty d\tau\, e^{-\tau}\nu_a(\tau). \tag{25}$$

At temperatures close to T_c, the parameter u is confined within the limits $\lambda(T_c/\epsilon_F)^{1/2} \ll u \ll 1$. It is clear that, as T_c is approached, the coefficient η tends to its value η_0 in the normal state and interest centers on the nature of the dependence on Δ of the first correction to η_0 at temperatures below T_c.

It is convenient to introduce a function $\Phi_a(\epsilon)$ of the dimensionless variable $\epsilon/T \to \epsilon$, connected with $\varphi_\sigma(\epsilon)$ by the relation

$$\varphi_\sigma(\xi) = \frac{4\pi^4 p_F^2}{\lambda^2 m^2 T^3}\frac{\xi}{\varepsilon}\sigma\Phi_a(\varepsilon) \tag{26}$$

The function $\Phi_a(\epsilon)$ satisfies the equation

$$-n_0(\varepsilon)n_0(-\varepsilon) = \hat{I}(u)\Phi_a, \quad \hat{I}(u)\Phi_a = -\sum_{\sigma_2}\int_u d\varepsilon_2 K(u;\varepsilon,\sigma_2\varepsilon_2)$$
$$\times\left[\left(1 - \sigma_2\frac{u^2}{\varepsilon\varepsilon_2}\right)\frac{\varepsilon_2}{\sqrt{\varepsilon_2^2-u^2}}\Phi_a(\varepsilon) - \frac{1}{5}\frac{\sqrt{\varepsilon^2-u^2}}{\varepsilon}\Phi_a(\varepsilon_2)\right], \tag{27}$$

$$K(u;x,y) = \sum_{\sigma_1}\int_u' d\varepsilon_1\left(1 - \frac{\sigma_1 u^2}{\varepsilon_1(x+y-\sigma_1\varepsilon_1)}\right)\frac{\varepsilon_1}{\sqrt{\varepsilon_1^2-u^2}}$$
$$\times\frac{|x+y-\sigma_1\varepsilon_1|}{\sqrt{(x+y-\sigma_1\varepsilon_1)^2-u^2}}n_0(-x)n_0(-y)n_0(\sigma_1\varepsilon_1)n_0(x+y-\sigma_1\varepsilon_1). \tag{28}$$

For u = 0, the equation is simplified:

$$-n_0(\varepsilon)n_0(-\varepsilon) = \hat{I}^0\Phi_a,$$

$$\hat{I}^0\Phi_a = -\sum_{\sigma_2}\int_0^\infty d\varepsilon_2 K(0;\varepsilon,\sigma_2\varepsilon_2)\left[\Phi_a(\varepsilon) - \frac{1}{5}\Phi_a(\varepsilon_2)\right]. \tag{29}$$

It can be shown[8] that the solution Φ_a^0 of this equation is positive and tends to a constant as $\epsilon \to 0$: $\Phi_a^0(0) = 5/2\pi^2$.

Substituting (26) into (21), we have

$$\eta = \frac{16\pi^2 p_F^7}{15\lambda^2 m^3 T^2}\int_u^\infty d\varepsilon\frac{\sqrt{\varepsilon^2-u^2}}{\varepsilon}n_0(\varepsilon)n_0(-\varepsilon)\Phi_a(\varepsilon). \tag{30}$$

For T = T_c, we have

$$\eta_0 = \frac{16\pi^2 C_1}{15}\frac{p_F^7}{\lambda^2 m^3 T_c^2}, \quad C_1 = \int_0^\infty d\varepsilon\, n_0(\varepsilon)n_0(-\varepsilon)\Phi_a^0(\varepsilon). \tag{31}$$

To find the correction, we differentiate (30) with respect to u and, making u tend to zero, obtain

$$\left(\frac{\partial\eta}{\partial u}\right)_{u=0} = -\frac{16\pi^2}{15}\left(C_2 + \frac{\pi}{8}\right)\frac{p_F^7}{\lambda^2 m^3 T_c^2},$$

$$C_2 = -\int_0^\infty d\varepsilon\, n_0(\varepsilon)n_0(-\varepsilon)\Phi_a'(\varepsilon), \tag{32}$$

where $\Phi_a'(\epsilon)$ is the u = 0 limit of the derivative with respect to u of the solution of Eq. (27). This function is determined from the equation obtained by differentiating (27) with respect to u and then making u tend to zero. The equation obtained has a cumbersome inhomogeneous part

$$\left[\frac{\partial}{\partial u}\hat{I}(u)\right]_{u\to 0}\Phi_a^0,$$

in which, however, there is only one term

$$-\frac{u}{5}\sum_{\sigma_2}\int_u^\infty d\varepsilon_2 K(0;\varepsilon,\sigma_2\varepsilon_2)\frac{\Phi_a^0(\varepsilon_2)}{\varepsilon_2\sqrt{\varepsilon_2^2-u^2}},$$

that does not go to zero for u = 0 (the remaining terms are linear in u). Keeping only this term, we obtain

$$\frac{P\Phi_a^0(0)}{10}n_0(\varepsilon)n_0(-\varepsilon)\frac{\varepsilon}{e^\varepsilon - 1} = \hat{I}^0\Phi_a'.$$

It is not difficult to convince oneself that the numerical function $\Phi_a'(\epsilon)$ that is the solution of this equation has no singularities and is negative.

Integrating (32) and combining it with (31), we finally obtain

$$\eta = \eta_0\left(1 - \alpha\frac{\Delta}{T}\right), \quad \alpha = (C_2 + \pi/8)/C_1. \tag{33}$$

In the calculation of the thermal conductivity coefficient κ occurring in the nonequilibrium correction to the energy flux (12)

$$Q_0^{(1)} = -\varkappa\nabla T,$$

only the term odd in ξ and proportional to ∇I in the left-hand side of (19) plays a part. The even term alongside it appears in the supplementary condition $j_0^{(1)} = 0$ and determines the vector constant of the general solution. In analogy with the preceding discussion, we introduce the function $\varphi_\sigma(\xi)$ by the relation

$$\varphi_\sigma(\mathbf{p}) = \mathbf{p}\nabla T\varphi_\sigma(\xi).$$

The calculation of $\varphi_\sigma(\xi)$ at temperatures close to zero is completely analogous to the calculation performed in the determination of η, and gives finally

$$\varkappa = \frac{16\pi^2 C_a}{3}\frac{p_F^5}{\lambda^2 m^2 T}. \tag{34}$$

Unlike η, which is finite at T = 0 (formula (25)), the thermal conductivity is inversely proportional to the temperature, just as in a normal Fermi liquid[8].

Near T_c, the equation for the function $\Phi_s(\epsilon)$, connected with $\varphi_\sigma(\xi)$ by the relation

$$\varphi_\sigma(\xi) = \frac{8\pi^4 p_F^2}{\lambda^2 m^2 T^3}\frac{\xi}{\varepsilon}\Phi_s(\varepsilon),$$

is analogous to (27):

$$-\varepsilon n_0(\varepsilon)n_0(-\varepsilon) = \hat{I}(u)\Phi_s;$$

$$\hat{I}(u)\Phi_s = -\sum_{\sigma_2}\int_u^\infty d\varepsilon_2 K(u;\varepsilon,\sigma_2\varepsilon_2)$$
$$\times\left[\left(1 - \sigma_2\frac{u^2}{\varepsilon\varepsilon_2}\right)\frac{\varepsilon_2}{\sqrt{\varepsilon_2^2-u^2}}\Phi_s(\varepsilon) + \frac{\sigma_2}{3}\frac{\sqrt{\varepsilon^2-u^2}}{\varepsilon}\Phi_s(\varepsilon_2)\right].$$

For u = 0, this equation has the solution $\Phi_s^0(\epsilon)$, which tends linearly to zero for small ϵ: $\Phi_s^0(\epsilon) = 9\epsilon/4\pi^2$. This behavior of $\Phi_s^0(\epsilon)$ leads to the result that, as investigation shows, the inhomogeneous part of the equation for $\Phi_s'(\epsilon)$ does not contain divergent integrals reducing its order in u, and therefore $\Phi_s'(\epsilon)$ is proportional to the first power of u. We shall not write out the explicit form of the equation for Φ'_S, in view of its unwieldiness; we point out only that Φ'_S has no non-integrable singularities. The thermal conductivity coefficient is expressed in terms of $\Phi_S(\epsilon)$:

$$\varkappa = \frac{16\pi^2 p_F^5}{3\lambda^2 m^2 T}\int\limits_u^\infty d\varepsilon \sqrt{\varepsilon^2 - u^2}\, n_0(\varepsilon) n_0(-\varepsilon)\Phi_s(\varepsilon),$$

and in the case u = 0 is equal to

$$\varkappa_0 = \frac{16\pi^2 C_3}{3}\frac{p_F^5}{\lambda^2 m^2 T_c}, \quad C_3 = \int\limits_0^\infty d\varepsilon\, n_0(\varepsilon) n_0(-\varepsilon)\Phi_s^0(\varepsilon).$$

Denoting $\Phi'_S = u\tilde{\Phi}_S$, where $\tilde{\Phi}_S$ is a numerical function, we obtain the first correction to κ_0 associated with the deviation of Δ from zero in the form

$$\varkappa = \varkappa_0(1 - \beta\Delta^2 / T_c^2),$$

$$\beta = \frac{1}{2C_3}\left[\int\limits_0^\infty d\varepsilon\, n_0(\varepsilon) n_0(-\varepsilon)\frac{\Phi_s^0(\varepsilon)}{\varepsilon} - \int\limits_0^\infty d\varepsilon\, n_0(\varepsilon) n_0(-\varepsilon)\varepsilon\tilde{\Phi}_s(\varepsilon)\right]. \quad (35)$$

2. THE SECOND-VISCOSITY COEFFICIENTS

To determine the second-viscosity coefficients, it is necessary to calculate the quantities defined by the equalities (18). These expressions depend on the scalar part of the general solution of Eq. (19). We first consider the case of zero temperature.

The function odd in ξ appearing in n_1 is determined by an equation whose inhomogeneous part, after the thermodynamic derivatives appearing in the left-hand side of Eq. (19) have been calculated[9] and the small quantities ($\sim T\epsilon_F^{-1}\exp(-\Delta/T)$) have been discarded, takes the form

$$n_0\left(\frac{\sigma\varepsilon}{T}\right) n_0\left(-\frac{\sigma\varepsilon}{T}\right)\frac{\xi}{\varepsilon}\sigma\frac{p_F^2}{3mnT}\operatorname{div}(\mathbf{j} - n\mathbf{v}_n).$$

The right-hand side contains a collision integral, which, being written for a function

$$\psi_a(\varepsilon) = -\frac{8\pi^4 p_F^4}{3\lambda^2 n m^2 \Delta^3}\operatorname{div}(\mathbf{j} - n\mathbf{v}_n)\sigma\varphi_a(\xi)$$

of the dimensionless variable $\epsilon/\Delta \to \epsilon$, has the same form as the right-hand side of Eq. (22), apart from the factor 1/5 multiplying $\psi_a(\epsilon_2)$, which originates from the angular part. Substituting the solution of this equation

$$\psi_a(\varepsilon) = ue^{-u}\nu_a(\tau)$$

into the expression (18a), we obtain

$$n_1 = \frac{8\pi^2 C_a p_F^5}{3\lambda^2 n m \Delta^2}\operatorname{div}(\mathbf{j} - n\mathbf{v}_n). \quad (36)$$

It is found that the second viscosity, in the approximation used, is determined entirely by the expression (36). To convince ourselves of this, we estimate the order of F_1. The function occurring in F_1 that is symmetric in ξ is determined from an equation whose left-hand side, after the appropriate transformations of the expression (19), is, to within a numerical factor (small terms $\sim\lambda$ and smaller are also omitted), equal to

$$n_0\left(\frac{\sigma\varepsilon}{T}\right) n_0\left(-\frac{\sigma\varepsilon}{T}\right)\frac{\xi^2}{\varepsilon}\sigma\frac{1}{Tn\lambda}\operatorname{div}\mathbf{j}.$$

The above-mentioned equation is reduced, by means of transformations analogous to those which led to Eq. (24), to an integral equation that does not contain any parameters and whose solution is smooth and determines the numerical constant, unimportant for us, in F_1. Omitting it, we obtain for F_1 the estimate:

$$F_1 \sim \frac{p_F^2}{\lambda^2 n m \Delta^2}\operatorname{div}\mathbf{j} + C,$$

where C is a certain dimensional constant associated with the existence of the solution $\sigma\epsilon$ of the homogeneous equation corresponding to Eq. (19). To eliminate it, we make use of the condition $E_0^{(1)} = 0$:

$$\frac{\partial E_0}{\partial\mu}\delta\mu + \frac{\partial E_0}{\partial\Delta}\delta\Delta = -E_1, \quad E_1 = -\int\frac{d\mathbf{p}}{(2\pi)^3}\sum_\sigma\frac{p^2}{2m}\frac{\xi}{\varepsilon}\sigma f_\sigma^{(1)}(\mathbf{p}). \quad (37)$$

We can represent the quantity E_1 in the form

$$E_1 = \mu n_1 + E_1'.$$

It is easy to convince oneself that E_1' goes to zero at T = 0. Substituting the expression for E_1 into (37) and solving this equation consistently with Eqs. (18), we obtain

$$\delta\mu = -\frac{p_F^2}{3mn}n_1, \quad \delta\Delta = 0. \quad (38)$$

Thus, only one second-viscosity coefficient ζ, defined by the relation

$$\delta\mu / m = -\zeta\operatorname{div}(\mathbf{j} - n\mathbf{v}_n)$$

is non-zero (the relative order of the other second-viscosity coefficients is $T\Delta_0/\lambda\mu^2$). If (36) and (38) are taken into account, it is equal to

$$\zeta = \frac{8\pi^2 C_a}{9}\frac{p_F^7}{\lambda^2 m^3 n^2 \Delta_0^2}. \quad (39)$$

We now find the part of the solution of Eq. (19) close to T_c that is odd in ξ. We introduce the function $\varphi(\xi)$ of the dimensionless variable $\xi/T_c \to \xi$, related to $\varphi_\sigma(\xi)$ by: $\varphi(\xi) = \sigma\varphi_\sigma(\xi)$.

Omitting the unimportant terms in the collision integral (20), we write it in the form:

$$\hat{I}_\sigma(u)\varphi = -\frac{\lambda^2 m T_c^2}{8\pi^4 p_F^2}\sum_{\sigma_2}\int\limits_0^\infty d\xi_2\, K(0; \sigma\xi, \sigma_2\xi_2)$$
$$\times\left[\left(1 - \sigma\sigma_2\frac{u^2}{\xi\xi_2}\right)\varphi(\xi) - \frac{\xi\xi_2}{\sqrt{\xi^2 + u^2}\sqrt{\xi_2^2 + u^2}}\varphi(\xi_2)\right], \quad (40)$$

where K(u; x, y) is given by expression (28). The behavior of the solution for small u can be understood qualitatively if we note that $\hat{I}_\sigma^0$ is made to vanish by the function φ_0 = const. In fact, this means that the resolvent of the operator (40) as a function of u has a pole at the the point u = 0 and, consequently, the solution of Eq. (19) obtained by the action of the resolvent on the inhomogeneous part increases as a function of u as u approaches zero, if the inhomogeneous term has a non-zero projection on φ_0. For $u \neq 0$, the inhomogeneous term is not orthogonal to φ_0, this being connected with the non-conservation of particle number by the collision integral and with the fact, noted above, that the continuity equation is independent of the kinetic equation (6). By calculating the thermodynamic derivatives in (19) and retaining the principal part in the inhomogeneous term, we write the equation for $\varphi(\xi)$ in the form

$$n_0(\sigma\xi) n_0(-\sigma\xi)\frac{p_F^2}{3mnT_c}\operatorname{div}(\mathbf{j} - n\mathbf{v}_n) = \hat{I}_\sigma^0\varphi + \delta\hat{I}_\sigma(u)\varphi, \quad (41)$$

where $\delta\hat{I}_\sigma(u) = \hat{I}_\sigma(u) - \hat{I}_\sigma^0$. Introducing the scalar product by the formula

$$(f, g) = \sum_\sigma\int\limits_0^\infty d\xi\, f_\sigma(\xi) g_\sigma(\xi),$$

we expand $\varphi(\xi)$ in the eigenfunctions φ_k of the operator

$\hat{I}^0_\sigma$. Then it is clear from what has been said that the dominant contribution to the solution for $u \to 0$ will be given by the term with φ_0, which is easily found by expanding both sides of the equation in the set of functions φ_k, and in the principal approximation we obtain

$$\varphi(\xi) = \frac{a_0}{(\varphi_0, \delta I_\sigma \varphi_0)} \varphi_0, \tag{42}$$

where a_0 is the projection of the inhomogeneous term on φ_0. Taking into account that $\varphi_0 = \text{const}$, after a simple calculation of the integrals occurring in (42), we find

$$\varphi(\xi) = -\frac{16 p_F^4}{3\lambda^2 n m^2 T_c^3 u} \operatorname{div}(\mathbf{j} - n\mathbf{v}_n).$$

Substituting the solution found into (18a), we obtain

$$n_1 = \frac{8 p_F^5}{3\lambda^2 n m T_c \Delta} \operatorname{div}(\mathbf{j} - n\mathbf{v}_n). \tag{43}$$

We shall show that the second viscosity is determined entirely by the expression (43), as for $T = 0$. To within numerical factors, the inhomogeneous term of the equation for the part of the solution even in ξ is equal to

$$n_0\left(\frac{\sigma\xi}{T_c}\right) n_0\left(-\frac{\sigma\xi}{T_c}\right) \frac{\sigma}{\xi} \frac{T_c}{\lambda n} \operatorname{div} \mathbf{j}$$

and is singular at zero. Analysis of the equation shows that its solution is also singular at zero: $\varphi(\xi) \sim \xi^{-1}$ as $\xi \to 0$, and this leads to divergence of the integral in (18b) as $u \to 0$. When the contribution of the homogeneous solution is taken into account, the estimate for F_1 has the form

$$F_1 \sim \frac{p_F^2}{\lambda^2 n m T_c \Delta} \operatorname{div} \mathbf{j} + \frac{\lambda T_c}{2} C. \tag{44}$$

The constant appearing here is again eliminated with the help of Eq. (37), in which E_1' is given by the estimate:

$$E_1' \sim \frac{p_F^3}{\lambda^3 n} \operatorname{div} \mathbf{j} - m p_F T_c^3 C.$$

As a result, Eq. (18b) acquires the form

$$\frac{\partial F}{\partial \mu} \delta\mu - \frac{|g|\mu}{2T_c^2} \frac{\partial n}{\partial \Delta} \delta\Delta \sim -\frac{p_F^2}{\lambda^2 n m T_c \Delta} \operatorname{div} \mathbf{j}.$$

Solving it consistently with Eq. (18a), we obtain

$$\delta\mu = -\frac{p_F^2}{3mn} n_1, \quad \delta\Delta \sim \frac{p_F^2 T_c}{\lambda^2 n m \Delta^2} \operatorname{div} \mathbf{j}. \tag{45}$$

In the expression for $\delta\mu$, we have omitted terms of order $T_c^3/\lambda\Delta\mu^2$ relative to the term describing the first viscosity, which are small by virtue of the inequality (4). From (45), the same estimate also follows for the second-viscosity coefficients occurring in the flux conservation law (9) and contained in the term $\Delta\delta\Delta/m|g|$. Substituting (43) into (45), we obtain the coefficient ζ in the form

$$\zeta = 8 p_F^7 / 9\lambda^2 m^3 n^2 T_c \Delta. \tag{46}$$

Attention is drawn to the singularity of this coefficient at the critical point. But, as a whole, the term describing the second viscosity goes to zero in proportion to Δ at the critical point (like the correction to the first-viscosity coefficient also), since it contains the factor $\operatorname{div}(\mathbf{j} - n\mathbf{v}_n)$, which decreases in proportion to Δ^2 near T_c.

In conclusion, we write out the complete system of first-order hydrodynamic equations, denoting the equilibrium fluxes by the index "zero":

$$\partial n/\partial t + \operatorname{div} \mathbf{j} = 0,$$

$$\frac{\partial J_i}{\partial t} + \frac{\partial}{\partial R_k} \Pi_{ik}^{(0)} = \frac{\partial}{\partial R_k}\left[\eta\left(\frac{\partial v_{ni}}{\partial R_k} + \frac{\partial v_{nk}}{\partial R_i} - \frac{2}{3}\delta_{ik} \operatorname{div} \mathbf{v}_n\right)\right],$$

$$\frac{\partial E}{\partial t} + \operatorname{div} \mathbf{Q}^{(0)} = \operatorname{div}[\varkappa \nabla T],$$

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla\left(\frac{v_s^2}{2} + \frac{\mu}{m} - \frac{|g|n}{2}\right) = \nabla[\zeta \operatorname{div}(\mathbf{j} - n\mathbf{v}_n)].$$

The transport coefficients occurring in this system are determined by the formulas (25), (33)—(35), (39) and (46). (46).

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