ROLE OF NONLINEAR EFFECTS IN THE PROBLEM OF THE ANOMALOUS RESISTANCE OF PLASMA

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Nonlinear effects in the problem of anomalous plasma resistance is investigated under asymptotic conditions. It is shown that the self-similar solution obtained in the quasilinear approximation formally leads to an explosive instability with characteristic time constants which are significantly smaller than the fundamental time scale in the self-similar problem. Allowance for the nonlinear effects leads to a definition of rigid limits of applicability for the quasilinear approximation.

VEKSHTEĬN et al.[1-3] have investigated the anomalous resistance and turbulent heating of plasma by a current under asymptotic conditions, i.e., when the thermal energy density in the plasma is substantially greater than the initial density, so that the particle distribution function and the noise spectrum do not depend on the initial plasma parameters. These workers consider the three-dimensional (for H = 0) and one-dimensional (for $\omega_{Hi} \gg \omega_{pi}$) problems. The solution of the latter leads to the following physical results: 1) practically all the ions and electrons are freely accelerated by the electric field under the asymptotic conditions; 2) in contrast to the initial stage of the heating process,[4,5] when only the ion-acoustic noise is developed in plasma, the Langmuir oscillations in the asymptotic state have a much higher energy density than the ion-acoustic oscillations.

The above results were obtained in the quasilinear approximation and nonlinear effects were assumed to be unimportant. We note, however, that this assumption is valid only when the characteristic time for the nonlinear interaction between the waves is much greater than the characteristic time scale in the linear theory (reciprocal of the growth rate):

$$\tau_n \gg \gamma_l{}^{-1}. \tag{1}$$

When this condition is not satisfied the nonlinear effects will substantially contribute to the heating of particles and the evolution of the wave spectrum. It will be shown below that the inequality (1) leads to very rigid conditions for the validity of the self-similar solutions[1-3] and these can hardly be satisfied experimentally.

Let us consider as an example the one-dimensional problem of the instability of a current in plasma, which was solved rigorously in the quasilinear approximation,[1-3] and consider the contribution of nonlinear effects to the evolution of the noise spectrum. Using the self-similar variables

$$\Omega := \frac{\omega}{\omega_{pe}}, \quad u = \frac{v_z}{u_T}, \quad q = \frac{k_z u_T}{\omega_{pe}}, \quad u_T = \frac{eEt}{m} \tag{2}$$

for the particle distribution function

$$f_{e,i}(v_z) = u_T^{-1} g_{e,i}(u) \tag{3}$$

and the spectral noise density

$$W(k_z, t) = \frac{E}{8\pi^2} \frac{m u_T{}^2}{e} w(q) \tag{3'}$$

the authors of[1] show that

$$g_e = 2\mu^{1/4} \frac{u}{u + \mu^2} + (1 - 2\mu^{1/4}) \delta(u - 1), \tag{4}$$

$$g_i = 2\mu^{3/2} \frac{1 - u}{u + \mu^2} + (1 - 4\mu^{1/2} \ln \mu^{-1}) \delta(u),$$

$$w(q) = \mu^{-2} (\Omega/q)^2 (1 - \Omega/q) |\Omega/q - d\Omega/dq|,$$
$$\mu = m/M.$$

The dispersion relation for the potential oscillations, taking the distribution functions given by Eqs. (3) and (4) into account, assumes the following form:

$$\varepsilon(\Omega, q) = 1 - q^{-2} G(u_p) = 0;$$
$$G(u_p) = \frac{1 - 2\mu^{1/4}}{(u_p - 1)^2} + \frac{2\mu^{1/4}}{u_p(u_p - 1)} + \frac{\mu}{u_p{}^2}, \quad u_p = \frac{\Omega}{q}. \tag{5}$$

To within small corrections of the order of $\mu^{1/2}$ the expressions given by Eq. (5) can be used to obtain explicit expressions for $\Omega = \Omega_n(q)$:

$$\Omega_1 = \mu^{1/2} q / (q + 1),$$

$$\Omega_2 = \begin{cases} \mu^{1/2} q / (1 - q) & q < 1 - 2\mu^{1/4} \\ q - 1 + \mu^{1/2} q / (q - 1), & q > 1 + 2\mu^{1/4} \end{cases},$$

$$\Omega_3 = q + 1 - \mu^{1/2} q / (q + 1),$$
$$\Omega_4 = \frac{1}{2}[q - 1 - ((q - 1)^2 + 4q\mu^{1/2})^{1/2}]. \tag{6}$$

It is readily shown from Eqs. (5) and (6) that the derivatives are given by

$$(\varepsilon_\omega')_1 \approx \frac{2}{\omega_{pi}} \left(\frac{q+1}{q}\right)^2, \quad (\varepsilon_\omega')_3 \approx \frac{2}{\omega_{pe}},$$

$$(\varepsilon_\omega')_2 \approx \begin{cases} -2(1 - q)^2/\omega_{pi} q^2, & q < 1 - 2\mu^{1/4} \\ -2/\omega_{pe}, & q > 1 + 2\mu^{1/4} \end{cases}. \tag{7}$$

Hence, it is clear that the branch $\Omega_2(q)$ represents oscillations with negative energy. Dikasov et al.[6] consider the nonlinear interaction of waves with positive and negative energies, including the explosive instability connected with the simultaneous generation of various types of wave. This instability can develop if

$$\omega_-(k) = \omega_+(k') + \omega_+(k - k'),$$

where ω_- and ω_+ are the frequencies of waves with negative and positive energies, respectively. In our case, interaction is possible only when three waves belonging to different branches of the dispersion curve participate in the process. The wave with negative energy can then have an arbitrary wave number $q > 2$, whereas for the remaining waves the conservation laws yield

$$\Omega_2(q) = \Omega_3(q') + \Omega_1(q''),$$

$$\Omega_3 \approx q - 1 + \mu^{1/6}\left(\frac{4}{3} - \frac{q-2}{q-1}\right), \quad \Omega_1 \approx \frac{2}{3}\mu^{1/6};$$

$$q' \approx q - 2 + {}^4\!/_3\mu^{1/6}, \quad q'' \approx 2 - {}^4\!/_3\mu^{1/6}. \tag{8}$$

The instability results in a growth of the wave with negative energy in a broad spectral range ($1/2 < \Omega/q < 1$), whereas waves belonging to branch 1 develop in a very narrow range ($\Delta q'' \ll \mu^{1/2}$).

The kinetic equations for the waves, which describe the explosive instability, can be written in the form

$$\dot{n}_{2k} = \int dk' V_{kk'}[n_{2k}n_{3k'} + n_{3k'}n_{1k-k'} + n_{1k-k'}n_{2k}],$$

$$\dot{n}_{3k} = \int dk' V_{k'k}[n_{2k}n_{3k} + n_{3k}n_{1k'-k} + n_{1k'-k}n_{2k'}],$$

$$\dot{n}_{1k} = \int dk' V_{k',k'-k}[n_{2k}n_{3k'-k} + n_{3k'-k}n_{1k} + n_{1k}n_{2k'}], \tag{9}$$

where $n_k = |W_k/\omega_k|$ and the kernel $V_{kk'}$ can be evaluated by standard methods:[6,7]

$$V_{kk'} = \frac{8\pi^2}{|\varepsilon_\omega'\varepsilon_{\omega'}'\varepsilon_{\omega''}'|}\delta(\omega_k - \omega_{k'} - \omega_{k''})|S_{kk'}|^2,$$

$$S_{kk'} = \omega_{pe}^2\frac{e}{m}\int\frac{dv}{(\omega - kv)(\omega' - k'v)(\omega'' - k''v)}\frac{\partial}{\partial v}(f^v - \mu^2 f^i),$$

$$\omega'' = \omega - \omega', \quad k'' = k - k'. \tag{9'}$$

Let us transform Eq. (9') to the variables given by Eq. (2), substitute Eq. (4) into it, and isolate the leading term which clearly corresponds to the electron contribution. This results in a considerable simplification of the above expression, namely:

$$V_{kk'} = \frac{\pi}{4}\omega_{pi}\frac{\omega_{pe}}{nmu_T^2}\left[1 + \frac{2}{(q-1)^2} - \frac{2}{3}q\mu^{1/6}\right]^2\delta(\Omega_q - \Omega_{q'} - \Omega_{q''}). \tag{10}$$

Let us first confine our attention to the spectral range $2 < q < \mu^{-1/2}$, in which Eq. (10) assumes a particularly simple form. Substituting Eq. (10) into Eq. (9), integrating with respect to k' and bearing in mind the fact that, according to[1], waves belonging to branch 3 do not grow in the quasilinear approximation, we find that during the initial stage of the explosive instability, Eq. (9) assumes the form

$$\dot{n}_2 = \dot{n}_3 = Vn_1n_2, \quad \dot{n}_1 = V\chi n_1n_2,$$

$$V = \frac{\pi}{4}\omega_{pi}\frac{\omega_{pe}^2}{nmu_T^3}, \quad \chi = \frac{3}{8}\frac{(q-1)^3}{\mu}. \tag{11}$$

The set of equations given by (11) can be used to estimate quite readily the characteristic time for the development of the explosive instability. Thus, the doubling time for the number of waves n_1 is given by

$$\tau_1 \approx (V\chi n_2|_{t=0})^{-1},$$

whereas that for n_2 is given by

$$\tau_2 \approx \left(\frac{1}{V\chi n_2}\ln\frac{\chi n_2}{n_1}\right)_{t=0}.$$

We shall take the initial noise density n_1 and n_2 from Eq. (4) (the results of Vekshtein et al.[1]), taking the dispersion relations given by Eq. (6) into account. The

final expressions are

$$\tau_1 \approx 4\pi\left(\frac{m}{M}\right)^{1/2}\tau, \quad \tau_2 \approx 8\pi\left(\frac{m}{M}\right)^{1/2}\ln\left(5^{1/2}\frac{M}{m}\right)\tau, \tag{12}$$

where $\tau = mu_T/eE$ is the fundamental time scale in the self-similar problem.[1] Thus, the noise density doubles during a time $\tau_2 \ll \tau$ in a broad spectral range ($q > 2$). The narrow line in the wave spectrum $\Omega_1(q)$, which leads to a singularity in the quasilinear diffusion coefficient, grows even more rapidly ($\tau_1 \ll \tau_2$).

It follows from the foregoing that the solution of the anomalous resistance problem demands, in general, the inclusion of nonlinear effects. The formal presence of the explosive instability in the solution given by Eq. (4) does not mean that a highly turbulent state is established under asymptotic conditions.[1-3] It is simply necessary to take into account the nonlinear terms in the oscillation growth rate:

$$\gamma_l + \gamma_n = 0, \tag{13}$$

where γ_l and γ_n are the characteristic linear and nonlinear growth rates, respectively. The question then is whether the corresponding asymptotic solution will, in some way, approach the solution given by Eq. (4).

Let us consider the ion-acoustic noise [the branch $\Omega_1(q)$]. If the true solution is not very different from Eq. (4), $\gamma_n \sim \tau_1^{-1}$ and $\gamma_1 \ll \omega_{pi}$. Substituting for τ_1 from Eq. (12), we can readily show that Eq. (13) can be satisfied if

$$E/(4\pi nmu_T^2)^{1/2} \ll 4\pi(m/M)^3, \tag{14}$$

and since the theory reported in[1-3] is constructed for nonrelativistic current velocities, we have from Eq. (14)

$$E^2 \ll (4\pi)^3 nmc^2(m/M)^6. \tag{15}$$

This inequality is practically never satisfied in plasma heating experiments and, therefore, the solution given by Eq. (4) is not valid under these conditions.

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