THEORY OF MAGNETOACOUSTIC RESONANCE

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The equations of motion of the magnetization and of the electric quadrupole-moment density are derived for an acoustically excited paramagnet. In the approximation of high temperatures and small correlation times, the possibility of hard excitation of nonlinear magnetoacoustic resonance is demonstrated.

1. It is well known that ultrasound can excite transitions in the Zeeman spectrum of a paramagnet located in an external magnetic field[1]. In the present paper the dynamics of this process, which is called magnetoacoustic resonance (MAR), will be treated on the basis of one of the methods of the statistical theory of nonequilibrium processes—the method of Robertson[2].

The Hamiltonian of an acoustically excited paramagnet in general has the form

$$\hat{\mathscr{H}}(t) = \int [\hat{\mathscr{H}}_l(\mathbf{r}) + \hat{\mathscr{H}}_{s_1}(\mathbf{r}) + \mathscr{H}_{s}(\mathbf{r}, t) + \hat{\mathscr{H}}_z(\mathbf{r}, t)]dV, \quad (1)$$

where the first term is the Hamiltonian of the crystal lattice, the second describes possible interactions of magnetic and nonmagnetic origin within the paramagnet, the third describes interactions of the sound with the magnetic subsystem, and the last is the Zeeman energy

$$\hat{\mathscr{H}}_z(\mathbf{r}, t) = -\mathbf{M}(\mathbf{r})\mathbf{H}(\mathbf{r}, t), \quad (2)$$

here $\hat{\mathbf{M}}(\mathbf{r})$ is magnetic-moment density operator, which is related to the spin-density operator $\hat{\mathbf{I}}(\mathbf{r})$ by

$$\mathbf{M}(\mathbf{r}) = \hbar g \mathbf{I}(\mathbf{r}) \quad (3)$$

(g is the gyromagnetic tensor); $\mathbf{H}(\mathbf{r}, t)$ is the external magnetic field.

Interaction of sound with the crystal lattice, which is accompanied by nonresonant absorption of the sound, is not considered in the case assumed.

For paramagnets with spin $I > 1/2$, the Hamiltonian $\mathscr{H}_s$ has the form[1]

$$\mathscr{H}_s(\mathbf{r}, t) = \hbar\omega_a \sum_{q=-2}^{2} (-1)^q \hat{T}_{2q}(\mathbf{r}) E_{-q}(\mathbf{r}, t). \quad (4)$$

Here $\hbar\omega_a$ is the coupling energy, $\hat{T}_{2q}$ are irreducible tensor operators of the second rank, normalized by the relation $\operatorname{Sp} \hat{T}_{2q}\hat{T}_{2q}^{+} = 1$, and $E_{-q}(\mathbf{r}, t)$ can be expressed in terms of linear combinations of the components of the deformation tensor.

We introduce the system of Hermitian tensor operators $\hat{D}_{Lm}$ defined by the equations

$$\begin{gathered}\hat{D}_{10} = 2^{-1/2}\hat{I}_z, \quad \hat{D}_{11} = 2^{-1/2}\hat{I}_x, \quad \hat{D}_{1-1} = 2^{-1/2}\hat{I}_y, \\ \hat{D}_{20} = \hat{T}_{20}, \quad \hat{D}_{21} = 2^{-1/2}(\hat{T}_{2-1} - \hat{T}_{21}), \quad \hat{D}_{2-1} = 2^{-1/2} i(\hat{T}_{21} + \hat{T}_{2-1}), \\ \hat{D}_{22} = -2^{-1/2}(\hat{T}_{22} + \hat{T}_{2-2}), \quad \hat{D}_{2-2} = 2^{-1/2} i(\hat{T}_{22} - \hat{T}_{2-2})\end{gathered} \quad (5)$$

and normalized by the condition

$$\operatorname{Sp} D_{Lm}\hat{D}_{Lm}^{+} = \operatorname{Sp} \hat{D}_{Lm}^{2} = 1 \quad (6)$$

(we note that the operators $\hat{D}_{2m}$ are, except for a numerical factor, the operators of the components of the quadrupole-moment tensor). The Hamiltonian (1), expressed in terms of the operators $\hat{D}_{Lm}$, has the form

$$\hat{\mathscr{H}}(t) = \int \Big[\hat{\mathscr{H}}_l(\mathbf{r}) + \hat{\mathscr{H}}_{\text{int}}(\mathbf{r}) + \hbar\Omega \sum_{L=1}^{2}\sum_{m=-L}^{L} A_{Lm}(\mathbf{r}, t)\hat{D}_{Lm}(\mathbf{r})\Big] dV. \quad (7)$$

For the case of spin I = 1, to which we shall restrict ourselves, the set of operators $\hat{D}_{Lm}$, supplemented by the unit operator, forms a complete orthonormal system.

In the theory of MAR, when the Hamiltonian (1) and (7) is used, the equations of motion of the magnetization and of the electric quadrupole moment are coupled[1]. As thermodynamic coordinates that guarantee a macroscopically complete description of the system, we choose the quantities

$$D_{Lm}(\mathbf{r}, t) = \operatorname{Sp} \hat{D}_{Lm}(\mathbf{r})\hat{\rho}(t), \quad (8)$$

$$U(t) = \operatorname{Sp} \int [\hat{\mathscr{H}}_l(\mathbf{r}) + \hat{\mathscr{H}}_{\text{int}}(\mathbf{r})]dV \cdot \hat{\rho}(t), \quad (9)$$

where $U(t)$ is the total internal energy of the specimen; $\hat{\rho}(t)$ is the statistical operator, which satisfies von Neumann's equation

$$d\hat{\rho}(t)/dt = -i\tilde{L}(t)\hat{\rho}(t), \quad (10)$$

and $\tilde{L}(t)$ is the evolution operator, which acts on an arbitrary operator $\hat{A}$ in the following manner:

$$\tilde{L}(t)\hat{A} = \hbar^{-1}[\hat{\mathscr{H}}(t), \hat{A}] \quad (11)$$

(here and hereafter, the tilde designates an operator that acts on operators defined in the Hilbert space of the system).

2. We introduce, in accordance with[2], the canonical density operator

$$\hat{\sigma}(t) = \alpha \exp\Big\{-\beta(t)\int\Big[\hat{\mathscr{H}}_l(\mathbf{r}) + \hat{\mathscr{H}}_{\text{int}}(\mathbf{r}) + \hbar\Omega\sum_{Lm} A_{Lm}^{*}(\mathbf{r}, t)\hat{D}_{Lm}(\mathbf{r})\Big]dV\Big\} \quad (12)$$

$$\alpha^{-1} = \operatorname{Sp}\exp\Big\{-\beta(t)\int\Big[\hat{\mathscr{H}}_l(\mathbf{r}) + \hat{\mathscr{H}}_{\text{int}}(\mathbf{r}) + \hbar\Omega\sum_{Lm} A_{Lm}^{*}(\mathbf{r}, t)\hat{D}_{Lm}(\mathbf{r})\Big]dV\Big\} \quad (13)$$

The values of $\beta(t)$ and $A_{Lm}^{*}(\mathbf{r}, t)$ are determined by the equations

$$D_{Lm}(\mathbf{r}, t) = \operatorname{Sp} \hat{D}_{Lm}(\mathbf{r})\hat{\sigma}(t), \quad (14)$$

$$U(t) = \operatorname{Sp}\int [\hat{\mathscr{H}}_l(\mathbf{r}) + \hat{\mathscr{H}}_{\text{int}}(\mathbf{r})]dV \cdot \hat{\sigma}(t). \quad (15)$$

Each of the thermodynamic coordinates $\Phi_n(\mathbf{r}, t)$ satisfies the following equation[2]:

$$\frac{\partial \Phi_n(\mathbf{r},t)}{\partial t} = -i\,\mathrm{Sp}\,\hat{\Phi}_n(\mathbf{r})\tilde{L}(t)\hat{\sigma}(t)$$

$$-\int_0^t \mathrm{Sp}\{\hat{\Phi}_n(\mathbf{r})\tilde{L}(t)\tilde{T}(t,t')[1-\tilde{P}(t')]\tilde{L}(t')\hat{\sigma}(t')\}dt', \qquad (16)$$

where the projection operator $\tilde{P}(t)$ acts on all operators to the right of it:

$$\tilde{P}(t)\hat{A} = \sum_m \int \left[\frac{\delta\hat{\sigma}(t)}{\delta\Phi_m(\mathbf{r},t)}\mathrm{Sp}\,\hat{\Phi}_m(\mathbf{r})\hat{A}\right]dV, \qquad (17)$$

the operator $\tilde{T}(t, t')$ is the solution of the equation

$$\partial\tilde{T}(t,t')/\partial t' = i\tilde{T}(t,t')[1-\tilde{P}(t')]\tilde{L}(t') \qquad (18)$$

with the initial condition $\tilde{T}(t, t) = 1$. It is assumed that at $t = 0$ the system was in equilibrium.

We shall transform the expression $\tilde{L}(t)\hat{\sigma}(t)$ that appears in (16). On using (11) and (12) and on supplementing (7) to form the expression in the argument of the exponential function in $\hat{\sigma}(t)$, we get

$$\tilde{L}(t)\hat{\sigma}(t) = \Omega\int\sum_{Lm}(A_{Lm}(\mathbf{r},t) - A_{Lm}^*(\mathbf{r},t))[\hat{D}_{Lm}(\mathbf{r}),\hat{\sigma}(t)]dV. \qquad (19)$$

The dynamic term of equation (16) for the thermodynamic coordinates (8) takes the form

$$-i\Omega\sum_{L'm'}\int(A_{L'm'}(\mathbf{r}',t) - A_{L'm'}^*(\mathbf{r}',t))\mathrm{Sp}[\hat{D}_{Lm}(\mathbf{r}),\hat{D}_{L'm'}(\mathbf{r}')]\hat{\sigma}(t')dV' \qquad (20)$$

(the identity $\mathrm{Sp}\,\hat{A}[\hat{B},\hat{C}] = \mathrm{Sp}\,[\hat{A},\hat{B}]C$ has been used). On expanding the commutators $[\hat{D}_{Lm}(\mathbf{r}), \hat{D}_{L'm'}(\mathbf{r}')]$ with respect to the complete system of operators $\hat{D}_{Lm}$, we get

$$[\hat{D}_{Lm}(\mathbf{r}),\hat{D}_{L'm'}(\mathbf{r}')] = i\delta_{\mathbf{r},\mathbf{r}'}\sum_{L''m''}\alpha_{Lm,L'm'}^{L''m''}\hat{D}_{L''m''}(\mathbf{r}'). \qquad (21)$$

Because of the orthonormality of the system of operators $\hat{D}_{Lm}$, the coefficients $\alpha_{Lm,L'm'}^{L''m''}$ are equal to

$$\alpha_{Lm,L'm'}^{L''m''} = -i\,\mathrm{Sp}[\hat{D}_{Lm},\hat{D}_{L'm'}]\hat{D}_{L''m''}^+ \qquad (22)$$

and are connected (because of (22)) by the relations

$$\alpha_{Lm,L'm'}^{L''m''} = \alpha_{L'm',L''m''}^{Lm} = \alpha_{L''m'',Lm}^{L'm'}, \quad \alpha_{Lm,L'm'}^{L''m''} = -\alpha_{L'm',Lm}^{L''m''}. \qquad (23)$$

Their values for spin $I = 1$ are given in the Appendix.

We define $K_{Lm}^{L'm'}(\mathbf{r},\mathbf{r}';t,t')$

$$= \mathrm{Sp}\,\hat{D}_{Lm}(\mathbf{r})\tilde{L}(t)\tilde{T}(t,t')[1-\tilde{P}(t')][\hat{D}_{L'm'}(\mathbf{r}'),\hat{\sigma}(t')]. \qquad (24)$$

Then, with (14) taken into account, equation (16) for the thermodynamic coordinates $D_{Lm}(\mathbf{r}, t)$ takes the form

$$\frac{\partial D_{Lm}(\mathbf{r},t)}{\partial t} = \Omega\sum_{L'm'}(A_{L'm'}(\mathbf{r},t) - A_{L'm'}^*(\mathbf{r},t))\sum_{L''m''}\alpha_{Lm,L'm'}^{L''m''}D_{L''m''}(\mathbf{r},t)$$

$$-\Omega\sum_{L'm'}\int_0^t dt'\int dV' K_{Lm}^{L'm'}(\mathbf{r},\mathbf{r}';t,t')(A_{L'm'}(\mathbf{r}',t') - A_{L'm'}^*(\mathbf{r}',t')). \qquad (25)$$

The equation for the thermodynamic coordinate $U(t)$ can be obtained by integrating (16) over the volume. It is easier to do this, however, by expressing dU/dt with dynamical exactness in terms of $\partial D_{Lm}/\partial t$. For this purpose we use the definition (9), differentiating it with respect to time; then with the aid of (10) and (8) we get

$$\frac{dU}{dt} = -\Omega\sum_{Lm}\int A_{Lm}(\mathbf{r},t)\frac{\partial D_{Lm}(\mathbf{r},t)}{\partial t}dV. \qquad (26)$$

Equations (25) and (26) together with equations (14) and (15) solve the problem posed exactly, without any assumptions. But the values of $\beta(t)$ and $A_{Lm}^*(\mathbf{r}, t)$ that occur in the equations are themselves functionals of the thermodynamic coordinates, implicitly defined by equations (14) and (15).

3. Let the temperature of the lattice be sufficiently high, and let its heat capacity be much greater than the heat capacity of the spin system. We shall identify $\beta(t)$ with the reciprocal of the constant temperature of the lattice. On expanding the operator $\hat{\sigma}(t)$ as a power series in β and on retaining terms of the expansion up to and including those of second order in β, we have

$$\hat{\sigma}(t) \approx \hat{\sigma}_l\left[1 + \beta\left(a - \int_0^\beta \hat{B}_\mu d\mu\right) + \beta^2\left(b - a\int_0^\beta \hat{B}_\mu d\mu + \int_0^\beta \hat{B}_\mu\int_0^\mu \hat{B}_\nu d\nu\, d\mu\right)\right], \qquad (27)$$

where

$$\hat{\sigma}_l = [\mathrm{Sp}\exp(-\beta\hat{\mathscr{H}}_l)]^{-1}\exp(-\beta\hat{\mathscr{H}}_l); \qquad (28)$$

$$a = \mathrm{Sp}\int_0^\beta \hat{B}_\mu d\mu, \quad b = \mathrm{Sp}\int_0^\beta \hat{B}_\mu\int_0^{\mu_1}\hat{B}_\nu d\nu\, d\mu,$$

$$\hat{B}_\mu = \beta^{-1}\exp(\mu\hat{\mathscr{H}}_l)\hat{B}\exp(-\mu\hat{\mathscr{H}}_l), \qquad (29)$$

$$\hat{B} = \int\left[\hat{\mathscr{H}}_{\mathrm{int}}(\mathbf{r}) + \hbar\Omega\sum_{Lm}A_{Lm}^*(\mathbf{r},t)\hat{D}_{Lm}(\mathbf{r})\right]dV.$$

We shall use (27) to determine the explicit functional dependence of $A_{Lm}^*(\mathbf{r}, t)$ on $D_{Lm}(\mathbf{r}, t)$. To the second order in β, we get

$$A_{Lm}^*(\mathbf{r},t) = -(\beta\hbar\Omega)^{-1}D_{Lm}(\mathbf{r},t) - \sum_{L'm'}N_{Lm}^{L'm'}D_{L'm'}(\mathbf{r},t), \qquad (30)$$

where

$$N_{Lm}^{L'm'} = (\hbar\Omega)^{-1}\,\mathrm{Sp}\,\hat{D}_{Lm}(\mathbf{r})\int_0^\beta d\mu\int_0^\mu d\nu(\hat{\mathscr{H}}_{\mathrm{B3}\,\mu}\hat{D}_{L'm'\nu} + \hat{D}_{L'm'\mu}\hat{\mathscr{H}}_{\mathrm{int}}) \qquad (31)$$

is the analog of the demagnetizing coefficients obtained earlier[3].

If in the calculation of the kernel $K_{Lm}^{L'm'}$ of equation (25) we restrict the external fields to a constant magnetic field, and do not allow for alternating magnetic and acoustic fields (the case of "normal saturation"), then $K_{Lm}^{L'm'}$ do not depend on t and t' separately but are rapidly decaying functions of $\tau = t - t'$[3]. Neglecting memory effects, we write (25) in the form (taking account of (30))

$$\frac{\partial D_{Lm}(\mathbf{r},t)}{\partial t} = \Omega\sum_{L'm'}\left[A_{L'm'}(\mathbf{r},t) + \sum_{L''m''}N_{L'm'}^{L''m''}D_{L''m''}(\mathbf{r},t)\right]$$

$$\times\sum_{L'''m'''}\alpha_{Lm,L'm'}^{L'''m'''}D_{L'''m'''}(\mathbf{r},t) - \sum_{L'm'}\int\varkappa_{Lm}^{L'm'}[D_{L'm'}(\mathbf{r}',t) + \beta\hbar\Omega A_{L'm'}(\mathbf{r}',t)]dV', \qquad (32)$$

where

$$\varkappa_{Lm}^{L'm'}(\mathbf{r},\mathbf{r}') = (\beta\hbar)^{-1}\int_0^\infty K_{Lm}^{L'm'}(\mathbf{r},\mathbf{r}';\tau)d\tau. \qquad (33)$$

Neglect of "memory", i.e., replacement of the integro-differential equation (26) by the differential equation (32), is possible when the set of inequalities[4]

$$\eta \ll 1, \quad \varkappa_{Lm}^{L'm'} \ll \omega_c \qquad (34)$$

is satisfied (η is defined by equation (38), below; ω_c is a representative value of the set of correlation frequencies). The quantities (33) can be calculated by the Green's function method, by means of the perturbation theory constructed for calculation of analogous quantities[3]. We note that relaxation parameters for the macroscopic quadrupole moment were first considered by Kessel'[5] (also for spin I = 1).

4. As one of the possible applications of the equations obtained above, we shall consider the question of the possibility of excitation of nonlinear magnetoacoustic resonance in paramagnets. It is well known that in certain ferromagnets and ferrites, by means of sufficiently powerful ultrasound, nonlinear (parametric) magnetic resonance can be excited in a threshold manner with respect to the amplitude of the sound wave. Measurement of the excitation threshold and of the threshold characteristics gives extensive information about magnetic crystals[6]. As a cause of instability of the uniform precession, we shall consider only the anisotropic dipole field. If the coefficients $N_{Lm}^{L'm'}$ are due to dipole-dipole interactions, then

$$N_{Lm}^{L'm'}=(2\hbar\Omega)^{-1}\operatorname{Sp}\hat{D}_{Lm}(\mathbf{r})\int dV'\{\hat{\mathscr{H}}_{\text{dip}}(\mathbf{r},\mathbf{r}'),\hat{D}_{L'm'}(\mathbf{r}')\}_{+}. \quad (35)$$

It is obvious that $N_{2m}^{L'm'} = N_{Lm}^{2m'} = 0$ (since the real spur of an odd number of spin operators vanishes). We write $N_{1m}^{1m'} = N_{mm'}$ and note that $N_{mm'}(m, m' = 0, \pm1)$ is a tensor of second rank (demagnetization tensor).

In equations (32) it is convenient to transform to the dimensionless variables

$$D_{Lm}'(\mathbf{r},t)=D_{Lm}(\mathbf{r},t)/\beta\gamma\hbar H_0, \quad (36)$$

where H_0 is the constant magnetic field, directed along the z axis, and γ is the gyromagnetic ratio. By means of (36) the components of the vector $\mathbf{M}(\mathbf{r}, t)$ are dedimensionalized by the equilibrium value M_0. The quantities $A_{2m}(\mathbf{r}, t)$ have the form[1]

$$A_{2m}(\mathbf{r},t)=\frac{1}{\sqrt{2}}\frac{2\pi A_0}{\lambda}\mathscr{E}_m(\mathbf{r})\cos\omega t, \quad (37)$$

where $\mathscr{E}_m(\mathbf{r})$ are linear combinations of the coordinate parts of the components of the deformation tensor produced by the sound, λ is the wavelength of the sound, and A_0 is the amplitude of the displacement. After the dedimensionalization (36), there appear in equations (32) the small dimensionless parameters

$$\eta=\frac{\omega_a}{\sqrt{2}\gamma H_0}\frac{2\pi A_0}{\lambda} \quad (\eta\ll 1) \quad (38)$$

and

$$\varepsilon_{mm'}=\Omega\beta\hbar N_{mm'} \quad (\varepsilon_{mm'}\ll 1), \quad (39)$$

which determine respectively the percentage modulation and the "detuning" of the mean dipole field of the heat motion. We write, as an illustration, the equation for $D'_{1-1}(\mathbf{r}, t)$ (i.e., for $M_y(\mathbf{r}, t)$) in explicit form:

$$\begin{aligned}\frac{\partial D'_{1-1}}{\partial t}&=-\omega_0 D_{11}'-\omega_0\eta\mathscr{E}_{-1}D'_{2-2}\cos\omega t+\omega_0\eta\mathscr{E}_{-2}D'_{2-1}\cos\omega t\\&-\omega_0\eta\mathscr{E}_1 D_{22}'\cos\omega t+\omega_0\eta\mathscr{E}_2 D'_{2-1}\cos\omega t-\sqrt{3}\,\omega_0\eta\mathscr{E}_1 D_{20}'\cos\omega t\\&+\sqrt{3}\,\omega_0\eta\mathscr{E}_0 D_{21}'\cos\omega t+\sqrt{2}\sum_{m,n,l=-1}^{1}a_{1-1,1m}^{1l}\,\varepsilon_{mn}D_{1n}'D_{1l}'\\&-\sum_{Lm}\int\varkappa_{1-1}^{Lm}(\mathbf{r},\mathbf{r}')\left[D_{Lm}'(\mathbf{r}',t)+\frac{\Omega}{\omega_0}A_{Lm}(\mathbf{r}',t)\right]\end{aligned} \quad (40)$$

(we shall not write all the equations because of their large size). Here $\omega_0 = \gamma H_0$ is the Larmor frequency.

Thus we have a system of eight quasilinear equations, which we shall solve by the Krylov-Bogolyubov method[7], generalized to the case of several small parameters. Because the system (32) is very complicated, we shall restrict ourselves to the following simple approximation in the calculation of the relaxation terms: in each of the equations, we keep only the "self-" relaxation term; that is, we set $\kappa_{Lm}^{L'm'} = \delta_{LL'}\delta_{mm'}\kappa_{Lm}^{Lm}$ (allowance for the cross-relaxation terms would lead only to small corrections for the characteristic frequencies of oscillations of the magnetization and of the quadrupole moment). If the chief relaxation mechanism is the spin-lattice mechanism (it is just this case that is of especial interest to us), then it can be shown that all the quantities κ_{Lm}^{Lm} are of a single order; and in the lowest order of perturbation theory in which they differ from zero, $\kappa_{Lm}^{Lm}(\mathbf{r}, \mathbf{r}') = \kappa\delta(\mathbf{r} - \mathbf{r}')$.

Under the assumptions made, for $\eta = 0$ and $\epsilon_{mm'} = 0$ equations (32) have the general solution

$$D_{Lm}'^{(0)}=e^{-\varkappa t}\left(A_{Lm}\sin|m|\,\omega_0 t-\frac{m}{|m|}A_{L-m}\cos m\omega_0 t\right)+\delta_{L1}\delta_{m0}, \quad (41)$$

where A_{Lm} are constants of integration. We set

$$\varepsilon_{mm'}=\varepsilon\nu_{mm'}. \quad (42)$$

We shall seek a solution of the system (32) in the form of a multiple power series in the small parameters:

$$D_{Lm}'(t)=D_{Lm}'^{(0)}(t)+\sum_{r,s=1}^{\infty}\varepsilon^r\eta^s u_{Lm}^{(r,s)}(t). \quad (43)$$

We carry out the elimination of the secular terms in the usual way[7]. Let A_{Lm} be functions of time, satisfying the equations

$$\frac{dA_{Lm}}{dt}=\sum_{r,s=1}^{\infty}\varepsilon^r\eta^s a_{Lm}^{(r,s)}(A_{L'm'}) \quad (L'=1,2;\ m'=-L'\ldots L') \quad (44)$$

In the course of the solution we shall choose the functions $a_{Lm}^{(r,s)}$ so as to eliminate the secular terms in (43).

On substituting (43) in (32) and on equating the expressions corresponding to like powers of ϵ and η, we get systems of equations for the functions $u_{Lm}^{(r,s)}$ in which functions of lower order occur. Without writing all the equations, because of their large size, we shall trace the role of the individual terms of the series (43). The functions $u_{Lm}^{(0,1)}$ give the "Bloch" solutions, which describe the usual resonance at characteristic frequencies. As an example,

$$u_{2-2}^{(0,1)}=\left[\frac{\omega_0(\omega-2\omega_0)\mathscr{E}_{-2}-\varkappa\omega_0\mathscr{E}_2}{\varkappa^2+(\omega-2\omega_0)^2}\cos\omega t\right]+\left[\begin{array}{c}\mathscr{E}_2\leftrightarrow\mathscr{E}_{-2}\\ \cos\omega t\leftrightarrow\sin\omega t\end{array}\right]. \quad (45)$$

Furthermore, in the functions $u_{Lm}^{(0,1)}(t)$ appear terms of the form $(\omega \pm \omega_0)^{-1}e^{-\kappa t}(n = 1, 2)$, indicating the inapplicability of these solutions near the frequencies $\omega \approx \pm\omega_0$ and $\omega \approx \pm 2\omega_0$. To investigate the corresponding regions, we successively set

$$\omega_0=\pm\frac{\omega}{n}+\eta\Delta \quad (n=1,2) \quad (46)$$

in (32)–(40) and, again seeking functions $u_{Lm}^{(0,1)}(t)$, arrive by elimination of the secular terms at equations (44). For example, for the case n = 1 these equations have the form

$$dA_{22}/dt=\alpha A_{11}+\beta A_{1-1}-2\omega\eta\Delta A_{2-2},$$

$$dA_{2-2}/dt = \alpha A_{1-1} - \beta A_{11} + 2\omega\eta\Delta A_{22},$$
$$dA_{11}/dt = \beta A_{11} - \alpha A_{22} + 3^{1/2}\beta A_{20} - \omega\eta\Delta A_{1-1},$$
$$dA_{1-1}/dt = -\alpha A_{2-2} - \beta A_{22} + 3^{1/2}\alpha A_{20} + \omega\eta\Delta A_{11}, \quad (47)$$
$$dA_{20}/dt = -3^{1/2}\beta A_{11} - 3^{1/2}\alpha A_{1-1},$$

where $\alpha = \frac{1}{2}\omega\eta\mathscr{E}_{-1}$, $\beta = \frac{1}{2}\omega\eta\mathscr{E}_1$. On seeking solutions of (37) $\sim e^{\lambda t}$, we obtain the result that for small Δ all five roots λ are pure imaginary; that is, the zero-order solution of (41) in the corresponding region is stable, there is no parametric resonance. Consideration of the remaining regions leads also to the same conclusion. The absence of parametric resonance in terms of order η is physically obvious (in this approximation there is no anisotropy).

In obtaining the functions $u_{Lm}^{(1,0)}$ we arrive, by elimination of the secular terms, at the equations

$$dA_{11}/dt = \omega_0\varepsilon((\nu_{11} + \nu_{-1-1})/2 - \nu_{00})A_{1-1},$$
$$dA_{1-1}/dt = -\omega_0\varepsilon((\nu_{11} + \nu_{-1-1})/2 - \nu_{00})A_{11}, \quad (48)$$

whose solutions lead to a shift of the characteristic frequency in the functions $D_{1-1}^{\prime(0)}$ and $D_{11}^{\prime(0)}$ caused by the demagnetizing fields.

Having obtained the functions $u_{Lm}^{(1,0)}$ and $u_{Lm}^{(0,1)}$, we seek a solution of second order. The functions $u_{Lm}^{(1,1)}$ are undefined in the frequency ranges $\omega \approx \pm n\omega_0$ ($n = 1, 2, 3, 4$). We successively set

$$\omega_0 = \pm\omega/n + \varepsilon\eta\Delta \quad (49)$$

in the original equations, eliminate the secular terms in the functions $u_{Lm}^{(1,1)}$, and obtain in each case equations (44). Thus for $n = 1$ we have

$$dA_{11}/dt = -\alpha A_{20} - \varepsilon\eta\Delta A_{1-1},$$
$$dA_{1-1}/dt = \beta A_{20} + \varepsilon\eta\Delta A_{11}, \quad (50)$$
$$dA_{20}/dt = \tfrac{1}{4}\beta A_{1-1} - \tfrac{1}{4}\alpha A_{11},$$

where

$$\alpha = \frac{3^{1/2}}{8}\omega\varepsilon\eta[(\nu_{11} - \nu_{-1-1})\mathscr{E}_1 + 2\nu_{1-1}\mathscr{E}_{-1}],$$

$$\beta = \frac{3^{1/2}}{8}\omega\varepsilon\eta[(\nu_{11} - \nu_{-1-1})\mathscr{E}_{-1} - 2\nu_{1-1}\mathscr{E}_1]$$

(the remaining equations are of no interest). On seeking a solution of equations (50) $\sim e^{\lambda t}$, we get

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm\tfrac{1}{2}\sqrt{\alpha^2 + \beta^2 - 4\Delta^2}. \quad (51)$$

Thus when $\Delta < \frac{1}{2}\sqrt{\alpha^2 + \beta^2}$, the solution of equations (50) increases exponentially with increase of t, as $\exp(\frac{1}{2}\sqrt{\alpha^2 + \beta^2 - 4\Delta^2})$, whereas the solution (41) increases exponentially with increase of t when

$$\sqrt{\alpha^2 + \beta^2 - 4\Delta^2} > 2\varkappa.$$

For $n \neq \pm 1$, a direct proof demonstrates the absence of parametric resonance; consequently, in the case of MAR the first parametric resonance band is near the Larmor frequency, while the threshold that must be surmounted for hard excitation of this resonance is determined by the expression

$$\eta_{cr} = \frac{16\varkappa}{3\varepsilon\omega_0}\{(\mathscr{E}_1^2 + \mathscr{E}_{-1}^2)[(\nu_{11} - \nu_{-1-1})^2 + 4\nu_{1-1}^2]\}^{-1}. \quad (52)$$

In the case of a magnetically isotropic crystal, for which the tensor $N_{\alpha\beta}$ is a multiple of the unit tensor, the threshold (52) is infinite (for then $\nu_{11} = \nu_{-1-1} = \nu_{00}$, $\nu_{1-1} = 0$). In the case of a uniaxial crystal, if the field H_0 is directed along the axis of symmetry, the tensor $N_{\alpha\beta}$ is diagonal in the coordinate system being used; obviously $\nu_{11} = \nu_{-1-1} \neq \nu_{00}$, $\nu_{1-1} = 0$, and the threshold (52) is again infinite. But if H_0 is directed at an angle to the axis of symmetry, then in order of magnitude

$$\eta_{cr} \sim \varkappa/\varepsilon\omega_0. \quad (53)$$

On estimating the values of $\epsilon_{mm'}$ by means of (35) (here we use the usual expression for the operator of the dipole-dipole interaction energy), we get in the nearest-neighbor approximation

$$\varepsilon \sim \frac{m}{3}I(I+1)\frac{\beta\gamma^2\hbar^2}{a^3},$$

where m is the number of nearest neighbors, and a is the distance between them.

The relation (53) imposes severe limitations on the choice of materials in which nonlinear magnetoacoustic resonance can be observed. Besides powerful ultrasound, the following are necessary: a) large magnetoelastic coupling constants; b) appreciable paramagnetic susceptibility; c) large relaxation times (this is also necessary for fulfillment of (34)). Suitable parameters are possessed, for example, by europium ethylsulfate at helium temperatures. For an amplitude of ion displacement $A_0 \sim 10^{-8}$ cm, which corresponds to sound power ~ 100 W/cm^2, by means of data on the absorption coefficient of sound entering perpendicularly to the trigonal axis of the crystal[8], we estimate $\eta \sim 10^{-4}$ for $\omega_0 \sim 10^{10}$ Hz. Under the same conditions, $\epsilon \sim 10^{-2}$. It is known that in such compounds the longitudinal and transverse relaxation times are of the same order (in accordance with our supposition) and are extremely dependent on temperature[9]. In the temperature range 2 to 4°K, $\kappa \sim 10^{-3}$ to 10^{-4} sec^{-1}; this is quite sufficient for fulfillment of the relation (53). The second of the inequalities (34) is also known to be satisfied.

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APPENDIX

Since the spur is invariant to the choice of representation, we calculate the quantities $\alpha_{Lm,L'm'}^{L''m''}$ directly by taking the operators $\hat{D}_{Lm}$ in the z-representation in matrix form. We get the following values for them:

$$\alpha_{1-1,10}^{11} = \alpha_{1-1,2-2}^{2-1} = \alpha_{21,1-1}^{22} = \alpha_{10,21}^{2-1} = \alpha_{2-1,11}^{22} = \alpha_{11,21}^{2-2} = 1/\sqrt{2},$$
$$\alpha_{1-1,20}^{21} = \alpha_{11,2-1}^{20} = \sqrt{3/2}, \quad \alpha_{2-2,10}^{22} = \sqrt{2}. \quad (54)$$

The remaining nonvanishing coefficients $\alpha_{Lm,L'm'}^{L''m''}$ are obtained from (54) by means of the relations (23).

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