NONLINEAR EVOLUTION OF A QUASIMONOCHROMATIC PACKET OF HELICAL WAVES IN A PLASMA

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Effects due to nonlinear interaction between resonant particles and quasimonochromatic wave packets of circularly polarized waves propagating in a plasma along a magnetic field are considered. It is shown that the finiteness of the packet results in some new interesting phenomena which are not manifest in a constant amplitude wave, viz., steepening of the packet fronts in a stable plasma and its compression, expansion of the packet in an unstable plasma, periodic amplitude modulation of a packet with a sufficiently steep leading front. The results are compared with experiments on the propagation of monochromatic whistlers in a magnetosphere.

1. INTRODUCTION

THE interaction of monochromatic helical waves with resonant particles and the ensuing nonlinear effects were discussed in a number of papers (cf., e.g.,[1-7]). In all these papers, they considered unbounded plane waves. Yet in many cases (this pertains, for example, to the propagation of whistlers in the magnetosphere) one deals with quasimonochromatic wave packets. This gives rise to new physical factors that can cause effects greatly different from those that follow from the theory of unbounded waves. A numerical investigation of the trajectories of particles in a packet of whistlers was carried out in[8]. In the present paper we use analytic methods that make it possible to explain a number of interesting features of the nonlinear evolution of a packet propagated in a stable or unstable plasma. This evolution depends on the shape of the leading front of the packet: the effects will be different for abrupt and for gradual fronts.

We assume that the results obtained in the present paper can be useful, in particular, for the interpretation of experiments on large-amplitudes whistlers emitted by terrestrial transmitters and received in magnetically-conjugated points (cf.[9] and a large number of analogous papers).

2. EVOLUTION OF WAVE PACKET

We consider a quasimonochromatic wave packet of circularly polarized waves (whistlers, to be specific) propagating in a collisionless plasma along an external magnetic field[1]:

$$B_x = B \sin (kZ - \omega t), \quad B_y = B \cos (kZ - \omega t), \quad (2.1)$$

$$k^2 c^2 / \omega^2 = \omega_p^2 / \omega (\omega_c - \omega), \quad (2.2)$$

where ω_p and ω_c are the plasma and cyclotron frequencies of the electrons. Here B(Z, t) is a sufficiently slowly varying function, such that the interaction between the waves and the particles can be considered

on the basis of the theory of a monochromatic wave. Packets of this type will be called quasimonochromatic. The electric field of the wave is determined from Maxwell's equation

$$\text{rot } \mathbf{E} = -c^{-1} \partial \mathbf{B} / \partial t \quad (2.3)$$

and can be written in the form

$$E_x = \frac{\omega}{kc} B_y + \delta E_x, \quad E_y = -\frac{\omega}{kc} B_x + \delta E_y,$$

where δE_x or δE_y is the part of the electric field connected with the variation of the amplitude of the wave packet in space and in time. In reaches the maximum value at the boundaries of the packet (if the latter are sufficiently abrupt). However, in this case, just as for a packet with gradual shape, the terms δE can be neglected, since they lead to effects $\sim B$, whereas the principal effect of interest to us are $\sim B^{1/2}$.

We start from the kinetic equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{R}} - \frac{e}{mc} \{ c\mathbf{E} + [\mathbf{v}(\mathbf{B} + \mathbf{B_0})] \} \frac{\partial f}{\partial \mathbf{v}} = 0, \quad (2.4)$$

where B_0 is the intensity of the constant magnetic field. We introduce cylindrical variables in velocity space:

$$v_x = W \cos \varphi, \quad v_y = W \sin \varphi, \quad v_z,$$

and then change over in (2.4) to the independent variables that are most convenient for our problem

$$t, \ z = -(Z - v_g t), \ 2\xi = kZ - \omega t + \varphi - \tfrac{3}{2}\pi,$$
$$u = v_z - (\omega - \omega_c) / k, \ w^2 = W^2 - 2u\omega_c / k. \quad (2.5)$$

The quantity z is the coordinate of the particle in a system where the wave packet is at rest, reckoned from the leading edge inside the packet, and v_g is the group velocity. The variable u is the deviation of the particle velocity v_z from the exact resonant value $v_R = (\omega - \omega_C)/k$, i.e., particles with sufficiently small u interact resonantly with the wave. The wave acts on them with a longitudinal force that varies slowly over distances on the order of the wavelength. An analysis of the motion of the particle in the wave field (2.1) (cf.[1-7]) shows that the characteristic time of variation

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[1] The extension of the corresponding results of the case of ion-cyclotron waves is obvious.

of the velocity u for the resonant particles is

$$\tau = (hk\omega_c W)^{-\frac{1}{2}}, \qquad (2.6)$$

where h is the amplitude of the wave field in units of the constant field $h = B/B_0$. In order for the particle to have a velocity in the resonant region, it is necessary that u satisfy the condition $u \lesssim 1/k\tau$. We shall henceforth be interested only in the motion of resonant particles. For such particles, the kinetic equation in terms of the variables (2.5) is given by

$$\frac{\partial f}{\partial t} + v_0 \frac{\partial f}{\partial z} + \frac{ku}{2}\frac{\partial f}{\partial \xi} - \frac{\sin 2\xi}{k\tau^2}\frac{\partial f}{\partial u} = 0, \qquad (2.7)$$

where

$$v_0 = v_s - v_R = v_s(1 + \omega_c/2\omega). \qquad (2.8)$$

In the derivation of (2.7) from (2.4) it was assumed that the wave amplitude is sufficiently small, namely

$$\beta \equiv (h\omega_c^3 / k^3 W_T^3)^{\frac{1}{2}} \ll 1$$

(W$_T$ is the average transverse thermal velocity of the particle). Outside the wave packet, the kinetic equation is obtained from (2.7) by assuming there $\tau = \infty$.

The characteristics of Eq. (2.7) are determined by the following system of ordinary differential equations:

$$dt = \frac{dz}{v_0} = \frac{2}{ku}d\xi = -\frac{k\tau^2}{\sin 2\xi}du \qquad (2.9)$$

and coincide formally with the equations of motion of the electron in the field of a longitudinal electrostatic wave $E = E_0 \sin(kx - \omega t)$, if the following correspondence is established:

$$2\xi \to kx, \quad u \to \frac{dx}{dt} - \frac{\omega}{k}, \quad z \to v_0 t, \quad (hk\omega_c W)^{-\frac{1}{2}} \to (eE_0 k/m)^{-\frac{1}{2}}.$$

After finding the characteristic particle trajectories from the system (2.9), we obtain the general solution of the kinetic equation. As a simple example, but one of importance in what follows, we present the solution for a packet of rectangular form and of length l:

$$f = f\left(t - \frac{z}{v_0}, \, \mathrm{am}\left[F(\xi, \varkappa) - \frac{z}{v_0 \varkappa \tau}, \varkappa\right],\right.$$
$$\left. \frac{2}{k\varkappa\tau}\mathrm{dn}\left[F(\xi, \varkappa) - \frac{z}{v_0 \varkappa\tau}, \varkappa\right], \, w\right), \quad 0 < z < l, \qquad (2.10)$$

where f is an arbitrary function of its arguments, $F(\xi, \varkappa)$ is an incomplete elliptic integral of the first kind, and $\mathrm{am}[F, \varkappa]$ and $\mathrm{dn}[F, \varkappa]$ are elliptic functions with modulus $\varkappa$; the variable $\varkappa(u, \xi)$ is defined by the expression

$$\varkappa^2\tau^2 = (^1/_4 k^2 u^2 + \tau^{-2}\sin^2\xi)^{-1}, \qquad (2.11)$$

and w is defined in (2.5). Outside the limits of the packet we should put $\tau \to \infty$. Then $\varkappa \to 0$, but the product $\varkappa\tau$ remains finite and tends to the value $\varkappa\tau \to 2/ku$ ($z \to 0$, $z \to l$).

The quantity $1/\varkappa^2\tau^2$, apart from a constant factor, is the total "energy" of the particle; it remains constant when the particle moves in the field of the packet (if the amplitude of the latter is constant). By analogy with the motion in the field of an electrostatic wave, we call the particle trapped if it has $|\varkappa| > 1$. In this case the angle variable ξ takes on values in the range $-\arcsin|1/\varkappa| < \xi < \arcsin|1/\varkappa|$, and the mean value of the velocity u is equal to zero. Accordingly, a particle for which $|\varkappa| < 1$ will be called untrapped. For such particles $\bar{u} \sim 2/k\tau \neq 0$ and $-\infty < \xi < \infty$.

To determine the distribution function inside the packet, it suffices to know its value at the right-hand boundary, i.e., at $z \to -0$. We confine ourselves here to the simplest case when the distribution function $f_0(t, \xi, u, w)$ in front of the packet is stationary, i.e., it does not depend on t, and furthermore, does not depend on the angle variable ξ. We assume also that the distribution function in the resonant region ahead of the packet, expressed in terms of u and w, can be expanded in powers of u and we can confine ourselves only to the term linear in u (this is valid if the distribution function is sufficiently smooth and the width of the resonant region is sufficiently smooth and the width of the resonant region is sufficiently small, i.e., $2/k\tau \ll v_T$, where v_T is the longitudinal thermal velocity). Taking into account the connection between w and W, which is determined by the last formula in (2.5), we obtain

$$f_0(u, \, w) = f_0 + f_0' u,$$

where

$$f_0 = f_0(v_z, W)_{v_z \to v_R}, \quad f_0' = \left(\frac{\partial f}{\partial v_z} + \frac{\omega_c}{kW}\frac{\partial f}{\partial W}\right)_{v_z \to v_R}. \qquad (2.12)$$

Bearing in mind expression (2.10), we obtain the distribution function in the resonant region inside the rectangular packet in the following form:

$$f = f_0 + \frac{2f_0'}{k\varkappa\tau}\mathrm{dn}\left[F(\xi, \varkappa) - \frac{z}{v_0\varkappa\tau}, \varkappa\right], \qquad (2.13)$$

where $\tau = \tau(W)$ is determined by expression (2.6). This distribution function does not depend on the time t, by virtue of the assumption that the amplitude h of the wave field and the distribution function ahead of the packet are stationary. It is seen from (2.13) that the distribution function is an oscillating function on the coordinate z. The character of these oscillations can be visualized by expanding the function $\mathrm{dn}[F, \varkappa]$ in a Fourier series (see[10]). A contribution independent of z is made to the distribution function by its ergodic part[11]. It determines the main properties of the distribution function inside the packet at sufficiently large z ($z \gg v_0\tau$). Denoting the ergodic distribution function by f_E, we obtain after averaging (2.13) over z:

$$f_E(\varkappa, W) = f_0(W) + \pi f_0'(W)/k\tau\varkappa K(\varkappa), \quad |\varkappa| < 1,$$
$$f_E(\varkappa, W) = f_0(W), \quad |\varkappa| > 1, \qquad (2.14)$$

which accordingly coincides with the ergodic distribution function in an unbounded wave[7b] ($K(\varkappa)$ is a complete elliptic integral of the first kind: $K(\varkappa) = F(\pi/2, \varkappa)$).

3. PACKET WITH SLOWLY VARYING AMPLITUDE

We consider a packet with an amplitude that varies sufficiently slowly, such that

$$\partial\tau/\partial t \ll 1, \quad v_0\partial\tau/\partial z \ll 1. \qquad (3.1)$$

Then the ergodic distribution function of the resonant particle at the beginning of the packet is given by

$$f_E(z) = \begin{cases} f_0 + \pi f_0'/k\tau_0\varkappa_0 K(\varkappa_0), & |\varkappa_0| < 1, \\ f_0, & |\varkappa_0| > 1, \end{cases} \qquad (3.2)$$

where $\varkappa_0$ and τ_0 correspond to the initial amplitude

$h_0{}^{2)}$. To determine the distribution function for arbitrary z, we consider first the variation of the quantity κ, which is an integral of the motion at constant amplitude (τ = const).

Differentiating (2.11) with respect to the coordinate z and using the equation of motion (2.9), we obtain

$$-\frac{1}{\kappa^3}\frac{d\kappa}{dz} = \tau\frac{d\tau}{dz}\frac{k^2 u^2}{4},$$

where κ is a certain slowly-varying function. Averaging the quantity u^2 over the rapid variations we obtain (for untrapped particles)

$$\overline{u^2} = \frac{4}{k^2\kappa^2\tau^2}\frac{E(\kappa)}{K(\kappa)},$$

where $E(\kappa)$ is a complete elliptic integral of the second kind. Substituting $\overline{u^2}$ in the preceding equation, we obtain after simple transformations

$$\frac{d}{dz}\left[\frac{E(\kappa)}{\kappa\tau}\right] = 0. \tag{3.3}$$

From (3.3) we get, in particular, a connection between κ, τ, and the corresponding quantities κ_0, τ_0 at the beginning of the packet:

$$E(\kappa)/\kappa\tau = E(\kappa_0)/\kappa_0\tau_0 \quad (|\kappa| < 1, \ |\kappa_0| < 1). \tag{3.4}$$

The left-hand side of (3.4) is an adiabatic invariant for the untrapped particles. Similar reasoning yields for the trapped particles the relation

$$\tau^{-1}[E(1/\kappa) - (1 - \kappa^{-2})K(1/\kappa)] = \text{const} \quad (|\kappa| > 1), \tag{3.5}$$

which is the condition for the conservation of the adiabatic invariant for the trapped particles. Formula (3.5) was obtained earlier by Laval and Pellat[12]. It will not do for us, however. It should be noted that the conservation of the adiabatic invariants is violated when the particles go over, with changing amplitude, from the untrapped region to the trapped region and vice versa. It is therefore important to apply relation (3.4) only to particles which remain untrapped during the entire time of field variation.

To write down the distribution function for arbitrary z with the aid of (3.2) and (3.4), it is convenient to change over from κ to a new independent variable

$$\mu = E(\kappa)/\kappa \quad (|\kappa| < 1). \tag{3.6}$$

It is easy to verify that $\mu(\kappa)$ is a monotonically decreasing function. Obviously, μ increases to take over the particle energy, and assumes for the untrapped particles values in the interval $1 < |\mu| < \infty$ (the sign of μ coincides with the sign of the velocity).

We now use the function $R(\mu)$ introduced in[13] and defined in parametric form by the relation

$$R(\mu) = 1/\kappa K(\kappa), \tag{3.7}$$

where $\kappa(\mu)$ is uniquely defined in (3.6). It is easy to verify that $R(\mu)$ is a monotonically growing odd function, $R(1) = 0$, and $R(\infty) = \infty$. The asymptotic representations for $R(\mu)$ are as follows:

$$\begin{aligned}R(\mu) &= -2/\ln(\mu - 1) + \ldots, \quad \mu - 1 \ll 1,\\ R(\mu) &= 4\mu/\pi^2 - \pi^2/128\mu^3 + \ldots, \quad \mu \gg 1.\end{aligned} \tag{3.8}$$

Relation (3.4) now takes the form

$$\mu/\tau = \mu_0/\tau_0. \tag{3.9}$$

It follows from it, in particular, that if the field amplitude increases and τ decreases accordingly, then μ decreases (the particle goes over to a lower energy level). Some of the untrapped particles then become trapped. The distribution function of the untrapped particles at the point z is expressed in terms of the corresponding distribution function at the start of the packet (3.2) in accordance with the Liuville theorem

$$f(\mu, W, z) = f_0(W) + \frac{\pi f_0'(W)}{k\tau_0}R\left(\frac{\tau_0}{\tau(h)}\mu\right), \tag{3.10}$$

$$|\kappa| < 1, \quad |\mu| > 1, \quad h = h(z),$$

where we have used relation (3.9).

As to the trapped particles, it is obvious that their distribution function should not depend on the direction of the velocity and consequently on the sign of κ. The following condition should therefore be satisfied on the boundary between the regions of the trapped and untrapped particles;

$$f_T(|\kappa| \to 1) = \frac{1}{2}[f_{UT}(\kappa \to 1) + f_{UT}(\kappa \to -1)],$$

where the distribution function of the trapped particles is on the left and that of the untrapped ones on the right (cf.[14]). Substituting (3.10) in the right-hand side of this relation and recognizing that $R(\mu)$ is an odd function of μ and consequently of κ, we find that in the vicinity of the separatrix, meaning everywhere, the distribution function of the trapped particles does not depend on $\kappa^{3)}$:

$$f_T(\kappa, W, z) = f_0(W), \quad |\kappa| > 1, \tag{3.11}$$

Formulas (3.10) and (3.11) determine, however, the distribution function only in the front part of the packet, where the resonant particle, which always moves in a direction of increasing z, "sees" a field that increases in amplitude. Thus, (3.10) and (3.11) hold when

$$z < z_m, \quad h(z_m) = h_m \quad (h_m = h_{max}), \tag{3.12}$$

We now determine the distribution function in the real part of the packet ($z > z_m$), where the field decreases in the direction of motion of the resonant electron. In this case μ increases when the particle moves in accordance with the relation $\mu(h)/\tau(h)$ $= \mu(h_m)/\tau(h_m)$, so that the particles that were untrapped at $h = h_m$ have at $z > z_m$ values of the parameter μ in the interval

$$1 < \tau(h)/\tau(h_m) < |\mu| < \infty.$$

Obviously, the distribution function of the untrapped particles has in this interval the same form as (3.10). The remaining untrapped particles, which at $z > z_m$ have values of μ in the interval

$$1 < |\mu| < \tau(h)/\tau(h_m),$$

$$f(\mu, W, z) = f_0(W) + \frac{\pi f_0'(W)}{k\tau_0}R\left(\frac{\tau_0}{\tau(h)}\mu\right)$$

$^{2)}$Of course, in order for the use of the ergodic distribution function to be valid at the beginning of the packet, the distance from the point z to the beginning of the packet should be large in comparison with $v_0\tau_0$.

$^{3)}$This, however, is connected with the fact that we take into account only the linear term in the expansion of the distribution function with respect to u (see the formula preceding (2.12)). In the next approximation we would obtain in the right-hand side of (3.11) an additional term that depends on κ^2.

were trapped at $z = z_m$, and consequently their distribution function is the same as for the trapped particles, i.e., it coincides with (3.11). Thus, behind the maximum of the packet ($z > z_m$) the distribution function of the untrapped particles is given by

$$\text{for } |\mu| > \tau(h)/\tau(h_m) > 1,$$

$$f(\mu, W, z) = f_0(W) \text{ for } 1 < |\mu| < \tau(h)/\tau(h_m). \quad (3.13)$$

For the trapped particles we have as before

$$f(\varkappa, W, z) = f_0(W), \quad |\varkappa| > 1. \quad (3.14)$$

The irreversibility that appears in the difference of the distribution functions in the front part of the packet, where the particle moves in an increasing field, and in the rear part, where the field decreases, is connected with the fact that when the field varies the particles cross the boundary between the regions of the trapped and untrapped particles. The adiabatic invariant is not conserved for this crossing.

4. ENERGY CONSERVATION EQUATION

We consider first the difference between the average kinetic energy of the particles in the field of the wave and in the absence of the wave. This difference is given by the formula

$$\delta T = \frac{mn}{9} \int dv (v_z^2 + W^2) \delta f, \quad (4.1)$$

where δf is the change induced in the distribution function by the wave. The variation of δf is significant only in the resonant region, i.e., for particles having a velocity $v_z \approx v_R$, $u \lesssim 1/k_T$. Outside the resonant region, it vanishes rapidly.

If we now denote the average density of the electromagnetic energy in the whistler by

$$U = \frac{B^2}{8\pi(1 - \omega/\omega_c)} = \frac{B_0^2}{8\pi(1 - \omega/\omega_c)} h^2, \quad (4.2)$$

then U and δT are connected by the following equation:

$$\frac{\partial U}{\partial t} + v_g \frac{\partial U}{\partial Z} + \frac{\partial \delta T}{\partial t} + v_R \frac{\partial \delta T}{\partial Z} = 0. \quad (4.3)$$

Equation (4.3) takes into account the fact that U propagates with the group velocity v_g of the packet, whereas δT propagates with the average velocity v_R of the resonant particles. Changing over to a reference frame that moves together with a packet, we obtain in place of (4.3)

$$\partial(U + \delta T)/\partial t + v_0 \partial \delta T/\partial z = 0, \quad (4.4)$$

where the coordinate z is defined in (2.5). Relations (4.3), (4.2), and (4.1), together with the expression for δf, determine in principle the evolution of the packet due to the interaction of the wave with the resonant particles.

5. EVOLUTION OF A PACKET WITH SMOOTHLY VARYING AMPLITUDE

Using the expressions (3.10), (3.11), and (3.13), (3.14) for the distribution function, we obtain from (4.1) after simple calculations

$$\delta T = -A h^{3/2} \frac{128 mn\omega}{9k} \left(\frac{\omega_c}{k}\right)^{3/2} \int_0^\infty f_0'(W) W^{5/2} dW, \quad (5.1)$$

where A(h) is defined by

$$A = 1 - \frac{9}{2} \frac{h_0}{h} \int_{(h/h_0)^{1/2}}^\infty dy\, R\left(\sqrt{\frac{h_0}{h}}\, y\right) \left[\frac{\pi^2}{4} R(y) - y\right] \quad (5.2)$$

at $z < z_m$ and

$$A = 1 + \frac{9}{2} \int_1^{(h/h_m)^{-1/2}} dy\, y R(y)$$
$$- \frac{9}{2} \frac{h_0}{h} \int_{(h_m/h_0)^{1/2}}^\infty dy\, R\left(\sqrt{\frac{h_0}{h}}\, y\right) \left[\frac{\pi^2}{4} R(y) - y\right] \quad (5.3)$$

at $z > z_m$.

The expression (5.2) is valid in the front part of the packet, where the field increases, and (5.3) is valid in the region where the field decreases.

We shall consider henceforth the case of greatest interset, when $h \gg h_0$. We can then neglect in the integrals of (5.2) and (5.3) the terms containing h_0, and (4.4) yields

$$[1 - \psi(h)] \frac{\partial h}{\partial t} - v_0 \psi(h) \frac{\partial h}{\partial z} = 0, \quad (5.4)$$

where we have ahead of the maximum of the packet

$$\psi(h) = a(h_m/h)^{1/2} \quad (5.5)$$

and behind the maximum of the packet

$$\psi(h) = a\left(\frac{h_m}{h}\right)^{1/2} \left\{1 + \frac{9}{2} \int_1^{(h_m/h)^{1/2}} dy\, y R(y) - \frac{3}{2} \frac{h_m}{h} R\left(\sqrt{\frac{h_m}{h}}\right)\right\}. \quad (5.6)$$

The constant a is determined in this case by the expression

$$a = \frac{64}{3\pi} \frac{\gamma_L}{\omega_c} \left(\frac{\omega_c}{k}\right)^{1/2} \frac{1}{h_m^{1/2}} \int_0^\infty f_0'(W) W^{5/2} dW \Big/ \int_0^\infty f_0'(W) W^3 dW, \quad (5.7)$$

where γ_L is the increment (decrement) of the wave in the linear approximation:

$$\gamma_L = \frac{\pi^2 \omega_p^2 \omega}{k^2 c^2} \left(1 - \frac{\omega}{\omega_c}\right) \int_0^\infty f_0'(W) W^3 dW.$$

If we introduce the characteristic nonlinear time

$$\tau_m = (k\omega_c W_T h_m)^{-1/2},$$

which differs from (2.6) in that the variables W and h are replaced by constant values W_T and h_m, then we can easily verify from (5.7) that

$$a \sim \gamma_L \tau_m. \quad (5.8)$$

Relation (5.4) has the form of the equation of a simple wave in hydrodynamics, where the propagation velocity C(h) is given by

$$C(h) = v_0 \psi(h) / [1 - \psi(h)]. \quad (5.9)$$

The solution of (5.4) is

$$h = \Phi[z + C(h)t], \quad (5.10)$$

where the function $\Phi(z)$ describes the profile of the wave at $t = 0$. In the investigation of the evolution of the the packet we assume that $(h/h_m)^{1/2} \gg a$ ($a \ll 1$), so that $\psi(h) \ll 1$, and we can write in place of (5.9)

$$C(h) = v_0 \psi(h). \quad (5.11)$$

The general character of the evolution is determined by the sign of the derivative dC/dh. It follows from (5.5), (5.6), and (5.7) that

$$\text{sign}\frac{dC}{dh} = -\text{sign}\,\gamma_L, \quad z < z_m;$$

$$\text{sign}\frac{dC}{dh} = \text{sign}\,\gamma_L; \quad z > z_m.$$

A plot of $C(h)$ is shown in Fig. 1.

Assume now that the medium is stable ($\gamma_L < 0$). It follows then from (5.5) and (5.6) that both the leading front and the "tail" of the packet should become steeper in time. When the leading front becomes sufficiently steep, relation (3.4), which is valid only in the adiabatic approximation, no longer holds. Accordingly, Eq. (5.4) ceases to be valid. In this case we must use for the description of the packet evolution a different model, which will be discussed qualitatively in the next section.

In an unstable medium ($\gamma_L > 0$), the evolution of the packet will be opposite that considered above, and the packet will stretch out.

We consider now certain important details of the described processes. Integrating the equation $h_t - C(h)h_z = 0$ over the entire length of the packet, we obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty} h(t,z)\,dz = \int_0^{h_m} C_1(h)\,dh + \int_{h_m}^0 C_2(h)\,dh, \quad (5.12)$$

where $C_1(h)$ and $C_2(h)$ are the propagation velocities of the perturbations ahead and behind the maximum of the amplitude. Owing to the irreversibility that leads to different expressions for $C_1(h)$ and $C_2(h)$ ahead of the maximum h_m and behind it, the integrals in the right-hand sides of (5.12) do not cancel each other, so that the area of the profile of the packet envelope is not conserved. It is easy to verify that

$$\frac{d}{dt}\int_{-\infty}^{\infty} h(t,z)\,dz \begin{cases} < 0, & \gamma_L < 0 \\ > 0, & \gamma_L > 0, \end{cases}$$

i.e., the area of the profile of the envelope decreases in a stable medium and increases in an unstable one. It is easy to verify here that the value of the maximum amplitude is conserved. Indeed, it follows from (5.4) and from the equation $\partial h_m/\partial z_m = 0$ that

$$\frac{dh_m}{dt} = \frac{\partial h_m}{\partial z_m}\frac{dz_m}{dt} + \frac{\partial h_m}{\partial t} = 0.$$

This, however, is valid only until the fronts of the packet become sufficiently steep.

The order of magnitude of the characteristic time of steepening of the leading front in a stable medium is

$$t \sim l(v_0\gamma_L\tau_m)^{-1}, \quad (5.13)$$
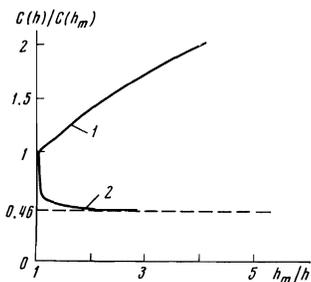
where l is the length of the packet. The packet can then become so steep, that we can regard it as rectangular with sufficient accuracy.

6. EVOLUTION OF A RECTANGULAR PACKET

We consider the following simple model. Assume that at the initial instant of time the envelope of the packet has a rectangular form, and the distribution function ahead of the packet is stationary. Then the distribution function in the resonant region, for the time interval until the amplitude of the packet manages to be significantly altered by the interaction with the resonant particles, is determined by expression (2.13), and at large distances from the leading front by expression (2.14) (if the length of the packet is $l \gg v_0\tau$). Substituting (2.13) in (4.1), we obtain the following expression for the change of the particle energy δT as a result of their interaction with the wave:

$$\delta T = \frac{16mn\omega}{k^4}\int_0^{\infty}\frac{f_0'W\,dW}{\tau^3}\iint_S\left\{ \text{dn}\left[F - \frac{z}{v_0\varkappa\tau},\varkappa\right] \right. \quad (6.1)$$

$$\left. - \text{dn}[F,\varkappa]\right\}\frac{\text{dn}(F,\varkappa)}{\varkappa^4}dF\,d\varkappa,$$

where the integration region S is shown in Fig. 2. We see that $\partial\delta T/\partial t = 0$ in the reference frame in which the packet is at rest, so that (4.4) takes the form

$$\frac{\partial U}{\partial t} + v_0\frac{\partial\delta T}{\partial z} = 0, \quad (6.2)$$

where U is determined by (4.2). Introducing the quantity

$$\gamma(z) = -\frac{v_0}{2U}\frac{\partial\delta T}{\partial z}, \quad (6.3)$$

we obtain from (6.2)

$$h(t,z) = h\exp[\gamma(z)t] \quad (0 < z < l). \quad (6.4)$$

Substituting (4.2) and (6.1) in (6.3) we obtain for the increment the following expression:

$$\gamma(z) = -\frac{8}{\pi^2}\gamma_L\iint_S\frac{\sin(2\text{am}[F - z/v_0\varkappa\tau,\varkappa])}{\varkappa^3}.$$

$$\times \text{dn}[F,\varkappa]dF\,d\varkappa \equiv \frac{8}{\pi^2}\gamma_L\iint_S\frac{\sin(2\text{am}[F,\varkappa])}{\varkappa^3}.$$

$$\times \text{dn}\left[F - \frac{z}{v_0\varkappa\tau},\varkappa\right]dF\,d\varkappa. \quad (6.5)$$

We see that (6.5) coincides with the well known expression obtained by O'Neil[11] for an unbounded plane wave, where t is replaced by z/v_0. The main contribution to the integral (6.5) is made by the region in which

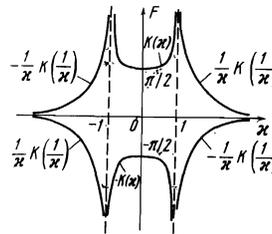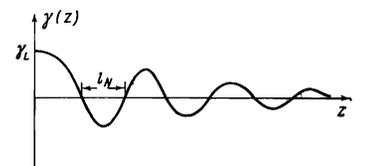FIG. 1. Plot of the function $C(h)$: $1-z < z_m$, $2-z > z_m$.

FIG. 2

FIG. 3

FIG. 2. Integration region S in Eqs. (6.1) and (6.5).

FIG. 3. Dependence of wave increment $\gamma(z)$ in a rectangular packet on the distance z reckoned from its leading edge; $l_N \sim v_0\tau$.

$\kappa \lesssim z/v_0\tau$. At small z, i.e., near the leading front of the packet, the last expression in (6.5) can be easily integrated, and we obtain $\gamma(z) = \gamma_L (z \ll v_0\tau)$. When $z \gtrsim v_0\tau$, the increment oscillates (with a period on the order of $v_0\tau$) and tends to zero at $z \gg v_0\tau$ (see Fig. 3). These results have a simple physical meaning. The resonant particles moving with velocity v_0 towards the packet from the unperturbed region ahead of the packet enter into the wave region. At distances $z \ll v_0\tau$, their distribution function varies in accordance with the linear-approximation formulas, so that the change of the wave amplitude is determined by the linear increment γ_L. At large distances $z \gg v_0\tau$, the character of the motion of the resonant particles is significantly altered and their distribution function comes close to being ergodic in the means; the increment $\gamma(z)$ then vanishes. It is seen from the foregoing that in the course of time there should appear in the region $z \lesssim v_0\tau$ quasiperiodic oscillations of the amplitude with a characteristic period $v_0\tau$. At large $t (t \gtrsim 1/\gamma_L)$, when the amplitude has become noticeably modulated, the expressions (6.4) and (6.5) are no longer valid. It is clear, however, that as the modulation develops in the front part of the packet, the latter should propagate with time and into the interior of the packet. A more detailed investigation of these processes at $\gamma_L t \gtrsim 1$ requires, however, numerical simulation[4].

For a strongly modulated packet, the period of the modulation is determined by the time τ corresponding to a certain effective value of the amplitude of the packet envelope, and this value is generally speaking several times smaller than the maximum amplitude. Indeed, the deviation from the average resonant velocity of the particles interacting with the wave is of the order of $(k\tau)^{-1}$ so that the particles for which τ is calculated from the maximum amplitude are relatively rarely at resonance with the wave.

In addition to amplitude modulation, there should take place, naturally, also frequency modulation characterized by the quantity $\Delta f \sim \tau^{-1}$, where τ corresponds to the local amplitude (see[7]). This modulation is due to the broadening of the spectrum as a result of the oscillations of the resonant particles with period τ. Thus, the period T of the amplitude-frequency modulation observed when signals are received

on earth should be connected with the effective broadening of the spectrum in the packet $(\Delta f)_{\text{eff}}$ by the following relation:

$$T(\Delta f)_{eff} \sim v_0/v_g = 1 + \omega_c/2\omega \qquad (6.6)$$

(generally speaking, $(\Delta f)_{\text{eff}}$ is several times smaller than $(\Delta f)_{\text{max}}$). Relation (6.6) is in perfectly satisfactory agreement with the experiment described in[15] (see also [16]).

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[1] C. S. Roberts and S. T. Buchsbaum, Phys. Rev. 135, A381 (1964).

[2] N. S. Erokhin and R. K. Mazitov, Prikl. Mat. Teor. Fiz. 5, 11 (1968).

[3] V. D. Shapiro and V. I. Shevchenko, Ukr. Fiz. Zh. 13, 1989 (1968).

[4] A. Roux and J. Solomon, Ann. Geophys. 26, 279 (1970).

[5] R. N. Sudan and E. Ott, J. Geophys. Res. 76, 4463 (1971).

[6] P. Palmadesso and G. Schmidt, Phys. Fl. 14, 1411 (1971).

[7] N. I. Bud'ko, V. I. Karpman, and O. A. Pokhotelov, a) ZhETF Pis. Red. 14, 469 (1971) [JETP Lett 14, 320 (1971); b) Cosmic Electrodynamics 3 (1972) (in press).

[8] M. Ashour-Abdalla, Planet Space Sci. 18, 1799 (1970).

[9] R. A. Helliwell, J. Katsufrakis, M. Trimpi, and N. Brice, J. Geophys. Res. 69, 2391 (1964).

[10] I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov, summ, ryadov i proizvedeniĭ (Tables of Integrals, Sums, Series, and Products), Fizmatgiz, 1962 [Academic, 1966].

[11] T. M. O'Neil, Phys. Fl. 8, 2255 (1965).

[12] G. Laval and P. Pellat, J. Geophys. Res. 75, 3255 (1970).

[13] V. I. Karpman and D. R. Shklyar, Zh. Eksp. Teor. Fiz. 62, 944 (1972) [Sov. Phys.-JETP 35, 500 (1972)].

[14] A. V. Gurevich, ibid. 53, 953 (1967) [26, 575 (1968)].

[15] Ya. I. Likhter, O. A. Molchanov, and V. M. Chmyrev, ZhETF Pis. Red. 14, 475 (1971) [JETP Lett. 14, 325 (1971)].

[16] T. F. Bell and R. A. Helliwell, J. Geophys. Res. 76, 8414 (1971).

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[4] In the numerical calculations of [8], z and t were not large enough to observe these phenomena.