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Structure of the Mixed State Near the Boundary of a Semi-Infinite Type II Superconductor of the Second Kind

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The equilibrium equations for vortex lines are solved for a number of particular cases by taking into account interaction of the lines with the boundary of an ideal type II superconductor. It is shown that the vortex-line lattice is almost undeformed for a certain position with respect to the boundary (Fig. 1a). The maximum and minimum values of the external field are determined for which a mixed state can still exist with the prescribed induction values.

1. INTRODUCTION

ACCORDING to the theory of Abrikosov,[1] a magnetic field arises in the interior of a type II superconductor ($\kappa > 1/\sqrt{2}$) in the form of vortex lines, each of which carries a single flux quantum $\varphi_0 = hc/2e$. In ideal samples of cylindrical shape, placed in an external magnetic field $\mathbf{H}$ parallel to the surface (in what follows, we shall use a set of coordinates whose z axis is directed along H), the vortex lines are straight lines parallel to the external field. In equilibrium, the centers of the lines in the plane perpendicular to the external field form a triangular lattice.[2-4]

A regular (in the interior of the sample) distribution of vortex lines is a consequence of the equilibrium conditions[4,5]

$$j_x = 0, \quad j_y = 0 \tag{1}$$

at the center of each vortex line (j_x and j_y are the components of the current density $\mathbf{j}$). However, inasmuch as a screening current flows near the surface of the sample, and the number of nearest neighbors for each vortex line located near the surface is different than for those in the interior, it must be expected that the lattice of vortex lines will be deformed near the surface of the sample. The present paper is devoted to this problem (see also[6]).

Another aspect of this problem is the determination of the absolute boundaries of stability of the mixed state, i.e., the maximum H_{max} and minimum H_{min} values of the external field for which the mixed state can still exist with a given value of B. Bean and Livingston[7] and De Gennes[4] have shown that $H_{max} = H_s \approx H_c > H_{c1}$ for $B = 0$, since at external field values $H < H_s$ there exists a surface potential barrier that does not allow the vortex lines to penetrate into the superconductor. We shall show that this potential barrier is preserved even for $B \neq 0$, and shall compute the corresponding value of $H_{max}(B)$ for $B \ll H_{c2}$ (see (38)).

As is well known, the vortex lines repel each other. The surface barrier also prevents their emergence from the superconductor. The quantity $H_{min}(B)$ is the value of the external field for which this surface barrier vanishes. The values of $H_{max}(B)$ and $H_{min}(B)$ depend strongly on the structure of the surface layer of the superconductor and its roughness. It will be assumed below that the surface of the superconductor (the plane $x = 0$) is ideally smooth (the size of the roughnesses is much less than the distance x_1 to the first layer of vortex lines), and the parameters of the superconductor (κ, T_c, λ, etc) near the surface have the same values as in its interior. Under these conditions, the boundary conditions for $x = 0$ can be written in the form

$$h = H, \quad j_x = 0, \tag{2}$$

where $h = h_z$ is the magnetic field intensity inside the superconductor. We shall also assume that $\kappa \gg 1$ and $B \ll H_{c2}$, and use the modified equation of F. and H. London for the determination of h:[1,4]

$$h - \lambda^2 \Delta h = \varphi_0 \sum_{n,m} \delta(\mathbf{r} - \mathbf{r}_{nm}), \tag{3}$$

where $\mathbf{r}_{nm}(x_m, y_{nm})$ are the radius vectors of the centers of the vortex lines, λ the penetration depth of the weak magnetic field.

It is natural to suppose that the vortex lines in equilibrium are arranged parallel to the boundary in layers $x = x_m$ ($m = 1, 2, 3, \ldots$) and that the distances between the neighboring vortex lines inside the limits of each layer are identical for all values of m, so that

$$y_{nm} = a(n + m/2), \quad n = 0, \pm 1, \pm 2, \ldots, \quad m = 1, 2, 3, \ldots \tag{4}$$

Farther from the boundary, the distances between neighboring layers must also be equal:

$$\lim_{m \to \infty} (x_{m+1} - x_m) = b. \tag{5}$$

For $b = a\sqrt{3}/2$ and for $b = a/2\sqrt{3}$, a triangular lattice with period $d = a$ in the first case and $d = 2b$ in the second is determined by Eqs. (4) and (5). For the same value of the induction

$$B = \varphi_0 / ab = 2\varphi_0 / d^2\sqrt{3} \tag{6}$$

these two cases differ by the location of the vortex lines relative to the boundary (see Fig. 1). For $b = a/2$ a square lattice is obtained with period $d = a\sqrt{2}$. In the following, when it is possible, all the calculations will be carried out for arbitrary value of the ratio a/b.

2. ANALYSIS OF THE EQUILIBRIUM CONDITIONS

The conditions of equilibrium (1) follow from the requirement that the Gibbs thermodynamic potential G of the system under consideration have a minimum. As was shown in the work of one of the authors.[5] upon

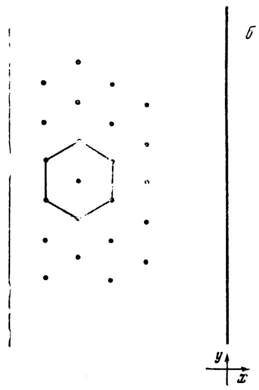

FIG. 1

variation of the shape and location of the axial lines of the vortex lines (see expressions (2.6) and (2.13) from from[5]), the Gibbs potential reduces to[1)]

$$\delta G = -\frac{\varphi_0}{c}\sum_{n,m}\int_{\Gamma_{nm}}[\mathbf{j}\,d\mathbf{l}_{nm}]\delta\mathbf{r}_{nm}, \qquad (7)^*$$

where Γ_{nm} is the axial line of the vortex line nm, $d\mathbf{l}_{nm}$ the element of length of this line, $\delta\mathbf{r}_{nm}$ the displacement of this element upon variation; it is assumed that all the other variables on which G depends are fixed. In particular, if the vortex lines are straight lines, and $\delta\mathbf{r}_{nm} = (\delta x_m, \delta y_{nm}, 0)$, we obtain

$$\delta G = \frac{\varphi_0 l}{c}\sum_{n,m}(j_x(\mathbf{r}_{nm})\delta y_{nm} - j_y(\mathbf{r}_{nm})\delta x_m),$$

where l is the length of the vortex lines. Then, taking into account the Maxwell equation curl $\mathbf{h} = 4\pi\mathbf{j}/c$, we find

$$\frac{\partial G}{\partial x_m} = \frac{\varphi_0 l}{4\pi}\frac{\partial h}{\partial x}\bigg|_{\mathbf{r}=\mathbf{r}_{nm}}, \qquad \frac{\partial G}{\partial y_{nm}} = \frac{\varphi_0 l}{4\pi}\frac{\partial h}{\partial y}\bigg|_{\mathbf{r}=\mathbf{r}_{nm}}. \qquad (8)$$

Derivatives with respect to the magnetic field $h = h_z$ must be calculated for fixed positions of all the vortex lines. The force acting on the straight vortex line (n, m) can be represented in the form

$$\mathbf{F}_{nm} = -\frac{\partial G}{\partial \mathbf{r}_{nm}} = -\frac{\varphi_0 l}{4\pi}\nabla h\bigg|_{\mathbf{r}=\mathbf{r}_{nm}}. \qquad (9)$$

The equilibrium conditions $\mathbf{F}_{nm} = 0$ (m = 1, 2, 3, ..., n = 0, ±1, ±2, ...) are obviously identical with (1).

Solving Eq. (3) with the boundary conditions (2), we find

$$h(x, y) = h_L(x) + h_v(x, y), \quad h_L = He^{-x/\lambda},$$

$$h_v(x, y) = \frac{\varphi_0}{2\pi\lambda^2}\sum_{n,m}\left[K_0\left(\frac{[(x-x_m)^2+(y-y_{nm})^2]^{1/2}}{\lambda}\right) - K_0\left(\frac{[(x+x_m)^2+(y-y_{nm})^2]^{1/2}}{\lambda}\right)\right] \qquad (10)$$

where K_0 is the Macdonald function. The divergence as $x \to x_m$ and $y \to y_{nm}$ can be removed using cutoff at the distances $[(x - x_m)^2 + (y - y_{nm})^2]^{1/2} = \xi$, as in the case of a single vortex line.[4,8] Using (8) and (10), it is not difficult to establish the fact that the equilibrium conditions

$$\frac{\partial G}{\partial y_{\mu\nu}} = 0, \qquad \mu = 1, 2, \ldots, \qquad \nu = 0, \pm 1, \pm, 2, \ldots \qquad (11)$$

[1)]An error was committed in writing down (2.6) and (2.8) in[5]. The positions of j and d_k under the signs of the vector product must be exchanged.

*$[\mathbf{j}\, d\mathbf{l}_{nm}] \equiv \mathbf{j} \times d\mathbf{l}_{nm}$.

are satisfied for all values of x_m, because of the regular distribution chosen in (4) for the vortex lines in all the planes $x = x_m$,

Proceeding to the derivation of the second group of equilibrium conditions, we introduce the notation

$$\frac{\varphi_0}{2\pi\lambda^2}\sum_{n=-\infty}^{+\infty}K_1\left(\frac{[(x_\mu \pm x_m)^2+(y_{\mu\nu}-y_{mn})^2]^{1/2}}{\lambda}\right) \times \frac{(x_\mu \pm x_m)}{[(x_\mu \pm x_m)^2+(y_{\mu\nu}-y_{mn})^2]^{1/2}} = {}^1/_2\beta B S(x_\mu \pm x_m). \qquad (12)$$

The function $S(x_\mu \pm x_m)$ also depends on the parameters b/λ and a/λ and has different values depending on whether the number $\mu - m$ is even or odd. If $\mu - m$ is an even number and $|x_\mu - x_m| \to 0$, then

$$S(x_\mu - x_m) \approx \frac{a}{\pi|x_\mu - x_m|}. \qquad (12a)$$

As $|x_\mu - x_n| \to \infty$,

$$S(x_\mu - x_m) \propto \exp(-|x_\mu - x_m|/\lambda). \qquad (12b)$$

The expression (12b) is valid even if $\mu - m$ is odd, but the quantity $S(x_\mu - x_m)$ is bounded in this case as $|x_\mu - x_m| \to 0$. The condition of equilibrium of the forces acting in the direction of the x axis can now, according to (8), (10), and (12), be represented in the form

$$\frac{\partial G}{\partial x_\mu} = \frac{\varphi_0^2 l}{8\pi\lambda^2 a}\left\{-\frac{2H}{\beta B}\exp\left(-\frac{x_\mu}{\lambda}\right) - \sum_{m=1}^{\mu-1}S(x_\mu - x_m) + \sum_{m=\mu+1}^{\infty}S(x_m - x_\mu) + \sum_{m=1}^{\infty}S(x_m + x_\mu)\right\} = 0, \qquad \mu = 1, 2, 3, \ldots. \quad (13)$$

We note the following general implications of this set of equations.

1) The first and last components in Eqs. (13) take into account the interaction of the vortex lines with the boundaries. For $x_\mu \gg \lambda$, they become negligibly small and we get

$$\sum_{k=1}^{\mu-1}S(x_\mu - x_{\mu-k}) = \sum_{k=1}^{\infty}S(x_{\mu+k} - x_\mu).$$

These equations are approximately satisfied (asymptotically accurately as $\mu \to \infty$) if

$$x_\mu - x_{\mu-k} = x_{\mu+k} - x_\mu = kb, \quad k = 1, 2, \ldots,$$

where b is an arbitrary constant. Thus, the distances between successive layers of vortex lines become uniform, far from the boundary. It is important to emphasize that the quantities a, b and $B = \varphi_0/ab$ appear as arbitrary constants in the solution of Eqs. (12) and (13). Therefore, in the general case, the solutions of these equations correspond to metastable states of the superconductor as a whole. Solving Eqs. (12) and (13), we can express the Gibbs potential in terms of B and a/b and then so choose these parameters that the Gibbs potential has a minimum value for a given H. We shall call such a state one of macroscopic equilibrium (cf.[5]).

2) The following equation holds:

$$\lim_{N\to\infty}\sum_{\mu=1}^{N+1}\left[\sum_{m=\mu+1}^{\infty}S(x_m - x_\mu) - \sum_{m=1}^{\mu-1}S(x_\mu - x_m)\right] = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}S(b(k+l+1)) = \sum_{k=1}^{\infty}kS(kb). \qquad (14)$$

Actually, it is not difficult to verify that

$$\sum_{\mu=2}^{N+1}\sum_{m=1}^{\mu-1} S(x_\mu - x_m) \equiv \sum_{\mu=1}^{N}\sum_{m=\mu+1}^{N+1} S(x_m - x_\mu).$$

Using this identity and recognizing that the lattice of vortex lines is not deformed far from the boundary, we arrive at the proof of the relation (14). Currents flowing along the boundary of the superconductor act on each vortex line with a certain force, which is counterbalanced by the forces of its interaction with the other vortex lines (the second and third components in (13)). Using the relation (14), we can calculate the total force (per unit area of the boundary) which acts on the vortex lines,

$$p_{xx} = \frac{\varphi_0^2}{8\pi a^2\lambda^2}\sum_{k=1}^{\infty} kS(kb). \tag{15}$$

The quantity p_{xx} is one of the diagonal elements of the energy-momentum tensor of the superconductor and is identical with the pressure in the isotropic approximation.

3) The set of equations (13) is equivalent to the system

$$\exp\left(\frac{x_\mu}{\lambda}\right)\frac{\partial G}{\partial x_\mu} - \exp\left(\frac{x_{\mu+1}}{\lambda}\right)\frac{\partial G}{\partial x_{\mu+1}} = 0, \qquad \mu = 1, 2, \ldots, \tag{16}$$

which does not contain H, and one other equation (see (18)). When account is taken of the obvious inequalities

$$0 < x_1 < x_2 < \ldots < x_m < x_{m+1} < \ldots.$$

Equations (16) are solved uniquely (see Secs. 3 and 5) and we obtain

$$x_\mu = f_\mu(x_1, a, b), \quad \mu = 2, 3, \ldots. \tag{17}$$

To determine x_i, we can use the equation

$$\frac{\partial G}{\partial x_1} \propto -\frac{2H}{\beta B}\exp\left(-\frac{x_1}{\lambda}\right) + \sum_{m=2}^{\infty}[S(x_m - x_1) + S(x_m + x_1)] + S(2x_1) = 0, \tag{18}$$

in which we must substitute the expressions (17). As a result, we obtain

$$H/B = F(x_1, a, b), \tag{19}$$

$$F(x_1, a, b) = \frac{1}{2}\beta\exp\left(\frac{x_1}{\lambda}\right)\left[S(2x_1) + \sum_{m=2}^{\infty}(S(x_m - x_1) + S(x_m + x_1)\right]. \tag{20}$$

Both for $x_1 \to 0$ and for $x_1 \to \infty$, the sum in this expression tends toward a finite limit. Inasmuch as

$$1/2\beta S(2x_1) \approx ab/4\pi\lambda x_1 \to \infty, \tag{21}$$

as $x_1 \to 0$ (see (12a)), the function $F(x_1, a, b)$ increases without limit when $x_1 \to 0$ and when $x_1 \to \infty$. Consequently, for some value $x_1 = x_0$, this function reaches a minimum and Eq. (19) does not have solutions for

$$H < H_{min}(B) = BF(x_0, a, b). \tag{22}$$

In accord with (18) and (21), the derivative $\partial G/\partial x_i > 0$ for sufficiently small values of x_1, so that there exists a potential barrier for entry of the vortex lines into the superconductor (cf.[4-7]). The location of the maximum potential barrier is determined by the smaller of the solutions of Eq. (19): $x_1 = x_1' < x_0$. The other solution of this equation $x_1 = x_1'' > x_0$ corresponds to a relative minimum in the Gibbs potential. In the range $x_1' < x_1 < x_1''$, we have $\partial G/\partial x_1 < 0$. Thus the surface barrier acts in both directions. When the external field decreases to the value $H_{min}(B)$,

$$x_1' = x_0 = x_1'',$$

and consequently the Gibbs potential falls off with decrease in x_1 over the entire range $0 < x_1 < x_2$, the surface barrier disappears, the vortex lines in the first layer approach the surface and are cancelled by their images. This is then repeated with the vortex lines of the second layer, and so on. The induction should decrease by a finite value, so that a surface barrier is again formed, capable of containing the lines inside the superconductor.

4) The surface barrier disappears also if the maximum of the external field emerges to the boundary when the field increases. Since our analysis does not hold when $x_1' \sim \xi$, the corresponding external field $H = H_{max}(B)$ can be determined only approximately by assuming $x_1 \approx \xi$ in (19). In accord with (19), (20), (21), we find

$$H_{max}(B) = BF(\xi, a, b) \approx H_s + \beta B\sum_{m=2}^{\infty} S(x_m), \tag{23}$$

where $H_s = \varphi_0/4\pi\lambda\xi \approx H_c$ (cf.[4]). For $H = H_{max}(B)$, the induction should increase by a finite jump before a potential barrier is formed, which hinders the further penetration of the vortex lines into the superconductor.

3. THE REGION $H_{c1} \ll B \ll H_{c2}$

In the case $a \ll \lambda$ and $b \ll \lambda$, and we need take a large number of terms in the sum into account. Using the known[9] relations

$$\sum_{n=-\infty}^{\infty} f_n = \int_{-\infty}^{+\infty} f_n\,dn + 2\sum_{k=1}^{\infty}\int_{-\infty}^{+\infty} f_n\cos(2\pi kn)\,dn,$$

$$\int_{-\infty}^{+\infty} K_1(q(p^2+y^2)^{1/2})\frac{\cos ay}{(p^2+y^2)^{1/2}}\,dy = \frac{\pi}{q|p|}\exp[-|p|(q^2+a^2)^{1/2}],$$

we get from Eq. (12)

$$S(x_\mu - x_m) = \exp(-|x_\mu - x_m|/\lambda) + 2\sigma(x_\mu - x_m), \tag{24}$$

$$\sigma(x_\mu - x_m) = -[1 - (-1)^{\mu+m}\exp(2\pi|x_\mu - x_m|/a)]^{-1}. \tag{25}$$

The corrections to the expression (24) are of the order $(a/\lambda)^2\exp[-2\pi|x_\mu - x_\nu|/a]$. Equation (16) can now be rewritten in the form

$$Q_\mu\exp(-2y_\mu) + (1 + 2\varphi_\mu)\exp(-y_\mu) - (2\varphi_{\mu+1} - 1) - Q_\mu = 0; \tag{26}$$

$$\lambda y_\mu = x_{\mu+1} - x_\mu, \qquad Q_\mu = \sum_{m=\mu+1}^{\infty}\exp\left[-\frac{x_m - x_{\mu+1}}{\lambda}\right],$$

$$\varphi_\mu = -\sum_{m=1}^{\mu-1}\sigma(x_\mu - x_m) + \sum_{m=1}^{\infty}[\sigma(x_{\mu+m} - x_\mu) + \sigma(x_\mu + x_m)]. \tag{27}$$

As becomes clear from what follows, $Q_\mu \sim \beta^{-1}$ and in the most important cases,

$$\beta\varphi_\mu \ll 1, \quad y_\mu \ll 1. \tag{28}$$

Under these conditions, the set of equations (26) can be solved approximately, expanding y_μ and Q_μ in power series in $\beta = b/\lambda \ll 1$. As a result, we find

$$y_\mu = \beta(1 + \varphi_\mu - \varphi_{\mu+1}) + O(\beta^3),$$

and consequently, for $\mu \geq 1$,

$$x_\mu = x_1 + b[\mu - 1 + \varphi_1 - \varphi_\mu + O(\beta^2)]. \tag{29}$$

For the determination of the quantities φ_μ, we obtain the set of equations (see (27))

$$\begin{aligned}\varphi_\mu = &\sum_{m=1}^{\mu-1} [1-(-1)^m \exp(k(m+\varphi_{\mu-m}-\varphi_\mu))]^{-1} \\ &- \sum_{m=1}^{\infty} \{[1-(-1)^m \exp(k(m+\varphi_\mu - \varphi_{\mu+m}))]^{-1} \\ &+ [1-(-1)^{m+\mu} \exp(k(m+\mu-2+2t+2\varphi_1-\varphi_m-\varphi_\mu))]^{-1}\},\end{aligned} \tag{30}$$

where $k = 2\pi b/a$, $t = x_1/b$, $\mu = 1, 2, \ldots$, For $t \to \infty$, the values of φ_μ tend to the finite limits $\varphi_{\mu\infty}$. Using a relation similar to (14), we obtain

$$\varphi_\infty = \sum_{\mu=1}^{\infty} \varphi_{\mu\infty} = -\sum_{m=1}^{\infty} m[1-(-1)^m e^{km}]^{-1}. \tag{31}$$

As $t \to 0$,

$$\varphi_1 \approx (e^{2kt}-1)^{-1} \to \infty, \quad \varphi_{\mu+1} \to \varphi_{\mu\infty}. \tag{32}$$

If $k \gg 1$ (for the case of Fig. 1a, $k = \pi\sqrt{3} = 5.44$) for all values of μ and t,

$$\varphi_\mu \approx (-1)^\mu \{e^{-k\mu} - [e^{k(\mu-1+2t)} - 1]^{-1}\}. \tag{33}$$

For the case of Fig. 1b, we have $k = \pi/\sqrt{3}$, and the expression (33) is too rough an approximation. Solution of (30) by means of a computer at $k = \pi/\sqrt{3}$ led to the results shown in Table I. Thus, in the case of Fig. 1a, the lattice of vortex lines is much less deformed near the boundary. The reason for this is the fact that in this case, even in the second layer, each vortex line has the normal number (six) of nearest neighbors. It will be shown below that a smaller value of the free energy also corresponds to this case.

For the determination of the quantity $x_1 = bt$, we get in accord with (19), (20), (24), (28) and (29) the equation

$$H/b = F(x_1, a, b) = 1 + \beta^2[{}^1\!/_2(t+\varphi_1-{}^1\!/_2)^2 + \varphi - {}^1\!/_{24}], \tag{34}$$

where (cf. (14), (27), (31))

$$\varphi = \sum_{\mu=1}^{\infty} \varphi_\mu = \varphi_\infty + \sum_{\mu=1}^{\infty}\sum_{m=1}^{\infty} \sigma(x_\mu + x_m). \tag{35}$$

Following the program set forth in Sec. 2, we find

$$H_{min}(B) = B + \frac{\varphi_0^2}{\lambda^2}\frac{b}{a}\gamma, \tag{36}$$

where $\gamma \sim 1$ is the maximal value of the function $\frac{1}{2}(t + \varphi_1 - \frac{1}{2})^2 + \varphi - \frac{1}{24}$. For $k \gg 1$, the minimum is reached for $t \approx \frac{1}{2}$, when $\varphi_1 \approx \varphi \approx 0$ (see (35)) and $\gamma \approx -\frac{1}{24}$.

In accord with (32)

$$\beta\varphi_1(\xi) \approx ba/4\pi\lambda\xi = H_s/B,$$

so that

$$H_{max} = BF(\xi, a, b) = B + H_s^2/2B. \tag{37}$$

This expression is valid only for $B \gg H_S$, since we used the inequality (28) in the derivation of Eq. (34). More exact expressions (see the Appendix) give the result:

$$H_{max}^2 = H_s^2 + B^2 + \beta B H_s + B^2\beta^2(2\varphi_\infty + {}^1\!/_6). \tag{38}$$

Of the two solutions of Eqs. (34), the smaller corresponds to the maximum of the surface barrier and the larger to

$$x_1 = x_1'' = b({}^1\!/_2 - \varphi_1) + \lambda[2(H/B-1) + \beta^2({}^1\!/_{12} - 2\varphi)]^{1/2} \tag{39}$$

is the stable position of the first layer. If $2(H/B - 1) \gg \beta^2$, we can set in place of $\varphi_1(t)$ and $\varphi(t)$ their values as $t \to \infty$ (see (31) and (33), which are given in Table II, where the values of the parameters a and b are also listed for the two values $k = 2\pi b/a$ and the single value $ab = d^2\sqrt{3/2}$.

The structure of (39) is such that the total number of vortex lines inside the superconductor for $x_1'' \gg b$ does not depend on the manner of location of the lattic relative to the boundary. We shall prove this. Let the superconductor have the dimension $X \gg b$ along the x axis and $Y \gg X$ along the y axis, and let M be the number of layers parallel to the y axis. Using (29), we get from the condition $x_m = X - x_{M-m+1}$, selecting $m \gg 1$ and $M - m + 1 \gg 1$

$$M = b^{-1}\{X - 2[x_1 + b(\varphi_1 - {}^1\!/_2)]\},$$

and, consequently, the total number of vortex lines

$$N = \frac{YM}{a} = \frac{1}{ab}\left\{XY - 2Y\lambda\left[2\left(\frac{H}{B}-1\right) + \beta^2\left(\frac{1}{12} - 2\varphi_\infty\right)\right]^{1/2}\right\} \tag{40}$$

as the data of Table 2 show, in the limit $x_1 \gg b$ does not depend on the value of the parameter k. This is connected with the fact that the elastic properties of the lattice of vortex lines are almost identical for different directions in the xy plane. Actually, using (24) and (31), we can represent the expression (15) in the form

$$p_{xx} = \frac{\beta^2 B^2}{8\pi}\left[\frac{e^{-\beta}}{(1-e^{-\beta})^2} + 2\varphi_\infty\right] = \frac{B^2}{8\pi}\left[1 - \beta^2\left(\frac{1}{12} - 2\varphi_\infty\right)\right].$$

The values $k = \pi/\sqrt{3}$ correspond to deformations of the vortex line lattice in two mutually perpendicular directions. According to the data of Table I, $p_{xx} \approx p_{yy}$. In

Table I

t	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ
0.05	+5,025	−0,235	+0.197	−0.109	+0 077	−0.043	+4.912
0,1	+2,283	−0,231	+0.192	−0.103	+0.069	−0.032	+2.168
0,2	+0,915	−0,201	+0.162	−0,089	+0.059	−0,026	+0.834
0,4	+0,221	−0.083	+0.061	−0.036	+0,023	−0,011	+0.173
0,6	−0,014	+0.022	−0.014	+0.009	−0.005	+0.003	+0.001
0.8	−0.124	+0.098	−0.058	+0.039	−0.022	+0,012	−0,056
1.0	−0,180	+0,144	−0,083	+0.057	−0.032	+0.017	−0.076
1,5	−0.226	+0.187	−0,104	+0.073	−0.041	+0,022	−0.088
2.0	−0,233	+0,195	−0.108	+0.076	−0,042	+0.023	−0.089
∞	−0,241	+0,198	−0.112	+0.077	−0,045	+0,025	−0.096

Table II

k	b	a	$\varphi_{1\infty}$	φ_{∞}	$\beta^2\left(\frac{1}{12}-2\varphi_{\infty}\right)\frac{4\lambda^2}{d^2}$
$\pi\sqrt{3}$	$d\sqrt{3}/2$	d	−0,00435	−0,00428	0,2759
$\pi/\sqrt{3}$	$d/2$	$d\sqrt{3}$	−0.241	−0,0967	0,2757

the isotropic approximation (see[4]) the pressure is

$$p = {}^1/_2(p_{xx}+p_{yy}) = -g + B\partial g/\partial B,$$

where g is the Gibbs potential per unit volume. In the considered region

$$g = \frac{B^2}{8\pi} + \frac{\varphi_0 B}{16\pi^2\lambda^2}\ln\frac{e^{1/2}\beta' d}{\xi} - \frac{BH}{4\pi}, \tag{41}$$

so that

$$n = \frac{B^2}{8\pi}\left[1-\frac{\varphi_0}{4\pi\lambda^2 B}\right] = \frac{B^2}{8\pi}\left[1-\frac{\sqrt{3}}{2\pi}\frac{d^2}{4\lambda^2}\right].$$

The number $\sqrt{3}/2\pi = 0.2758$ agrees with the data of Table II.

For macroscopic equilibrium ($\partial G/\partial B = 0$), according to[2-4],

$$H = H_0(B) = B + \frac{\varphi_0}{4\pi\lambda^2}\ln\frac{\beta' d}{\xi}, \tag{41a}$$

where $\ln\beta' = -1.893$ for the triangular lattice. The corresponding value of x_1'' for the case of Fig. 1a, when $\varphi_1 \approx \varphi = 0$, can be written in the form

$$x_1'' \approx \frac{d}{2}\left[1+\left(\frac{1}{3}+\frac{4}{\pi\sqrt{3}}\ln\frac{\beta' d}{\xi}\right)^{1/2}\right]. \tag{42}$$

The expressions (41) and (42) are valid if $\beta' d/\xi > 1$.

4. SURFACE ENERGY

Using the results given in Appendices 1 and 4 of the work of Shmidt,[6] we can represent the Gibbs potential for the superconductor in the form of a parallelepiped with dimensions X, Y, and l, placed in an external field H parallel to the z axis in the form (cf. (10))

$$G = \frac{\varphi_0 l}{8\pi}\sum_{n,m}(h_v(\mathbf{r}_{nm}) + 2h_L(\mathbf{r}_{nm}) - 2H). \tag{43}$$

As we go further away from the boundary, the value of $h_v(\mathbf{r}_{nm})$ approaches $h_v(0)$—the value of the magnetic field at the centers of the vortex lines in an unbounded superconductor. Setting

$$h_v(\mathbf{r}_{nm}) = h_v(0) + \delta h(\mathbf{r}_{nm}) \tag{44}$$

and using the expression (40), we find

$$G = Vg + \alpha S, \tag{45}$$

where

$$g = (B/8\pi)(h_v(0) - 2H) \tag{46}$$

is the Gibbs potential density in the unbounded superconductor, $V = XYl$ is the volume and $S = 2Yl$ is the lateral surface area of the superconductor,

$$\alpha = -g[x_1 + b(\varphi_1 - 0{,}5)] + \frac{Bb}{8\pi}\sum_{m=1}^{\infty}(\delta h(x_m) + 2h_L(x_m)) \tag{47}$$

is the surface energy. In the considered case, $h_L(x) = He^{-x/\lambda}$. Setting

$$Q = \sum_{m=1}^{\infty}\exp(-x_m/\lambda)$$

and summing Eq. (15) with account of the relations (14), (24) and (31), we obtain

$$\frac{2HQ}{\beta B} = \frac{e^{-\beta}}{(1-e^{-\beta})^2} + Q^2 + 2\varphi, \tag{48}$$

whence

$$\beta Q = \frac{H}{B} - \left[\frac{H^2}{B^2} - 1 + \beta^2\left(\frac{1}{12} - 2\varphi\right)\right]^{1/2} \approx \frac{H}{B} - \frac{x_1 + b(\varphi_1 - 0.5)}{\lambda}. \tag{49}$$

Calculation of the other sum in (47) presents no difficulty in principle, but is rather cumbersome. We shall give only the final result here:

$$\beta\sum_{m=1}^{\infty}\delta h(x_m) = -\frac{\beta B}{2}\left[\frac{e^{-\beta}}{(1-e^{-\beta})^2} + \frac{2\beta\varphi}{e^{\beta}-1}\right].$$

Equation (47) can now be represented in the form

$$\alpha = -F(B)\bar{x} + 3\lambda H^2/16\pi + B^2\bar{x}^2/16\pi\lambda. \tag{50}$$

Here

$$F(B) = g + BH/4\pi = Bh_v(0)/8\pi$$

is the free energy density, $\bar{x} = x_1'' + b(\varphi_1 - 0.5)$. In the approximation used (the corrections to (50) are of the order of units of $\lambda B^2\beta^3/8\pi$), the surface energy does not depend on the manner of location of the lattice relative to the boundary, when $H/B - 1 \gg \beta^2$ (see Table II). For smaller values of the external field, the case Fig. 1a corresponds to a somewhat smaller value of the surface energy. In order to prove this, we rewrite Eq. (34) in the form

$$2\left(\frac{H}{B}-1\right) + \beta^2\left(\frac{1}{12} - 2\varphi_\infty\right) = \frac{\bar{x}^2}{\lambda^2} + 2\beta^2(\varphi - \varphi_\infty). \tag{51}$$

According to (30) and (35), the quantity $\beta^2(\varphi - \varphi_\infty)$ is always nonnegative and falls off with increase in k. Inasmuch as the left hand side of Eq. (51) is the same for $k = \pi\sqrt{3}$ and $k = \pi/\sqrt{3}$, the larger of these values corresponds to the larger value of $\bar{x}$, and consequently, the smaller surface energy.

5. THE REGION $B \ll H_{c1}$

In this region, $a \gg \lambda$ and $b \gg \lambda$, and it is necessary to take into account the interaction of the vortex lines only with nearest neighbors. In the case of Fig. 1a, even in the second layer, each vortex line has the usual number of nearest neighbors, so that the lattice of the vortex lines is practically undeformed. For the determination of the location of the first layer, in accord with (12) and (18), we obtain the equation

$$H = \frac{\varphi_0}{2\pi\lambda^2}F = \frac{\varphi_0}{2\pi\lambda^2}\exp\left(\frac{x_1}{\lambda}\right)\left[K_1\left(\frac{2x_1}{\lambda}\right) + \sqrt{3}K_1\left(\frac{d}{\lambda}\right)\right]. \tag{52}$$

In the case of Fig. 1b, the distance between the first and second layers $x_2 - x_1$ should be somewhat less than b. Eliminating x_2 from the corresponding set of equations, we get for the determination of x_1 a relation that differs from (52) only in that instead of $\sqrt{3}$

Table III

$\tilde{d}$	$\tilde{B}$	$\tilde{H}_0$, $\varkappa = 10$	$\tilde{k} = \sqrt{3}$			$\tilde{k} = 3$		
			$\tilde{H}_{min}$	$\tilde{x}_0$	$\tilde{x}_1$, $\varkappa = 10$	$\tilde{H}_{min}$	$\tilde{x}_0$	$\tilde{x}_1$, $\varkappa = 10$
2	1.815	1.953	1,080	1.202	2.050	1.468	0.825	1,402
3	0.806	1,476	0.490	1.403	3.075	0.677	1,200	2.510
4	0.453	1.300	0.249	1,910	4,052	0.338	1.685	3,550
5	0.290	1.233	0.134	2,512	5.151	0.180	2,100	4,610
6	0.201	1.208	0.073	2,803	6.102	0,098	2,625	5.752
7	0.148	1,198	0.040	3.402	7,110	0,052	2,975	6.851
8	0.113	1.194	0,022	3.801	8.105	0,030	3,500	7,850
9	0.089	1,193	0.013	4.102	9.100	0,017	3,875	8.950
10	0.073	1.192	0,005	5,001	10,000	0,011	4.200	10,000

we have the coefficient 3 before $K_1(d/\lambda)$. Setting $x_1 = \xi$ in (52), we find

$$H_{max} = \frac{\varphi_0}{4\pi\lambda\xi} + \frac{\varphi_0}{2\pi\lambda^2} K_1\left(\frac{d}{\lambda}\right) \approx H_s. \tag{53}$$

The magnetic field and the induction below in this section and in Table III will be measured in units of $\varphi_0/2\pi\lambda^2$, and the distances in units of λ. For example,

$$\tilde{B} = 2\pi\lambda^2 B / \varphi_0 = 4\pi / \tilde{d}^2 \sqrt{3}. \tag{54}$$

The function $F(x_1)$ has a minimum for $x_1 = x_0$. The quantity x_0 is the solution of the equation

$$2K_0(2\tilde{x}_0) + (\tilde{x}_0^{-1} - 1) K_1(2\tilde{x}_0) = \tilde{k} K_1(\tilde{d}), \tag{55}$$

where $\tilde{k} = \sqrt{3}$ for the case of Fig. 1a and $\tilde{k} = 3$ for the case of Fig. 1b. Furthermore, we find

$$\tilde{H}_{min} = \exp(\tilde{x}_0) [2K_0(2\tilde{x}_0) + \tilde{x}_0^{-1} K_1(2\tilde{x}_0)]. \tag{56}$$

The relations (54)--(56) determine the dependence of $\tilde{H}_{min}(\tilde{B})$ in parametric form. The values of $\tilde{H}_{min}$, $\tilde{x}_0$ and $\tilde{x}_1$ ($\lambda\tilde{x}_1$ is the distance of the first layer to the boundary at macroscopic equilibrium) are given in Table III.

General expressions for the surface energy (44)–(47) are valid also in the case under consideration (in (47) we must formally set $\varphi_1 = 0.5$). Without going through the corresponding calculations, we note that in the case of Fig. 1a the quantity α is smaller at macroscopic equilibrium.

6. DISCUSSION OF THE RESULTS

The above analysis of the equilibrium conditions of (12) shows that:

1) the lattice of vortex lines is deformed only at distances from the boundary of the order d (and not λ as was to have been expected); 2) there exists a position of the vortex lines relative to the boundary (Fig. 1a) for which the deformation is minimal; 3) for a given value of B, the balance equations have solutions for values of the external field H in the range $H_{min}(B) < H < H_{max}(B)$ (see (36), (38), (53), (56) and Fig. 2); 4) the case Fig. 1a corresponds to the smallest values of the surface energy and $H_{min}(B)$ and the largest value of $H_{max}(B)$; however, the dependence of these quantities on the orientation of the lattice is weak.

The results obtained show that considerable hysteresis will be observed even in ideal type II superconductors in the experimental determination of the magnetization curves H(B). This is due to the interaction of the vortex lines with the surface. Such a "surface hysteresis" is actually observed (see, for example,[10-13]) but the experimental values of $H_{max} - H$ and $H - H_{min}$ are significantly less (sometimes by an order of magnitude) than is predicted here. This divergence can be connected both with the obvious experimental difficulties of obtaining metastable states far from macroscopic equilibrium, and also with the limited applicability of our results for small values of the parameter κ. It should also be noted that, at finite temperatures, a thermally activated transition of the vortex lines through the surface barrier are possible, so that the absolute boundaries of stability of the mixed state computed here may turn out to be unattainable for $T \neq 0$.

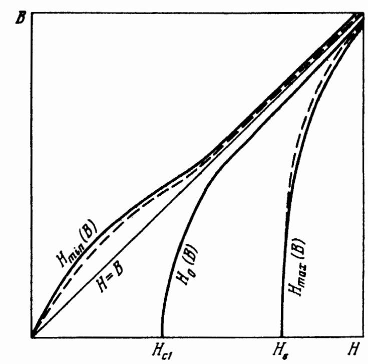

Fig. 2—Stability limits of mixed state $H_{min}(B)$ and $H_{max}(B)$ (shown schematically). The solid lines pertain to the case of Fig. 1a and the dashed to the case Fig 1b. $H_0(B)$ is the equilibrium magnetization curve.

In conclusion, we consider the case in which the external field on the lateral surface of a thick ($X \gg \lambda$) layer has different values H_1 and H_2, i.e., a current flows along the layer. For definiteness, let $H_1 > H_2$ and let the value of the induction be B. The quantity B is determined from the history of the sample. Relatively stable current states exist when

$$H_{max}(B) > H_1 > H_2 > H_{min}(B).$$

If, upon increase in the current, H_1 reaches the value $H_{max}(B)$, or H_2 is shifted to the value $H_{min}(B)$, instabilities arise (see Sec. 2), which can lead to a disruption of the superconductivity. The maximum critical current is evidently proportional to $H_{max}(B) - H_{min}(B)$.

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APPENDIX

The proof of Eq. (38) is given below. Using (24) and

(27), we can represent (20) in the form

$$F(x_1,a,b)=\beta\exp\left(\frac{x_1}{\lambda}\right)\left[\varphi_1+\frac{1}{2}\exp\left(-\frac{x_1}{\lambda}\right)+\operatorname{ch}\frac{x_1}{\lambda}\sum_{m=2}^{\infty}\exp\left(-\frac{x_m}{\lambda}\right)\right].\qquad\text{(A.1)}$$

If the first layer is very close to the boundary ($x_1\approx\xi$), occupying a position at the maximum of the surface barrier, its magnetic field is almost entirely cancelled by the images. Therefore the vortex lines of the first layer practically do not interact with the other vortex lines. It then follows that the vortex lines of the second layer does not exist at all, i.e., $x_2 = x_1''$ (see Sec. 2). Therefore, in (A.1)

$$\sum_{m=2}^{\infty}\exp\left(-\frac{x_m}{\lambda}\right)=Q,$$

where Q is determined by the relations (48) and (49), in which one must set $H = H_{max}(B)$. Recognizing also that $\beta\varphi_1(\xi)\approx H_sB$, in accord with (19) and (A.1), we find

$$\frac{H_{max}}{B}=\frac{H_s}{B}+\frac{\beta}{2}+\frac{H_{max}}{B}-\left[\frac{H_{max}^2}{B^2}-1+\beta^2\left(\frac{1}{12}-2\varphi_\infty\right)\right]^{1/2}.\qquad\text{(A.2)}$$

Solving this equation relative to H_{max}, we obtain the expression (38). This result is valid for any value of B.

If $H\sim H_{max}$, the quantity x_1'' can reach values of the order of λ. In this case the expression (39) should be made more precise. Using (29), we get

$$\beta Q=\beta\exp\left[-\frac{x_1}{\lambda}-\beta\left(\varphi_1-\frac{1}{2}\right)\right]\sum_{m=1}^{\infty}\exp[-\beta(m-1)+\beta\varphi_m]\qquad\text{(A.3)}$$

$$=\exp\left[-\frac{x_1}{\lambda}-\beta\left(\varphi_1-\frac{1}{2}\right)\right]\left[1-\frac{\beta^2}{24}+\beta^2\varphi\right]$$

whence

$$x^1=x_1''=b(1/2-\varphi_1)+\lambda\ln\left[(1-\beta^2/24+\beta^2\varphi)\beta^{-1}Q^{-1}\right],\qquad\text{(A.4)}$$

where βQ is given by the expression (49). For $H/B - 1 \ll 1$, Eq. (A.4) is identical with (39).

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