Contribution to the Theory of Stochastic Heating of Particles in a Turbulent Plasma

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Attention is called to the fact that the solution of the problem of turbulent plasma heating requires that account be taken of the evolution of the particle distribution function in the matrix element of the nonlinear wave interaction. It is shown that with increasing temperature there is established an ion distribution that decreases in power-law fashion with increasing velocity v. It is found that stochastic ion heating cannot be described by a self-similar solution.

THE purpose of the present study was to investigate a number of effects connected with the scattering of waves by plasma particles and leading to heating of the particles. The results of the theory of weakly-turbulent plasma have already been used many times to estimate the influence of nonlinear effects on two-stream instability[1] and on the current instability in the plasma[2,3], to calculate the stationary spectra of the plasma turbulence[4], etc. In these cases, however, allowance for the nonlinear effects reduces as a rule to the introduction of corresponding corrections to the oscillation increment. Yet the heating of the plasma particles by the nonlinear processes changes the increment itself, and this can play an important role in such problems as the calculation of the stationary turbulence spectra or in the theory of turbulent plasma heating by a current.

To describe the stochastic heating of the particles in many problems of practical interest, it suffices to calculate the moments $\langle v^2\rangle(t)$ and $\langle(v^2-\langle v^2\rangle)^2\rangle(t)$ (see, e.g.,[5]), but in some cases a kinetic analysis is essential. The point is that the time dependence of the mean-squared velocity gives a correct idea of the heating only in the case of Maxwellian (or near-Maxwellian) particle distribution functions, but entirely different solutions are also possible, as will be shown below. Plasma heating may be accompanied by a significant change in the matrix elements of the nonlinear interaction, and this change cannot always be taken into account by means of the time dependence of the average plasma parameters.

By way of an example, we consider the heating of ions when they scatter long-wave ion-acoustic noise ($\omega = kc_s$). As is well known, stochastic heating of particles by nonlinear damping of oscillations is described by the equations

$$\frac{\partial f^\alpha}{\partial t}=\frac{\partial}{\partial v_i}D_{ij}\frac{\partial f^\alpha}{\partial v_j},\qquad \alpha=i,e. \tag{1}$$

The diffusion coefficient D_{ij} in velocity space can be calculated by perturbation theory[5,6]. If the waves are potential, it is given by

$$D_{ij}^\alpha=4\pi\int d\mathbf{k}\,d\mathbf{k}'\frac{W_\mathbf{k}W_{\mathbf{k}'}}{(m_\alpha n)^2}(\omega\varepsilon_\omega')^{-1}(\omega'\varepsilon_{\omega'}')^{-1}\frac{\omega_{0\alpha}^4}{k^2k'^2}$$
$$\times\left|\frac{\mathbf{kk}'}{\tilde\omega\tilde\omega'}+\frac{S^\alpha_{k'k}}{\varepsilon(\mathbf{k}'',\omega'')}\right|^2k_i''k_j''\delta(\tilde\omega''), \tag{2}$$

where $W_\mathbf{k}$ is the spectral energy density of the noise, $\epsilon(\mathbf{k},\omega)$ is the dielectric constant,

$$\mathbf{k}''=\mathbf{k}-\mathbf{k}',\quad \omega''=\omega-\omega',\quad \tilde\omega=\omega-\mathbf{kv},\quad \tilde\omega'=\omega'-\mathbf{k}'\mathbf{v},$$
$$\omega_{0\alpha}^2=4\pi ne_\alpha^2/m_\alpha,$$

$$S^\alpha_{k'k}=\left(\frac{m}{e}\right)_\alpha\sum_{\beta=i,e}\frac{\omega_{0\beta}^2}{k^2}\left(\frac{e}{m}\right)_\beta\int\frac{d\mathbf{v}}{\tilde\omega''}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\frac{\mathbf{k}'\partial/\partial\mathbf{v}}{\tilde\omega'}-\mathbf{k}'\frac{\partial}{\partial\mathbf{v}}\frac{\mathbf{k}\partial/\partial\mathbf{v}}{\tilde\omega}\right)f^\beta.$$

In our case of scattering os sound by ions, expression (2) can be transformed into

$$D_{ij}^{(i)}=\pi\int d\mathbf{k}\,d\mathbf{k}'\frac{W_\mathbf{k}W_{\mathbf{k}'}}{(Mn)^2}\frac{\delta(k-k')}{c_s}\cos^2\widehat{\mathbf{kk}'}\,k_i''k_j''\left|\left(\frac{\omega\omega'}{\tilde\omega\tilde\omega'}-1\right)\right.$$
$$\left.+\frac{1-\cos\widehat{\mathbf{kk}'}}{c_s^2}\Omega^{-1}_{\mathbf{kk}'}-\int d\mathbf{v}\frac{\mathbf{k}''\,\partial f/\partial\mathbf{v}}{\tilde\omega''}\left(\frac{\omega\omega'}{\tilde\omega\tilde\omega'}-1\right)\Omega^{-1}_{\mathbf{kk}'}\right|^2,$$
$$c_s^2=T_e/M,\qquad M\equiv m_i,\qquad \Omega_{\mathbf{kk}'}=\int d\mathbf{v}\frac{\mathbf{k}''\,\partial f/\partial\mathbf{v}}{\tilde\omega''}. \tag{2$'$}$$

We turn first to the model problem. We assume that the distributions of the particles and of the noise are isotropic. In this case Eq. (1) with diffusion coefficient (2′) yields

$$\frac{\partial f^i}{\partial t}=\frac{1}{v^2}\frac{\partial}{\partial v}D(v)v^2\frac{\partial f}{\partial v},$$
$$D(v)=\pi^3\int dk\,k^6c_s\left(\frac{W_\mathbf{k}}{nT_e}\right)^2\begin{cases}\eta v^2, & v\gg\langle v^2\rangle/c_s\\ \zeta\langle v^2\rangle^2/c_s^2; & v\ll\langle v^2\rangle/c_s\end{cases}, \tag{3}$$

$\eta,\zeta\sim 1$.

For arbitrary values of v, it is convenient to investigate the following approximate equation:

$$\frac{\partial f}{\partial t}=\frac{d(t)}{v^2}\frac{\partial}{\partial v}(v^2+v_0^2)v^2\frac{\partial f}{\partial v},$$
$$d(t)\sim\omega(W/nT_e)^2,\quad v_0^2\sim\langle v^2\rangle^2/c_s^2\ll\langle v^2\rangle. \tag{3$'$}$$

Equation (3) admits of a self-similar separation of the variables. In the asymptotic regime, only a power-law increase of the ion temperature is possible, from which it follows that $d(t)=\delta/t$. We note that under conditions that are typical for the turbulent plasma heating problem we have[7,8]

$$W/nT_e\sim t^{-1/2},\quad \omega\approx\text{const},\quad d(t)\sim t^{-1}$$

We seek the distribution function f(v) in the form

$$f(v)=v_s^{-3}F(u),\qquad u=v/v_s.$$

The condition for the separation of the variables takes the form $v_0^2/v_s^2=\text{const}$, i.e., $\langle v^2\rangle\sim c_s^2$; $T_i\sim T_e$. We obtain the self-similar solution

$$F=\text{const}/(u^2+u_0^2)^{1/\delta},\qquad u_0=v_0/v_s,$$

from which it follows, however, that $\langle v^2\rangle\approx v_0^2$, which contradicts the condition $v_0^2\ll\langle v^2\rangle$. Thus, the ion-heating process must be non-self-similar.

For $v\gg v_0$, the solution (3′) can be represented in the form

$$f(v)=\sum_\alpha C_\alpha v^{-\alpha}\exp\left\{\alpha(\alpha-3)\int d(t)\,dt\right\}. \tag{4}$$

If we confine ourselves to solutions with $\alpha>5$ (the condition for the convergence of the integral $\int f(v)v^2dv$), then we get from (3′)

$$\partial\langle v^2\rangle / \partial t \approx 10\, d(t) \langle v^2\rangle$$

and, for example at $d(t) = \delta/t$, the solution takes the form

$$f(v) = \frac{\text{const}}{v^{8-1/\delta}} t^{\delta(5-1/\delta)(8-1/\delta)}. \tag{4'}$$

There exist, however, solutions with $\alpha \leq 5$, for which the quantity $\langle v^2\rangle$ is determined by the upper limit of integration $v \sim c_s$. Of course, both the nonlinear oscillation increment and the expression for the nonlinear diffusion coefficient change in this case. The qualitative result, however, remains in force: the function $f(v)$ decreases with increasing v like a power function (in any case, slower than the exponential function), and the rms velocity can be determined by a small group of high-energy particles. In the opposite case, Eq. (3′) would remain valid.

We note that slowly-decreasing power-law solutions of the diffusion equation mean an appreciable flux of particles into the resonance region $v > c_s$. This can cause the evolution of the wave spectrum in the asymptotic regime to be determined in the self-consistent problem of stochastic heating only by the quasi-linear effects.

Let us consider a problem closer to the real experimental conditions for turbulent plasma heating. Let the spectrum of the ion-acoustic wave be strongly anisotropic[7,3]. For simplicity we shall assume henceforth that the noise in k-space is distributed in a narrow cone with aperture angle ϑ_0 (the symmetry axis is the direction of the external electric field). We put

$$W_k = w_k W(\vartheta), \quad f(\mathbf{v}) = f_1(v_\parallel) f_2(v_\perp^2), \quad \mathbf{v}_\parallel \| \mathbf{E}, \quad \mathbf{v}_\perp \perp \mathbf{E}. \tag{5}$$

Since $k' \approx k$ in the case of scattering (see (2)), i.e., $\mathbf{k}'' \perp \mathbf{k}$, and the diffusion equation contains the quantities $\mathbf{k}''\partial/\partial\mathbf{v}$, the scattering leads in first approximation in v/c only to an increase of the value of $\langle v_\perp^2\rangle$. We retain in the matrix element (2′) only the terms that are principal in some velocity interval:

$$\left(\frac{\mathbf{k}\mathbf{k}'}{\tilde{\omega}\tilde{\omega}'} + \frac{S^i_{k'k}}{\varepsilon(\mathbf{k}'', \omega'')}\right) \approx \frac{1}{2c_s^2}\left[\frac{(\omega+\omega')(\mathbf{k}+\mathbf{k}', \mathbf{v})}{\omega\omega'} - \frac{3}{2}\frac{(k+k')^2\langle v_\parallel^2\rangle}{\omega\omega'} + \frac{k''^2/2 + 2k^2 r_{De}^2 k'^2}{\omega\omega'\Omega_{\mathbf{k}\mathbf{k}'}}\right], \tag{6}$$

where r_{De} is the Debye radius. The first term in the matrix element (6) is the principal one if any of the following three inequalities is satisfied:

$$v_\perp > \frac{k''}{k}\frac{\langle v_\perp^2\rangle}{c_s}, \quad v_\perp > \frac{k^2}{k''}\frac{\omega}{\omega_{pi}^2}\langle v_\perp^2\rangle, \quad v_\parallel > \frac{\langle v_\parallel^2\rangle}{c_s}. \tag{7}$$

The velocity diffusion equation takes the following form in the first non-vanishing approximation in v/c_s:

$$\frac{\partial f}{\partial t} = \frac{d(t)}{v_\perp}\frac{\partial}{\partial v_\perp}(v_\perp^2 + v_0^2) v_\perp \frac{\partial f}{\partial v_\perp},$$

$$d(t) = \pi^3 \int dk \left(\frac{w_k}{nT_e}\right)^2 k^6 c_s \left[{}^3/_2 \int_0^{\vartheta_0} \vartheta^5 W(\vartheta)\, d\vartheta \int_0^{\vartheta} \vartheta' W(\vartheta')\, d\vartheta' - \int_0^{\vartheta_0} \vartheta^3 W(\vartheta)\, d\vartheta \int_0^{\vartheta} \vartheta'^3 W(\vartheta')\, d\vartheta'\right],$$

$$v_0 \sim \vartheta_0 \langle v_\perp^2\rangle / c_s. \tag{8}$$

Solving (8), we arrive at the same results as were obtained earlier in the isotropic model. Namely, $f_2(v_\perp)$ is a power function as $t \to \infty$, which in turn can alter significantly the matrix element. The decrement of the nonlinear damping and the diffusion coefficient in velocity space turn out to be larger than could be expected by starting from the generally employed expressions.

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253