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Model of a Stationary Stellar Cluster with a High Binding Energy

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An hierarchical stationary model of a stellar cluster with a spherical symmetric density distribution and a binding energy close to the rest-mass energy of the system is proposed. The model is valid for a whole set of distribution functions including that isotropic with respect to the velocities.

1. INTRODUCTION

THE problem of the gravitational mass defect is of great interest to physicists, both those engaging in general relativity theory (GRT) and those interested in applications to astrophysics. If we disregard specially stationary systems, then it is well known that the gravitational mass defect can be arbitrary. An example is a closed universe whose energy is equal to zero and whose gravitational mass defect is equal to M_0c^2, and a "semi-closed world" with a mass defect arbitrarily close to M_0c^2 [1,2]. As shown by Zel'dovich[3], a body of arbitrary rest mass M_0 can be reduced to a state with mass arbitrarily close to zero and a binding energy approaching M_0c^2. Such a state is nonstationary and is separated from the equilibrium state, if the latter exists, by a potential barrier.

The question of the binding energy of a stationary state, and all the more of a stable state, is not simple, although it is precisely such systems, namely stellar clusters and superstars, which are of interest to astrophysicists in connection with the explanation of the processes occurring in quasars and in galactic cores (nuclei). The model of a stationary stellar cluster is being intensively investigated of late in connection with the possibility of obtaining in it large red shifts and large binding energies. In view of the destabilizing effects of GRT, it was proposed[4,5] that a stable model of a stellar cluster with an isotropic distribution function with respect to the momenta p has limited values of the red shift at the center, $z_c \lesssim 0.5$; the binding energy of such a stellar cluster is always bounded and does not exceed $B \lesssim 0.036M_0c^2$.

It was shown in[6] that the first part of this assumption is incorrect. In the same paper, models were constructed of stellar clusters with infinitely large z_c for distribution functions that are isotropic in p, and even a model of an isothermal star, in which z_c is also infinitely large. In the present paper it is shown that the second assumption is also incorrect. Stable models of stellar clusters of different hierarchical structures are constructed, capable of possessing a binding energy arbitrarily close to M_0c^2.

The stability of the models of[6] was considered in[7], where it was shown that the star model is stable; the stability of the stellar-cluster model could not be proved exactly, but arguments favoring its stability were advanced. However, the binding energy of such a model remained bounded, not exceeding $0.006M_0c^2$. A stable spherically-symmetrical model of a cluster with arbitrary z_c, in which the stars move in circular orbit, was constructed by Einstein[8]. In his model, the binding energy can reach $B \sim 0.06M_0c^2$, corresponding to the binding energy of a particle moving in a Schwarzschild metric along the last stable circular orbit[9].

The important role played by rotation in the stabilization of a relativistic system has been elucidated recently in a number of papers[10-12]. The uniformly-rotating disk considered in these papers remains stable up to $z_c \to \infty$, and its binding energy reaches $0.37M_0c^2$. An important role is played here by the influence of the angular momentum on the gravitational field of the disk. A disk with the same mass distribution but without angular momentum[13], with oppositely directed beams of stars, has a much lower limiting binding energy, approximately equal to the binding energy in Einstein's model. Examining the physical processes that occur during the evolution and fragmentation of such a disk, Salpeter[14] has shown qualitatively that it is possible to construct a rotating stable model in which the binding energy can come arbitrarily close to M_0c^2.

In the present paper, stationary models are constructed for clusters with binding energies arbitrarily close to M_0c^2, for the case of isotropy in the mean and spherical symmetry. A characteristic property of such a model is the hierarchy system employed in[14]. Each term of the hierarchy, at any level, constitutes the model considered in[4-6]. The properties of such a model, such as forces, velocities, potentials, and red shifts, are discussed. Thus, the possibility of obtaining a high binding energy is not always connected with rotation, and is typical for systems of quite arbitrary form.

2. CLUSTER MODEL

We consider a cluster of stars, where the mass of each star is m_0, and the momentum projected on the spherical coordinate system (r, θ, φ) is $p(p_r, p_\theta, p_\varphi)$, with $p_t^2 = p_\theta^2 + p_\varphi^2$. The equilibrium distribution function, which has a large red shift in the center[6], is of the form

$$f = \frac{A}{r^2}(p^2c^2 + m_0^2c^4)^{-k/2}p_t^{ka/2-2}. \tag{1}$$

Here the density is $\rho = m_0 \int f dp = \beta/r^2$, A is defined by the normalization condition, a is uniquely connected with β just as in[6], and the red shift $1 + z \sim r^{-a/2}$ can be arbitrarily large at the center. Strictly speaking, such a model has an infinite radius and an infinite central density, but by suitable choice of the cutoff we can eliminate the infinities without changing the essential properties of the model[7]. We consider two limiting cases of the model (1).

In case (a) we consider an isotropic distribution function with $k = 4a^{-1} = 2$:

$$f_0^{(a)} = \frac{A}{r^2}(p^2c^2 + m_0^2c^4)^{-\alpha}. \tag{2}$$

The binding energy of such a model, B_0, does not exceed $\sim 0.006 M_0c^2$, and stable models correspond to large α; we can therefore use the expansion

$$\frac{B_0}{M_0c^2} = \frac{1}{4\alpha} - \frac{81}{32}\frac{1}{\alpha^2}, \quad \alpha > 21. \tag{3}$$

The degree of relativism of the model is determined by the quantity $\gamma = r_g/r$, which is equal to

$$\gamma = (2\alpha - 1)/(\alpha^2 + 2\alpha - 1). \tag{4}$$

Case (b) corresponds to the Einstein model and is obtained from (1) as $k \to \infty$:

$$f_0^{(б)} = \frac{1}{\pi}\frac{\beta}{m_0 r^2}\delta(p_r)\delta(p_t^2 - p_0^2), \tag{5}$$
$$\frac{B_0}{M_0c^2} = 1 - \frac{1+x_0^2}{(1+3x_0^2)^{1/2}}, \quad \gamma = \frac{2x_0^2}{1+3x_0^2},$$
$$1 + z \sim r^{-x_0^2/(1+x_0^2)}, \quad x_0 = p_0/m_0c < 3^{-1/2}.$$

A characteristic feature of the foregoing models is the constancy of the degree of relativism γ over the entire cluster. Let us consider at the second stage of the hierarchy a cluster whose element is a cluster defined by the distribution function (2) or (5). We assume now that the distribution function of the second-degree cluster is also given by (2) or (5), and assume for simplicity that the degree of relativism is the same. Continuing such a hierarchical construction i times, we obtain

$$f_i^{(a)} = \frac{A}{r^2}(p^2c^2 + m_i^2c^4)^{-\alpha},$$
$$f_i^{(б)} = \frac{1}{\pi}\frac{\beta}{m_i r^2}\delta(p_r)\delta(p_t^2 - p_i^2). \tag{6}$$

Let now the cluster at each degree of the hierarchy consist of n elements, and then the rest mass of the cluster of first degree is $M_0 = nM_0$, while the total mass of this cluster, equal to the mass of the element m_1 of the cluster of second degree, is

$$m_1 = M_0 - B_0/c^2 = M_0(1-\delta) = nm_0(1-\delta); \tag{7}$$
$$\delta = \frac{1}{4\alpha} - \frac{81}{32}\frac{1}{\alpha^2} \text{ (a)}, \quad \delta = 1 - \frac{1+x_0^2}{(1+3x_0^2)^{1/2}} \text{ (б)}$$

We thus have for the mass of the element of i-th degree of the hierarchy and for its rest mass M_i the expressions

$$m_i = nm_{i-1}(1-\delta) = n^i m_0(1-\delta)^i, \quad M_i = n^{i+1}m_0. \tag{8}$$

Consequently, in such a model both the total mass and the rest mass increase with increasing level of the hierarchy, but the binding energy approaches M_ic^2:

$$\frac{B_i}{M_ic^2} = \frac{M_i - m_{i+1}}{M_i} = 1 - (1-\delta)^{i+1} \approx 1 - e^{-i\delta} \text{ for } i \gg 1. \tag{9}$$

An analogous model was constructed by Salpeter[14] for rotating disks. The model with the distribution function $f_i^{(b)}$ can be called the generalized Einstein model.

We note that to obtain large B it is not essential to use analytically-expressible distribution functions of type (1) for each member of the hierarchy. Each member of the hierarchy can be a cluster of quite general form, for example of the form in[4,5]; All that matters is that we have $\delta = B/Mc^2 > 0$ at each degree of the hierarchy.

3. PROPERTIES OF THE MODEL

Although the binding energy of the model (6) can be arbitrarily close to Mc^2, the dependence of the red shift on the radius is on the average the same as in model (2) and (5). In this case, however, this dependence has sharp local peaks at places occupied by the cluster elements. If $\gamma = r_g/r = 2GM/c^2R$ is the same for all the degrees of the hierarchy, then the volume occupied by the next degree of the hierarchy is n^3 times larger than the preceding one, whereas the total volume occupied by the elements of this degree is only n times larger than the volume of the preceding hierarchy. Therefore the radiation coming from such a hierarchic system in n places occupying a fraction n^{-2} of the total volume will experience a red shift greatly exceeding the mean value. At each such maximum there will be located, in turn, n more maxima of z, etc., to the very first degree of the hierarchy. Since the red shift is directly connected with the gravitational potential, it can be stated that the potential well corresponding to the cluster as a whole has a fine structure in the Newtonian interpretation, and the volume of each of the regions occupied by the narrowest minima is smaller by a factor n^{3i} than the volume of the entire cluster, while the total number of such minima is n^i.

We consider now the forces acting on each element of the cluster. The distance between the elements of the cluster of hierarchy j is n times larger than between the elements of the hierarchy $j + 1$, and therefore the Newtonian force exerted on an element of the cluster by all the elements of higher degrees of the hierarchy is smaller by a factor $\sim n$ than the force exerted by the elements of the given cluster; an independent consideration of the elements of each degree is therefore justified in the case of large n. Since the initial particle of mass m_0 participates in i independent motions, in each of which it has an rms velocity

$\sim(GM/R)^{1/2}$, the rms velocity of the particle relative to an immobile observer is larger by $\sqrt{i}$ times, so long as $iGM/R \ll c^2$. When this inequality is violated, the approach of the average velocity to the velocity of light proceeds in accordance with the rules for relativistic velocity addition.

Since each element of the hierarchy is independent, its stability can also be analyzed independently, and therefore the condition for the stability of the entire system coincides with the condition for the stability of each element[6,7]

$$\alpha > \sim 21 \quad (a), \qquad x_0 < 3^{-1/2} \quad (b). \tag{10}$$

Thus, the constructed cluster model is essentially relativistic with a binding energy arbitrarily close to the rest mass of the system, but is at the same time stationary and stable if condition (10) is satisfied.

The properties noted above are possessed also by hierarchical clusters based on the models of[4,5], but the average red shift for them is limited, although there are local maxima of z, at which the red shift can be very large. Obviously, the construction of a stable star with $B \sim M_0c^2$ is impossible, and the models of[6] can be generalized only for clusters of point-like masses.

Interest in the hierarchical model of the universe has been recently revived, but only nonstationary variants have been considered[15,16]. The existence of stationary hierarchical clusters of the type considered here is possible both on very large scales, of the order of the dimension of the visible part of the universe, and on scales of quasars and galactic cores.

[1]Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. **43**, 1037 (1962) [Sov. Phys.-JETP **16**, 732 (1963)].
[2]I. D. Novikov, Vestnik MGU, Ser. III **6**, 61 (1962).
[3]Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. **42**, 641 (1962) [Sov. Phys.-JETP **15**, 446 (1962)].
[4]Ya. B. Zel'dovich and M. A. Podurets, Astron. Zh. **42**, 963 (1965) [Sov. Astron.-AJ **9**, 742 (1966)].
[5]J. R. Ipser, Astrophys. J. **158**, 17 (1969).
[6]G. S. Bisnovatyĭ-Kogan and Ya. B. Zel'dovich, Astrofizika **5**, 223 (1969).
[7]G. S. Bisnovatyi-Kogan and K. S. Thorne, Astrophys. J. **160**, 875 (1970).
[8]A. Einstein, Ann. Math. **40**, 922 (1939).
[9]S. A. Kaplan, Zh. Eksp. Teor. Fiz. **19**, 951 (1949).
[10]J. M. Bardeen and R. V. Wagoner, Astrophys. J. **158**, 165 (1969).
[11]J. M. Bardeen and R. V. Wagoner, Astrophys. J. **167**, 359 (1971).
[12]E. E. Salpeter and R. V. Wagoner, Astrophys. J. **164**, 557 (1971).
[13]T. Morgan and L. Morgan, Phys. Rev. **183**, 1097 (1969).
[14]E. E. Salpeter, Nature, Phys. Sci. **233**, 5 (1971).
[15]J. K. Wertz, Astrophys. J. **164**, 227 (1971).
[16]M. J. Haggerty, Astrophys. J. **166**, 257 (1971).

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