Nonlinear Effects in the Interaction Between an Ultrarelativistic Electron Beam and a Plasma

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The problem of stationary injection of an ultrarelativistic electron beam into a half-space filled with a plasma is considered. The problem is solved within the framework of the weak turbulence theory for conditions when the interaction between oscillations excited by the beam is significant. The main interaction mechanism (induced scattering of the oscillations by plasma ions) is determined and the beam slowing-down length with allowance of the process is found. It is demonstrated that as relaxation proceeds the angular spreading of the beam increases up to $\Delta\theta \sim 1$. Variation of the plasma parameters under action of the beam is investigated. It is found that in a dense plasma disruption of relaxation due to the produced density inhomogeneities may lead to the formation of a so-called relaxation wave which propagates in the plasma with ultrasonic velocity.

1. INTRODUCTION

ONE of the methods of heating a plasma to thermonuclear temperatures is to impart to it the energy stored in a powerful beam of relativistic electrons. To estimate the prospects of such a heating method it is necessary to investigate the collective mechanisms of interaction between the beam and the plasma, since the deceleration of the beam by paired collisions is usually extremely ineffective. Recognizing that the characteristic time of variation of the plasma parameters under the influence of the beam under real conditions is much longer than the time of development of the two-stream instability, the theoretical description of the heating process can be divided into two parts. The first is to solve the problem of the relaxation of a beam in a plasma having parameters that are fixed in time and find the energy lost by the beam per unit length of its path in the plasma. The second is to investigate, on the basis of the results, the change in the state of the plasma when heated by the beam, and to ascertain how this change affects the character of the relaxation.

The first part of the problem has been solved within the framework of the quasilinear approximation in a number of papers[1-4], for both a homogeneous and an inhomogeneous plasma. In many cases of practical interest, however, the criterion for the applicability of the quasilinear approximation is not satisfied[3,5], and therefore particular interest attaches to an investigation of the nonlinear relaxation regime. As shown by Tsytovich and Shapiro[6], the processes of nonlinear transformation of the spectrum of the oscillations excited by the beam can lead to stabilization of the two-stream instability. The energy of the oscillations interacting with the beam electrons turns out in fact to be "frozen" at a very low level, and the beam deceleration length increases in comparison with the quasilinear one.

As applied to the problem of heating a dense plasma target by an ultrarelativistic beam, the role of the nonlinear effects was investigated by Rudakov[3]. He regarded the mechanism of nonlinear stabilization to be induced scattering of the oscillations by the plasma particles. The relaxation model constructed by Rudakov[3] is based on the assumption that the long-wave Langmuir oscillations ($k < \omega_p/c$, where k is the wave number, ω_p is the plasma frequency, and c is the speed of light), which do not interact with the beam, suppress the instability almost completely in the "resonant" region ($k \gtrsim \omega_p/c$), owing to the induced scattering of the "resonant" oscillations by the plasma electrons. The residual instability only compensates for the collision damping of the "nonresonant" oscillations. The role of the scattering of the oscillations by ions is actually disregarded in such a model, although scattering by ions leads to a rather faster transfer of the oscillations into the long-wave region than does scattering by electrons. As will be shown in the present paper, scattering by ions alters significantly the dynamics of the relaxation in comparison with Rudakov's model[3]. In particular, the relaxation ceases to be quasi-one-dimensional, and the beam slowing-down length increases appreciably.

The investigation of the nonlinear relaxation regime in a plasma with fixed parameters is the subject of Sec. 2. The change induced in the plasma parameters by the beam is considered in Secs. 3 and 4. It is shown there that under certain conditions the process of heating a dense plasma target consists of propagation in the plasma of a wave on whose front the beam energy is converted into heat. This phenomenon is called in the article a relaxation wave.

2. NONLINEAR RELAXATION REGIME

Assume that a monoenergetic ultrarelativistic beam ($E \gg mc^2$) is injected into a half-space $z > 0$ filled with a homogeneous plasma. The distribution function of the beam as it enters the plasma is given by

$$f = \frac{n_b g_0(\theta)}{2\pi p_0^2}\delta(p - p_0), \tag{1}$$

where n_b is the beam concentration, p_0 is the electron momentum, and $g_0(\theta)$ is the angular distribution of the particles. We assume that the angular spread $\Delta\theta$ of the beam is not too small:

$$1 \gg \Delta\theta \gg mc^2/E. \tag{2}$$

We can then neglect the difference between the absolute value of the beam particle velocity and c, and put $v = cp/p$. If in addition

$$\Delta\theta \gg \max\left\{\left(\frac{n_b}{n}\frac{mc^2}{E}\right)^{1/4}, \left(\frac{n_b}{n}\right)^{1/6}\left(\frac{mc^2}{E}\right)^{1/2}\right\} \tag{3}$$

(n is the plasma concentration), then the instability can be regarded as kinetic.

Within the framework of the quasilinear approximation, the quasistationary state of the plasma + beam system is established because the excitation of the Langmuir waves over the scale of the relaxation is compensated for by their drift into the interior of the plasma. At the same time, in the case of two-stream instability there is also another possible mechanism whereby the quasistationary state can set in, viz., generation of Langmuir oscillations in that region of k-space where they are at resonance with the beam, i.e., at

$$\left| k_\parallel - \frac{\omega_p}{c} \right| \lesssim \frac{\omega_p}{c} \Delta\theta^2 + k_\perp \Delta\theta \qquad (4)$$

($k_\parallel$ and $k_\perp$ are respectively the longitudinal and transverse components of the beam relative to the wave vector), can be offset by their transfer to the "nonresonant" part of the spectrum as a result of the nonlinear processes. To this end it is necessary to satisfy the condition $\gamma_{NL} \gtrsim \gamma$, where γ_{NL} is the reciprocal time of the spectral redistribution and γ is the instability increment.

In the investigation of such a (nonlinear) relaxation regime, we confine ourselves for concreteness to the case of an almost-isothermal plasma ($T_i \sim T_e$). Then, as shown by simple estimates, the main mechanism of nonlinear interaction is the scattering of the Langmuir oscillations by the plasma ions. Excitation of the oscillations by the beam and the evolution of their spectrum as a result of the induced scattering are described by the equation[1)]

$$\partial W/\partial t = 2(\gamma + \gamma_i) W, \qquad (5)$$

where $W(\mathbf{k}, z, t)$ is the spectral density of the oscillation energy and γ_i is the scattering frequency; according to[7], we have

$$\gamma_i = \frac{3(2\pi)^{1/2}}{16} \frac{T_e/T_i}{(1+T_e/T_i)^2} \int d^3k' \frac{W(\mathbf{k}')}{nmv_{Ti}} \frac{(\mathbf{kk}')^2}{k^2k'^2} \frac{k'^2-k^2}{|\mathbf{k}-\mathbf{k}'|} \times \exp\left(-\frac{1}{2}\left(\frac{3}{2}\frac{v_{Te}^2}{\omega_p v_{Ti}} \frac{k'^2-k^2}{|\mathbf{k}-\mathbf{k}'|}\right)^2\right). \qquad (6)$$

The scattering process is due to Cerenkov interaction of the plasma ions with the beats, each of which is made up of two Langmuir oscillations (ω, $\mathbf{k}$; ω', $\mathbf{k}'$). In order for an appreciable fraction of the ions to participate in the scattering, the phase velocity of the beats should be smaller than the ion thermal velocity v_{Ti}:

$$(\omega - \omega') / |\mathbf{k} - \mathbf{k}'| \lesssim v_{Ti}. \qquad (7)$$

If we recognize now that the characteristic value of the wave vector of the oscillations generated by the beam is equal to ω_p/c, then we can estimate with the aid of (7) the decrease of the frequency $\Delta\omega \equiv \omega - \omega'$ and the modulus of the wave vector $\Delta k \equiv k - k'$ in one scattering act

$$\Delta\omega \sim \omega_p v_{Ti}/c, \qquad (8)$$

$$\Delta k \sim \omega_p v_{Ti}/v_{Te}^2. \qquad (9)$$

In a hot plasma ($T_e > (m/M)^{1/2}(T_i mc^2)^{1/2}$) the ratio $\Delta k/k$ is small: $\Delta k/k \ll 1$. Scattering therefore produces primarily isotropization of the oscillation spectrum. In addition, the oscillations are transferred into a region of low k ($\omega/k > c$), where they cease to interact with the beam (see Fig. 1). Although this mechanism does limit the energy level of the "resonant" ($\omega/k < c$) oscillations, it need not necessarily lead to an establishment of a stationary noise spectrum. Indeed, the stationary spectrum should satisfy the equation

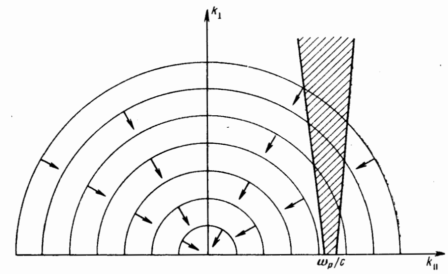

FIG. 1. Illustrating the theory of relaxation of an ultrarelativistic beam in a plasma. The wave-vector region in which there is interaction between the oscillations and the beam is shown shaded. The concentric circles represent the lines along which isotropization of the oscillation spectrum takes place. The arrows indicate the direction of the spectral transfer.

$$W(\gamma_i + \gamma) = 0.$$

This is an integral equation of the first kind and generally speaking has no regular solutions. Thus, the spectral transfer can cancel out the generation of the oscillations only in the mean, and the oscillation spectrum should in general be pulsating. To find the time-averaged W(k) dependence, we use the following reasoning.

We separate in **k**-space a spherical layer

$$k_0 - \Delta k < k < k_0 + \Delta k.$$

Since the change of the wave vector in one scattering act is equal to Δk, the oscillations inside the layer interact primarily with one another, and much more weakly with all the other oscillations. We note further that if $W(k_0)$ is sufficiently small, so that $-\gamma_i(k_0) < \gamma(k_0)$, then the energy density of the oscillations inside the layer will be increased by the two-stream instability. On the other hand, if $-\gamma_i(k_0) > \gamma(k_0)$, then $W(k_0)$ is decreased as a result of the transfer of the oscillation energy into the long-wave region. Therefore, even in the absence of a truly stationary solution, the following condition should be satisfied for each k_0 on the average (with respect to the time)

$$\overline{\gamma W + \gamma_i W} = 0 \qquad (10)$$

(the superior bar denotes averaging with respect to the time). It is impossible to find a formal solution of Eq. (10), but simple estimates enable us to find the form of the averaged function W.

In the absence of a true stationary solution, the estimate of γ_i depends essentially on the ratio of Δk to the width of the region of instability with respect to k, we shall denote by δk:

$$\gamma_i \sim \omega_p \frac{W}{nT_i}\left(1+\frac{T_e}{T_i}\right)^{-2} k^2 \begin{cases} \Delta k, & \Delta k > \delta k \\ \Delta k^2/\delta k, & \Delta k < \delta k \end{cases}. \qquad (11)$$

The quantity δk, in turn, is given by

$$\delta k = \begin{cases} \dfrac{k^2 \Delta\theta c}{\omega_p}, & \dfrac{\omega_p}{c} < k < \dfrac{\omega_p}{c\Delta\theta} \\ k, & k > \dfrac{\omega_p}{c\Delta\theta} \end{cases}. \qquad (12)$$

1)We neglect the drift of the oscillations, since their group velocity is very low.

Using the estimate of the increment of the two-stream instability

$$\gamma \sim \omega_p \frac{n_b}{n} \frac{mc^2}{E} \frac{\omega_p^2}{k^2c^2} \frac{1}{\Delta\theta^2}$$

and relation (10), we obtain[2)] W(k):

$$W \sim \frac{n_b}{n} \frac{mc^2}{E} \left(1+\frac{T_e}{T_i}\right)^2 nT_i \frac{\omega_p^2}{k^2c^2} \quad (13)$$

$$\times \begin{cases} \frac{1}{\Delta\theta^2} \frac{1}{k^2\Delta k}, & \frac{\omega_p}{c} < k < \left(\frac{\Delta k \omega_p}{c\Delta\theta}\right)^{1/2} \\ \frac{1}{\Delta\theta} \frac{c}{\Delta k^2 \omega_p}, & \left(\frac{\Delta k \omega_p}{c\Delta\theta}\right)^{1/2} < k < \frac{\omega_p}{c\Delta\theta} \\ \frac{1}{\Delta\theta^2} \frac{1}{k\Delta k^2} & k > \frac{\omega_p}{c\Delta\theta} \end{cases}$$

for $\Delta\theta < c\Delta k/\omega_p$, and

$$W \sim \frac{n_b}{n} \frac{mc^2}{E} \left(1+\frac{T_e}{T_i}\right)^2 nT_i \frac{\omega_p^2}{k^2c^2} \quad (14)$$

$$\times \begin{cases} \frac{1}{\Delta\theta} \frac{c}{\Delta k^2 \omega_p}, & \left(\frac{\Delta k \omega_p}{c\Delta\theta}\right)^{1/2} < k < \frac{\omega_p}{c\Delta\theta} \\ \frac{1}{\Delta\theta^2} \frac{1}{k\Delta k^2}, & k > \frac{\omega_p}{c\Delta\theta} \end{cases}$$

for $\Delta\theta > c\Delta k/\omega_p$. In the region $k < \omega_p/c$, the function W(k) is determined from the condition that the energy flux over the spectrum be constant:

$$W \sim \frac{n_b}{n} \frac{mc^2}{E} nT_i \left(1+\frac{T_e}{T_i}\right)^2 .$$

$$\times \begin{cases} \frac{1}{k^2\Delta k} \frac{1}{\Delta\theta^2}, & \Delta\theta < \frac{\Delta kc}{\omega_p} \\ \frac{1}{k^2\Delta k^2} \frac{\omega_p}{c} \frac{1}{\Delta\theta}, & \Delta\theta > \frac{\Delta kc}{\omega_p} \end{cases} \quad (15)$$

A spectrum of this type indicates that the Langmuir oscillations will accumulate in the region of small k. The problem of their dissipation will be considered later on.

We obtain now the spatial dependence of the angular spread $\Delta\theta(z)$ of the beam and of the average electron energy E(z). This can be done with the aid of the quasilinear equation

$$c\cos\theta \frac{\partial f}{\partial z} = \frac{1}{p^2} \frac{\partial}{\partial p} p^2 \left(D_{pp} \frac{\partial f}{\partial p} + \frac{D_{p\theta}}{p} \frac{\partial f}{\partial \theta}\right) + \frac{1}{p\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \left(D_{p\theta} \frac{\partial f}{\partial p} + \frac{D_{\theta\theta}}{p} \frac{\partial f}{\partial\theta}\right). \quad (16)$$

Knowing the spectrum of the oscillations interacting with the beam we can easily estimate the components of the diffusion tensor (see[4]):

$$D_{pp} \sim D_{p\theta} \sim D_{\theta\theta} \sim \omega_p \frac{n_b}{n} \left(\frac{E}{c}\right)^2 \left(\frac{mc^2}{E}\right)^3 .$$

$$\times \left(1+\frac{T_e}{T_i}\right)^2 \begin{cases} \left(\frac{M}{m}\right)^{1/2} \left(\frac{T_i}{T_e}\right)^{1/2} \left(\frac{T_e}{mc^2}\right)^{3/2} \frac{1}{\Delta\theta^2}, & \Delta\theta < \frac{\Delta kc}{\omega_p}. \\ \frac{M}{m} \left(\frac{T_e}{mc^2}\right)^2 \frac{1}{\Delta\theta}, & \Delta\theta > \frac{\Delta kc}{\omega_p}. \end{cases} \quad (17)$$

From the meaning of the quantity $D_{\theta\theta}$ we have

$$\left(\frac{c}{E}\right)^2 D_{\theta\theta} \sim c \frac{d}{dz} \Delta\theta^2(z).$$

Hence

$$\Delta\theta(z) \sim \begin{cases} \left(\Delta\theta_0^4 + \frac{zv_{Ti}c}{lv_{Te}^2}\right)^{1/4}, & 0 < z < l\left(\frac{v_{Ti}c}{v_{Te}^2}\right)^3 \\ \left(\frac{z}{l}\right)^{1/3}, & l\left(\frac{v_{Ti}c}{v_{Te}^2}\right)^3 < z < l \end{cases} \quad (18)$$

$$l = \frac{c}{\omega_p} \frac{n}{n_b} \frac{m}{M} \left(\frac{E}{mc^2}\right)^3 \left(\frac{mc^2}{T_e}\right)^2 \left(1+\frac{T_e}{T_i}\right)^{-2}, \quad (19)$$

where $\Delta\theta_0$ is the angular scatter of the beam on entering the plasma.

Since all the elements of the diffusion tensor are of the same order of magnitude if the spectrum is isotropic, the relative change of the energy of the electron beam during the relaxation process is approximately equal to the change of the angular scatter:

$$\frac{\Delta E}{E_0} \sim \begin{cases} \left(\Delta\theta_0^4 + \frac{zv_{Ti}c}{lv_{Te}^2}\right)^{1/4} - \Delta\theta_0, & 0 < z < l\left(\frac{v_{Ti}c}{v_{Te}^2}\right)^3 \\ \left(\frac{z}{l}\right)^{1/3}, & l\left(\frac{v_{Ti}c}{v_{Te}^2}\right)^3 < z < l \end{cases} \quad (20)$$

As seen from the foregoing estimates, l is the beam deceleration length in the target. At a distance l from the plasma boundary the beam loses an energy on the order of E_0, and its angular scatter reaches a value $\Delta\theta \sim 1$.

In the constructed relaxation scheme, the energy lost by the beam is transferred to the long-wave part of the spectrum. We shall indicate below some of the mechanisms that limit the level of the long-wave oscillations. The question of which of them is the principal one should be solved with the concrete experimental conditions taken into account. We emphasize, however, that if the removal of the energy from the long-wave part of the spectrum is efficient enough, then the results pertaining to the relaxation of the beam do not depend on the mechanism whereby the long-wave oscillations are annihilated. The words "efficient enough" mean that the characteristic time of annihilation of the Langmuir oscillations does not exceed the time during which they are transferred from the region $k > \omega_p/c$ into the region $k \ll \omega_p/c$ as a result of scattering by the ions (the transfer time is of the order of $\omega_p^{-1}(nT/U)(k/\Delta k)^2$, where U is the energy density of the short-wave ($k > \omega_p/c$) oscillations)[3)].

The absorption of the long-wave oscillations may be due, in particular, to pair collisions (if $\nu > \omega_p(U/nT) \times (\Delta k/k)^2$). There is also another possibility, namely transformation of Langmuir oscillations with $k < \omega_p/c$ and oscillations with $k > \omega_p/c$. The electromagnetic radiation has in this case a frequency $\sim 2\omega_p$.

An estimate of the rate of this process (an expression for the probability is given, for example, in the book by Tsytovich[7]) shows that it can hinder the accumulation of the oscillations in the long-wave region under the condition

$$\frac{T_e}{mc^2} \left(1+\frac{T_e}{T_i}\right) > \left(\frac{m}{M}\right)^{1/2} .$$

[2)]In the calculation of the oscillation energy density $U = 4\pi\int Wk^2dk$ with the aid of (13) and (14), we obtain an integral that diverges logarithmically at large values of k. This integral is cut off at the upper limit because the increment of the two-stream instability is very small at large k, and the instability can be suppressed by weak dissipative processes such as Coulomb collisions, emergence of the oscillations to the outside of the relaxation region, etc.

[3)]We note that in those cases when the time of the experiment is shorter than the characteristic transfer time, the question of the accumulation of the oscillations in the long-wave region of the spectrum does not arise at all.

The electromagnetic waves can either leave the plasma, or (in the case of a dense plasma) be absorbed as a result of pair collisions. If the removal of the energy from the long-wave ($k < \omega_p/c$) region is not efficient enough, then the spectral transfer in the course of scattering of the Langmuir oscillations by the ions causes these oscillations to be accumulated in the region $k \lesssim \Delta k$, where a very high energy concentration is produced. One of the mechanisms limiting this effect may be the buildup of sound, a process considered by Vedenov and Rudakov[8]. One cannot exclude the possibility that the buildup of the sound can change into a strong turbulence[10]. Then, as shown by Zakharov[10], local concentration perturbations (caverns) in which the Langmuir oscillations are "locked" can be produced in the plasma. The caverns collapse after a finite time, and the oscillation energy is transferred to the electrons and ions of the plasma. We note that excitation of the low-frequency oscillations can strongly distort the picture of the beam relaxation, since the acoustic wave modulates the concentration of the plasma and thereby causes the Langmuir oscillations to be diffused over the spectrum.

In concluding this section, let us indicate the condition for the applicability of the presented description of the relaxation. Formulas (18)—(20) were obtained by us under the assumption that the collective processes responsible for the relaxation of the beam can be considered within the framework of the theory of weak turbuelnce. To this end it is necessary that the beat frequency $\Delta\omega \sim \omega_p v_{Ti}/c$ exceed the frequency of the scattering of the oscillations by the ions

$$\omega_p v_{Ti}/c > \gamma_i.$$

Using the expression for γ_i (see (11)—(14)), this criterion can be written in the form of a limitation on the beam and plasma parameters:

$$\frac{n_b}{n}\frac{mc^2}{E}\frac{1}{\Delta\theta^2} < \frac{v_{Ti}}{c}. \tag{21}$$

On the other hand, if the inequality (21) is not satisfied, then an important role is assumed, besides the scattering, also by nonlinear processes of lower orders and the investigation of the relaxation becomes much more complicated.

3. THE RELAXATION WAVE (FORMULATION OF PROBLEM AND QUALITATIVE TREATMENT)

In the investigation of the relaxation of the electron beam, we have assumed that the plasma parameters (concentration profile, temperature) are fixed. If the beam is used to heat the plasma, we can confine ourselves to this approximation only for a sufficiently small time interval, until the plasma parameters become noticeably altered by the action of the beam. To describe the entire heating process it is necessary to solve the self-consistent problem of beam relaxation and motion of the plasma heated by the beam. This is the problem we proceed to consider.

We confine ourselves to an investigation of the heating of a dense plasma target in which the electrons and the ions exchange energy rapidly with each other via Coulomb collisions ($T_i = T_e = T$). In addition we bear in mind that the heating takes place in a quasi-stationary regime, i.e., the characteristic time of establishment of the stationary solution in the problem of beam relaxation is much smaller than the time of variation of the plasma parameters[4)].

Under the foregoing assumptions, the plasma motion can be described by the following system of gas dynamic equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{Mn}\nabla nT, \tag{22}$$

$$\frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{v} = 0, \tag{23}$$

$$nMT\left(\frac{\partial s}{\partial t} + \mathbf{v}\,\mathrm{grad}\, s\right) = \operatorname{div}(\chi\,\mathrm{grad}\, T) + Q. \tag{24}$$

Here s is the entropy per unit mass of the plasma, Q is the energy released by the beam per unit volume and per unit time; the remaining symbols are standard[5)].

We shall investigate with the aid of (22)—(24) the behavior of a plasma occupying the half-space $z > 0$, into which a symmetrical electron beam of radius R is injected. At the initial instant, the plasma is assumed to be immobile, and its concentration n and the temperature T are homogeneous and equal to n_0 and T_0, respectively. The dynamics of plasma heating is determined entirely by the properties of the heat source Q, which enters in Eq. (24). Denoting by l the deceleration length of the beam in the homogeneous plasma, we can estimate the characteristic value of Q with the aid of the formula[6)]

$$Q \sim n_b Ec/l. \tag{25}$$

We recall now that the relaxation of the beam in an inhomogeneous plasma is much less effective than in a homogeneous one[2,4]. Yet inhomogeneity of the concentration (even if absent initially) must set in during the course of the heating. As shown in[2], the relaxation is completely terminated if the plasma concentration gradient in the direction of the beam injection exceeds a certain critical value, which we denote here by $(\partial n/\partial z)_{max}$. From this we can estimate how much the concentration must drop over a scale l in order for the inhomogeneity to "turn off" the heating:

$$\left(\frac{\Delta n}{n}\right)_{max} \sim \frac{l}{n}\left(\frac{\partial n}{\partial z}\right)_{max}. \tag{26}$$

The role of the inhomogeneity becomes particularly pronounced if

$$(\Delta n/n)_{max} \ll 1. \tag{27}$$

We shall henceforth assume this inequality to be satisfied.

We consider first the case when the thermal conductivity of the plasma is low and the heating is produced with a broad ($R \gg l$) beam. It will then become easy to understand how the thermal conductivity of the plasma and the radial limits of the beam influence the result.

4)This should be the situation, in particular, in the experiments proposed in[11].

5)Generally speaking, it is necessary to include in (22) the momentum delivered by the beam to the plasma, but estimates show it to be negligibly small.

6)Depending on the concrete conditions, l is given by different relations. Thus, for example in the case of a nonlinear relaxation mode (see Sec. 2), formula (19) is valid.

Under the assumptions made, the dynamics of the heating is described by the linearized (with respect to the small parameter $(\Delta n/n)_{max}$) system of equations (22)–(24), in which all the quantities depend only on z and t:

$$\frac{\partial v_z}{\partial t} = -\frac{1}{M}\frac{\partial T}{\partial z}, \qquad (28)$$

$$\frac{\partial n}{\partial t} + n_0\frac{\partial v_z}{\partial z} = 0, \qquad (29)$$

$$n_0 M c_v \frac{\partial T}{\partial t} = Q \qquad (30)$$

(c_V is the specific heat per unit mass of the plasma at constant volume). To solve this system exactly we would need to know the exact form of the function A. Although we have only its "gross" characteristics, these suffice to obtain a qualitative description of the process. The physical picture of the heating is the following:

The beam, which is turned on at the instant t = 0, begins to heat the plasma inside a layer $0 < z < l$. Owing to the inhomogeneity of the heat release, the plasma temperature in the layer becomes inhomogeneous. The characteristic value of the temperature gradient can be easily estimated with the aid of (30):

$$\left|\frac{\partial T}{\partial z}\right| \sim \frac{tQ}{n_0 M c_v l} \sim \frac{n_b E c}{n_0 M c_v l^2} t.$$

Under the influence of the pressure gradient ($\partial P/\partial z = n_0 \partial T/\partial z$) the plasma is set in motion and its concentration also becomes inhomogeneous:

$$\left|\frac{\partial n}{\partial z}\right| \sim \frac{n_b E c}{M^2 c_v l^4} t^3.$$

After the lapse of a time

$$t \sim t_0 = \left(\frac{M^2 c_v}{n_b E c}\right)^{1/3} l^{4/3} \left[\left(\frac{\partial n}{\partial z}\right)_{max}\right]^{1/3}$$

the concentration differential inside the layer reaches the value $(\Delta n/n)_{max}$. Then the effectiveness of the heating in the layer actually drops to zero and the next section of the plasma

$$l < z < 2l$$

begins to heat up intensively. Inhomogeneity of the concentration sets in also in this section, and the region of where the heating power is maximal penetrates still farther into the initially homogeneous plasma.

We call such a phenomenon a relaxation wave. Behind the wave front, the inhomogeneity of the plasma concentration stops the relaxation of the beam almost completely. There is therefore no heat release here and the plasma temperature T is constant (see (30)):

$$T = T_0 + Mc^2\left(\frac{n_b E}{n_0 M c^2 M c_v}\right)^{2/3}\left[\frac{l}{n_0}\left(\frac{\partial n}{\partial z}\right)_{max}\right]^{1/3}, \qquad (31)$$

T_0 is the initial plasma temperature. Since the relaxation length l usually depends on the plasma temperature, the relation (31) determines the temperature behind the wave in implicit form. As seen from (31), the wave can be either strong ($T \gg T_0$) or weak ($T - T_0 \ll T_0$), depending on the beam and plasma parameters. The temperature behind the front of the strong wave is determined by the equation

$$T = Mc^2\left(\frac{n_b E}{n_0 M c^2 M c_v}\right)^{2/3}\left[\frac{l(T)}{n_0}\left(\frac{\partial n}{\partial z}\right)_{max}\right]^{1/3}. \qquad (32)$$

In a weak wave we have

$$\frac{T - T_0}{T_0} = \frac{Mc^2}{T_0}\left(\frac{n_b E}{n_0 M c^2 M c_v}\right)^{2/3}\left[\frac{l(T_0)}{n_0}\left(\frac{\partial n}{\partial z}\right)_{max}\right]^{1/3} \ll 1. \qquad (33)$$

We now obtain the wave propagation velocity u. To this end, we note that during the time $t \sim t_0$ the beam negotiates the distance $z \sim l$. It is convenient to express the wave velocity in terms of the speed of sound behind its front:

$$u = c_s\left(\frac{T - T_0}{T}\right)^{1/2}\left[\frac{l(T)}{n_0}\left(\frac{\partial n}{\partial z}\right)_{max}\right]^{-1/3}. \qquad (34)$$

We see from this that the propagation velocity of either a strong or a weak wave greatly exceeds the speed of sound under the conditions in question. Indeed, when linearizing (22), we assume $\Delta n/n \ll \Delta T/T$, but the ratio u/c is of the same order of magnitude as $n\Delta T/T\Delta n$ (see (34)), i.e., $u/c_s \gg 1$.

We consider now the influence of the thermal conductivity of the plasma on the character of the wave propagation. Owing to heat conduction, the energy released by the beam on entering the plasma penetrates into the interior in accordance with the law

$$z \sim (\chi t / n_0)^{1/2}.$$

If the temperature equalization is fast enough, then the temperature profile inside the layer $0 < z < l$, and consequently the concentration profile, remain homogeneous at all times. The described wave can obviously not occur in this case, and the mechanism whereby heat penetrates into the interior of the plasma is ordinary heat conduction. In other words, in order for the wave to arise it is necessary that the thermal conductivity of the plasma be low enough:

$$\chi \ll n_0 l^2 / t_0;$$

then the thermal conductivity will only smooth out somewhat the temperature profile behind the wave, without changing qualitatively the character of its propagation.

In concluding this section we note that the relaxation wave can exist also in the case when the radius of the beam is small in comparison with l ($R \ll l$). The entire picture of the phenomenon remains the same, except that the longitudinal inhomogeneity of the concentration results from radial motion of the plasma. We therefore confine ourselves only to an estimate of the propagation velocity of the strong wave, and of the temperature to which the plasma is heated behind the front:

$$u = \left(\frac{T}{M}\right)^{1/2}\frac{l}{R}\left[\frac{l(T)}{n_0}\left(\frac{\partial n}{\partial z}\right)_{max}\right]^{-1/2}, \qquad (35)$$

$$T = Mc^2\left(\frac{n_b E}{n_0 M c^2 M c_v}\right)^{2/3}\left(\frac{R}{l}\right)^{2/3}\left[\frac{l(T)}{n_0}\left(\frac{\partial n}{\partial z}\right)_{max}\right]^{1/3}. \qquad (36)$$

At the limit of its applicability (at $R \sim l$), these formulas, as expected, give velocity and temperature values corresponding to the case of an unbounded beam.

4. RELAXATION WAVE (QUANTITATIVE MODEL)

An attempt to construct a consistent quantitative description of the relaxation wave encounters the following difficulty: it is necessary to know first of all the detailed form of the function Q. It is therefore reasonable to obtain a model for the power released Q in accordance with those "gross" features of the beam relaxation process which are known to us, and then investigate such a model quantitatively. We consider here only one case, namely the weak relaxation wave. It is incidentally not difficult to apply a similar procedure to the strong wave.

We specify, to be specific, the concrete form of the radial distribution of the beam concentration on entering the plasma:

$$n_b(r) = \begin{cases} n_b^* \cos^2(\pi r/2R), & r < R \\ 0, & r > R \end{cases}$$

To abbreviate the notation, it is convenient to use in the solution of the system (22)—(24) dimensionless quantities introduced in the following manner:

$$\mathbf{r} \to l(T_0)\mathbf{r},$$

$$t \to \frac{l(T_0)}{2c}\left(\frac{n_b}{n_0 M c_v}\right)^{1/3}\left(\frac{Mc^2}{E}\right)^{1/3}\left[\frac{l(T_0)}{n_0}\left(\frac{\partial n}{\partial z}\right)_{max}\right]^{1/3} t,$$

$$\mathbf{v} \to 2c\left(\frac{n_b}{n_0 M c_v}\right)^{1/3}\left(\frac{E}{Mc^2}\right)^{1/3}\left[\frac{l(T_0)}{n_0}\left(\frac{\partial n}{\partial z}\right)_{max}\right]^{2/3} \mathbf{v},$$

$$T - T_0 \to 4E\left(\frac{n_b}{n_0 M c_v}\right)^{2/3}\left(\frac{Mc^2}{E}\right)^{1/3}\left[\frac{l(T_0)}{n_0}\left(\frac{\partial n}{\partial z}\right)_{max}\right]^{1/3} T,$$

$$Q \to 8\frac{n_b E c}{l(T_0)} Q, \qquad \frac{n - n_0}{n} \to \left[\frac{l(T_0)}{n_0}\left(\frac{\partial n}{\partial z}\right)_{max}\right] n,$$

$$\chi \to 2l(T_0) n_b c\left(\frac{n_b}{n_0 M c_v}\right)^{-2/3}\left(\frac{E}{Mc^2}\right)^{1/3}\left[\frac{l(T_0)}{n_0}\left(\frac{\partial n}{\partial z}\right)_{max}\right]^{-1/3} \chi.$$

The new variables **r**, t, **v**, Q, T, n, and χ, on the right hand sides of the equations, are chosen on the basis of the qualitative description such that all the main parameters for the propagating wave (the width of the front, the propagation, velocity, and the perturbations of the concentration and of the temperature) are of the order of unity. In terms of the new notation, the linearized system of equations (22)—(24) takes the form

$$\partial \mathbf{v}/\partial t = -\nabla T, \tag{37}$$

$$\partial n/\partial t + \operatorname{div} \mathbf{v} = 0, \tag{38}$$

$$\partial T/\partial t = \chi \Delta T + Q. \tag{39}$$

We now specify the heat release power Q(r, t):

$$Q(r, z, t) = \frac{1}{3}\cos^2 \pi\frac{r}{2R}\sum_{i=1,2,\ldots} \sin^4 \pi\frac{z - z_i}{z_{i+1} - z_i}\cdot$$

$$\cdot\, \theta(z - z_i)\theta(z_{i+1} - z)\theta\left(1 - \left|\frac{\partial n}{\partial z}\right|\right)\theta\left[1 - \int_0^z \theta\left(1 - \left|\frac{\partial n}{\partial z'}\right|\right)dz'\right],$$

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}, \tag{40}$$

where $z_i(r, t)$ are the roots of the equation

$$z\left(1 - \left|\frac{\partial n}{\partial z}\right|\right)\left(1 - \int_0^z \theta\left(1 - \left|\frac{\partial n}{\partial z'}\right|\right)dz'\right) = 0.$$

and are numbered in increasing order. All the multiple roots have the same subscript i. It is easy to verify that the function Q defined by the relation (40) reflects correctly the characteristic features of heat release in an inhomogeneous plasma. Thus, for example, at those points where $|\partial n/\partial z| > 1$, there is no heat release. This corresponds to the presence of a critical value of the plasma concentration gradient, at which the beam relaxation is interrupted. The fact that Q = 0 if

$$\int_0^z \theta\left(1 - \left|\frac{\partial n}{\partial z'}\right|\right) dz' > 1,$$

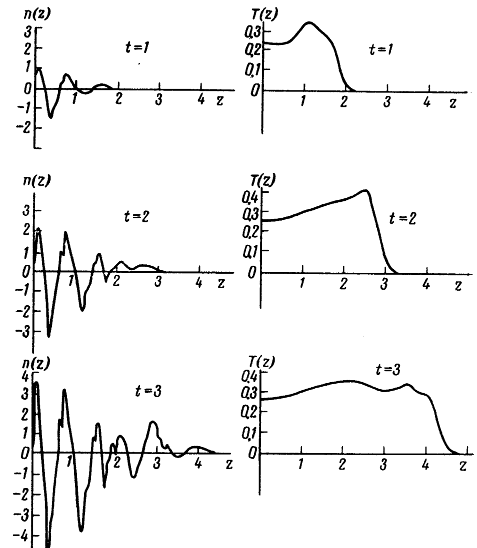

FIG. 2. Plasma temperature and concentration profiles in a one-dimensional relaxation wave. The dimensionless variables introduced in Sec. 4 are used, and χ = 1/10.

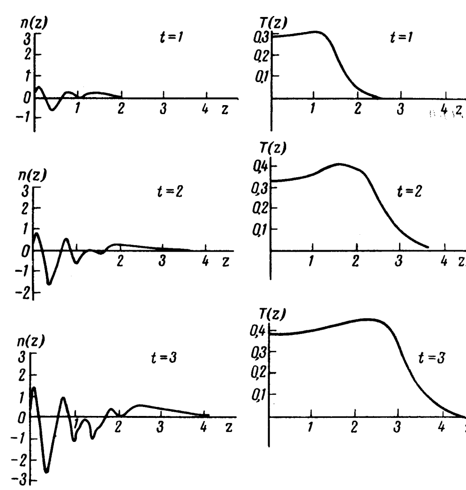

FIG. 3. Plasma temperature and concentration profiles in a one-dimensional relaxation wave, χ= 1/3.

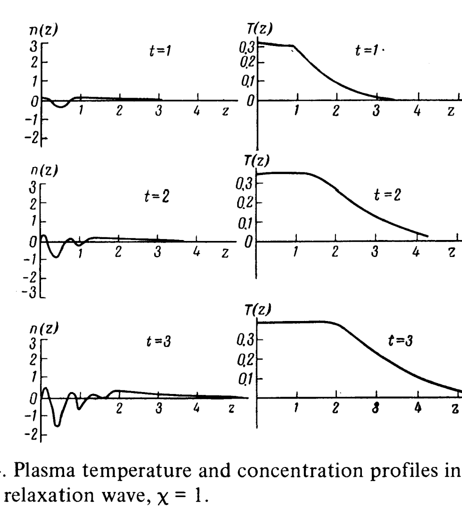

FIG. 4. Plasma temperature and concentration profiles in a one-dimensional relaxation wave, χ = 1.

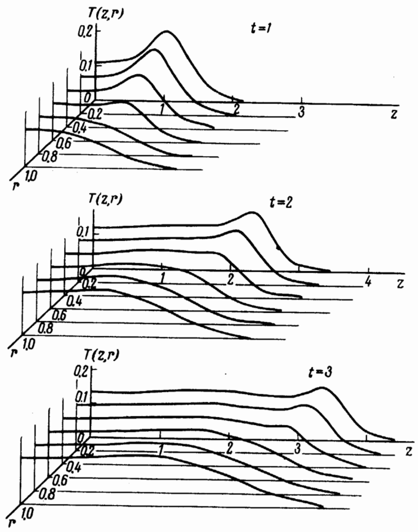

FIG. 5. Distribution of plasma temperature in relaxation wave. The beam radius R is chosen equal to unity, and $\chi = 1/3$.

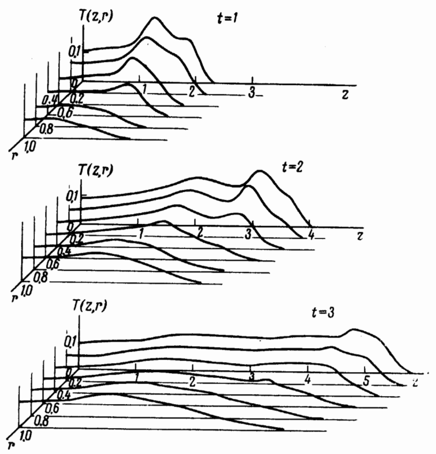

FIG. 6. Plasma temperature distribution in a relaxation wave, R = 1, $\chi = 1/10$.

also has a simple meaning: it means that the heating is most intense near the plasma boundary (of course, only in the case when the plasma is sufficiently homogeneous in this region). Finally, the factor $\cos^2(\pi r/2R)$, where R is the dimensionless radius of the beam, takes into account the radial inhomogeneity of the heat release.

We supplement the system (37)—(39) with boundary and initial conditions. We assume that the heat flux through the plasma boundary is equal to zero, i.e., $\partial T/\partial z = 0$ at $z = 0$. It is then seen from (37) that the particle flux also vanishes, $v_z = 0$, at $z = 0$. On the beam boundary ($r = R$) the heat flux is also assumed equal to zero: $\partial T/\partial r|_{r=R} = 0$. The initial conditions formulated above have the following form in the dimensionless variables: $n = 0$, $v = 0$, and $T = 0$ at $t = 0$.

In spite of the fact that the obtained system of equations is much simpler than the original one, it can be integrated only numerically. The calculations were performed both for the case of a one-dimensional model ($R \to \infty$) and with allowance for the radial limits of the plasma ($R \sim 1$). We chose here different values of the plasma heat conductivity in the interval $0 < \chi < 1$. The results of the calculations are shown in Figs. 2—6. Examination of these figures reveals the following regularities. At a low plasma temperature conductivity ($\chi = 1/10$, $\chi = 1/3$) the beam produces a relaxation wave with a steep temperature front[7)]. The temperature behind the front varies slowly (in space and in time), owing to the small amount of heat released near the extrema of the concentration profile. The concentration distribution behind the front has an oscillatory character, and the spatial scale of the oscillations decreases with time. The wave propagation velocity is of the order of unity. The temperature front becomes more and more smeared out with increasing temperature conductivity, and the wave velocity decreases.

All these conclusions are in full agreement with the result of the qualitative analysis.

[7)]It is easy to verify that the heating of a plasma of low heat conductivity must lead to a critical density gradient ($|\partial n/\partial z| = 1$) by putting $\chi = 0$.

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