Nonlinear Damping of Potential Monochromatic Waves in an Inhomogeneous Plasma

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Nonlinear interaction between a potential monochromatic wave and an inhomogeneous plasma is considered. The resonance particle distribution function is found for the case of weak inhomogeneity. An asymptotic expression is derived for the wave damping constant.

As is well known, collisionless mechanisms of damping and amplification of plasma waves are determined by particles whose interaction with the wave field has a resonant character. In the case, for example, of the Cerenkov mechanism, the resonant interaction takes place for those plasma particles whose velocity is sufficiently close to the phase velocity of the wave. In an inhomogeneous plasma, the phase velocity changes and this leads to new effects (in comparison with a homogeneous plasma), which are particularly important in the nonlinear stage of the process of resonant interaction. Several such effects were considered recently in [1, 2] from the point of view of the quasilinear theory. The latter, as is well known, is based on averaging over the phases for sufficiently broad wave packets, where this averaging is valid.[3, 4]

In the present paper we consider the nonlinear theory of resonant interaction of monochromatic waves with particles of an inhomogeneous plasma. The averaging method used in the quasilinear theory is not applicable to this case, and it is necessary to start with other methods developed in [5-8].

It should be noted that certain possible nonlinear effects in the propagation of monochromatic waves in an inhomogeneous plasma have already been discussed in a number of papers (see, for example, [9-12]). Thus it was indicated in [9] that particles whose velocity is close to that of the wave can be accelerated in an inhomogeneous plasma. This effect was then considered by a numerical simulation method.[9, 11] An expression for the adiabatic invariant of the particles "trapped" by the waves was obtained in [10]. Interesting results of numerical integration of the equations of motion of particles moving in the field of a monochromatic wave were recently obtained by Nunn.[12]

We shall use analytic asymptotic methods. The main result of the present paper is an analysis of the evolution of the distribution function in the field of the wave and the calculation of the damping coefficient (or growth increment) of a monochromatic longitudinal wave in an inhomogeneous plasma without a magnetic field. Inasmuch as this coefficient is determined by the work performed by the field of the wave on the resonant particles, it characterizes simultaneously the acceleration or deceleration of the resonant particles under the action of the field of a monochromatic wave propagating in an inhomogeneous plasma.

2. DISTRIBUTION FUNCTION OF RESONANT PARTICLES

Neglecting for simplicity the motion of the ions, we can write the fundamental equations in the form

$$\frac{\partial f}{\partial t} + \dot{x}\frac{\partial f}{\partial x} - \frac{e\mathcal{E}(x,t)}{m}\frac{\partial f}{\partial \dot{x}} = 0, \tag{2.1}$$

$$\mathcal{E} = \mathcal{E}_0 \sin\left[\int_0^x k(z)dz - \omega t\right], \tag{2.2}$$

where the wave frequency ω (assumed constant) is connected with the local wave number $k(x)$ by the same dispersion relation as in a homogeneous plasma; the dependence of k on x is due to the variation of the average plasma parameters with the coordinate x.

It follows from (2.2), in particular, that the "phase acceleration" of the wave, i.e., the acceleration of the point corresponding to the constant phase, takes the form

$$a = -\frac{d}{dt}\left[\frac{\omega}{k(x)}\right] = \frac{d}{dx}\left[\frac{\omega}{k(x)}\right]\frac{\omega}{k(x)} = -\frac{\omega^2}{k^3(x)}\frac{dk}{dx}. \tag{2.3}$$

Introducing the characteristic time that determines the phase acceleration

$$T(x) = (|ak|)^{-1/2} = \left(\left|\frac{\omega^2}{k^2}\frac{dk}{dx}\right|\right)^{-1/2}, \tag{2.4}$$

we shall see subsequently that the influence of the inhomogeneity of the plasma on the interaction of the wave with the resonant particles, (i.e., those having a velocity close to the phase velocity of the wave) will be determined by the ratio of this time T to the characteristic period τ of the oscillations of the particles trapped in the potential well of the wave field. The latter quantity is determined as follows:

$$\tau = (e\mathcal{E}_0 k / m)^{-1/2}. \tag{2.5}$$

We shall henceforth assume that the inhomogeneity of the plasma is weak, so that the dimensionless parameter

$$\varepsilon = -\frac{dk/dx}{|dk/dx|}\frac{\tau^2}{T^2} \tag{2.6}$$

is sufficiently small; we shall neglect terms of higher order of smallness in ϵ.

We now change over in (2.1) and (2.2) from the variables x and $\dot{x}$ to the new independent variables ξ and $\dot{\xi}$, in accordance with the relations

$$\int_0^x k(z)dz - \omega t = 2\xi, \quad k(x)\dot{x} - \omega = 2\dot{\xi}. \tag{2.7}$$

We then obtain, taking into account the notation in (2.4)–(2.6)

$$\frac{\partial f}{\partial t} + \dot{\xi}\frac{\partial f}{\partial \xi} - \frac{1}{2\tau^2}\left[\sin 2\xi + \varepsilon\left(1 + \frac{2\xi}{\omega}\right)^2\right]\frac{\partial f}{\partial \dot{\xi}} = 0. \qquad (2.8)$$

In a homogeneous plasma, where $dk/dx = 0$ and $\varepsilon = 0$, the variables ξ and $\dot{\xi}$ coincide, apart from constant factors, with the coordinate and velocity of the particle in the reference frame where the wave is at rest.

We shall be interested henceforth in the motion, in the wave field, only of resonant particles whose velocity differs from the phase velocity ω/k by an amount equal to the resonant velocity

$$v_r = 1/k\tau \equiv (e\mathscr{E}_0/mk)^{1/2} \qquad (2.9)$$

(v_r is of the order of magnitude of the characteristic velocity of the particle trapped in one of the potential wells of the wave). We shall thus be interested in particles for which $|\dot{x} - \omega/k| \lesssim v_r$, i.e.,

$$\dot{\xi} \lesssim kv_r = \tau^{-1}. \qquad (2.10)$$

Assuming, as was customarily done in all the papers cited above, that the wave amplitude is not too large, so that

$$\omega\tau \gg 1, \qquad (2.11)$$

we find from (2.10) that $\dot{\xi} \ll \omega$ for resonant particles, and consequently we can neglect the term $\dot{\xi}/\omega$ in comparison with unity in the kinetic equation for the resonant particles. As a result, the latter can be written in the form

$$\frac{\partial f}{\partial t} + \dot{\xi}\frac{\partial f}{\partial \xi} - \frac{1}{2\tau^2}(\sin 2\xi + \varepsilon)\frac{\partial f}{\partial \dot{\xi}} = 0. \qquad (2.12)$$

The coefficient preceding $\partial f/\partial \dot{\xi}$ in (2.12) has a simple meaning: it coincides, apart from a factor, with the sum of the force acting on the particle in the field of the wave (2.2) and the inertia force in the reference frame moving at the phase velocity of the wave.

To solve Eq. (2.12) it is necessary to change over to new independent variables ξ and λ, where the variable λ is connected with the "relative velocity" by the relation[1]

$$\dot{\xi} = \sqrt{1 - \lambda^2(\sin^2\xi + \varepsilon\xi)}/\tau\lambda. \qquad (2.13)$$

The kinetic equation (2.12) then takes the form

$$\left(\frac{\partial f}{\partial t}\right)_{\xi,\lambda} + \frac{1}{\tau\lambda}\sqrt{1 - \lambda^2(\sin^2\xi + \varepsilon\xi)}\left(\frac{\partial f}{\partial \xi}\right)_{t,\lambda} = 0. \qquad (2.14)$$

The characteristics of Eq. (2.14) are determined by the equations

$$\lambda\int_{\xi_0}^{\xi} dz[1 - \lambda^2(\sin^2 z + \varepsilon z)]^{-1/2} = \frac{t}{\tau}, \qquad (2.15)$$

where ξ_0 is the initial coordinate of the particle.

The distribution functions at any instant of time $f(\xi, \dot{\xi}, t)$ are determined from the initial distribution $f_0(\xi, \dot{\xi}) = f(\xi, \dot{\xi}, 0)$ in the following manner:

$$f(\xi, \dot{\xi}, t) = f_0[\xi_0(\xi, \dot{\xi}, t), \dot{\xi}_0(\xi, \dot{\xi}, t)],$$

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[1] The quantity λ^2 is inversely proportional to the total energy of the particle in the summary field of the wave and the inertia forces (in a system moving with phase velocity ω/k relative to the plasma), namely, $\tau^2\lambda^2 = [\xi^2 + (\sin^2\xi + \varepsilon\xi)\tau^{-2}]^{-1}$. A plot of the "potential" energy $(\sin^2\xi + \varepsilon\xi)/\tau^2$ is shown in Fig. 1.
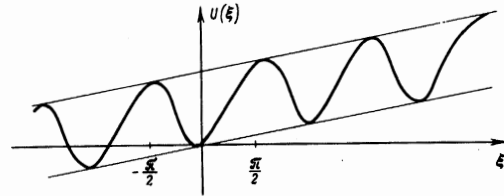

FIG. 1. Effective potential energy as a function of the coordinate ξ.

where $\xi_0(\xi, \dot{\xi}, t)$ and $\dot{\xi}_0(\xi, \dot{\xi}, t)$ are the initial coordinate and velocity of the particle, expressed with the aid of relations (2.13) and (2.15) as functions of the coordinates and velocities at the instant of time t.

This can be done approximately in the following manner. We assume that at the initial instant of time the distribution function in the system where the plasma is at rest does not depend on x, i.e., $f_0 = f_0(\dot{x}_0) = f_0(2\dot{\xi}_0/k + \omega/k)$. Expanding f_0 in powers of $2\dot{\xi}_0/k$ about the point ω/k, we write

$$f_0(\dot{x}_0) = f_0\left(\frac{\omega}{k}\right) + 2f_0'\left(\frac{\omega}{k}\right)\left(\frac{\dot{\xi}}{k}\right)_0 + \dots \qquad (2.16)$$

We neglect the remaining terms, taking into account the smallness of the quantity $v_r\omega/kv_T^2$. The problem now reduces to a determination of the function $\dot{\xi}_0(\xi, \dot{\xi}, t)$ from Eq. (2.15). To this end, we turn first to the "potential energy" plot corresponding to the equation of motion (2.13) (see Fig. 1, where it is assumed that $\varepsilon > 0$).

We assume, for concreteness, that at a given instant of time t the coordinate of the particle lies in the interval $-\pi/2 < \xi < \pi/2$. Particles with such coordinates can belong to one of two classes:

A. Transiting or untrapped particles, having one turning point. For such particles, obviously, the parameter λ should lie in the interval

$$\lambda^2 < 1 + |\varepsilon|\pi/2 \qquad (2.17)$$

(here and below the limits of the intervals are defined approximately, with allowance for the fact that $|\varepsilon| \ll 1$).

It is easy to verify that in order for the reflection point to lie in the interval $(2n - 1)\pi/2 < \xi < (2n + 1)\pi/2$, it is necessary that λ^2 lie between the limits

$$\frac{1}{1 + (2n+1)\pi|\varepsilon|/2} < \lambda^2 < \frac{1}{1 + (2n-1)|\varepsilon|\pi/2}. \qquad (2.18)$$

B. Trapped particles with two turning points. In this case

$$(1 + |\varepsilon|\pi/2) < \lambda^2 < \infty. \qquad (2.19)$$

We proceed now to solve Eq. (2.15) with respect to ξ_0 for each case. Introducing the notation

$$\Phi(x; \mu, \varepsilon) = \int_0^x \frac{dz}{\sqrt{\mu - (\sin^2 z + \varepsilon z)}}, \qquad (2.20)$$

we can rewrite (2.15) in the form

$$\Phi(\xi; 1/\lambda^2, \varepsilon) - \Phi(\xi_0; 1/\lambda^2, \varepsilon) = t/\tau. \qquad (2.21)$$

Since we assume that $|\xi| \lesssim \pi/2$ and $\varepsilon \ll 1$, we can put in the first of these terms, with sufficient accuracy, $\varepsilon = 0$, so that

$$\Phi(\xi; \, 1/\lambda^2, \, \varepsilon) \approx \Phi(\xi; \, 1/\lambda^2, \, 0) = \lambda F(\xi, \, \lambda), \qquad (2.22)$$

where $F(\xi, \lambda)$ is an incomplete elliptic integral of the first kind with modulus λ. We can proceed in similar fashion in the calculation of the second term in (2.21) for trapped particles, for in this case the quantity ξ_0 is also small, $|\xi| \lesssim \pi/2$.

Thus, the solution of (2.15) for the trapped particles differs insignificantly from the corresponding solution at $\epsilon = 0$, i.e., we can write

$$F(\xi, \lambda) - F(\xi_0, \lambda) = t/\lambda\tau \quad (\lambda^2 > 1 + |\epsilon|\pi/2), \qquad (2.23a)$$

$$\xi_0 = \mathrm{am}\left\{F(\xi, \lambda) - \frac{t}{\lambda\tau}, \, \lambda\right\}, \qquad (2.23b)$$

where $x = \mathrm{am}\{z, \lambda\}$ is a function inverse to $z = F(x, \lambda)$. Substituting (2.23b) in (2.13) we obtain

$$(\dot\xi)_0 \approx \frac{1}{\tau\lambda} \mathrm{dn}\left\{F(\xi, \lambda) - \frac{t}{\tau\lambda}, \, \lambda\right\}. \qquad (2.24)$$

If we use the general relation[13]

$$\mathrm{dn}\,(u, \lambda) = \mathrm{cn}\,(\lambda u, \, 1/\lambda), \qquad (2.25)$$

then (2.24) can be rewritten in another more customary form, which is convenient for expansion in a trigonometric series:

$$(\dot\xi)_0 = \frac{1}{\tau\lambda} \mathrm{cn}\left\{\lambda F(\xi, \lambda) - \frac{t}{\tau}, \, \frac{1}{\lambda}\right\}. \qquad (2.26)$$

We consider now the untrapped particles. If the time t is small compared with τ $(t \lesssim \tau)$, then the distance traversed by the resonant particle within the time interval $(0, t)$ is smaller than or of the order of the wavelength. Thus, in this case $\xi - \xi_0 \lesssim \pi$ and in the integral of (2.15) we can neglect terms of order ϵ, as was done for the trapped particles. We thus arrive again at formulas that have the same form as (2.23) and (2.24), but are now valid only for $t \lesssim \tau$.

When $t \gg \tau$ we can assume as before, for concreteness, that $-\pi/2 < \xi < \pi/2$, so that we can write in (2.21)

$$\Phi(\xi; \, 1/\lambda^2, \varepsilon) = \lambda F(\xi, \lambda) + O(\varepsilon). \qquad (2.27)$$

As to the second term, the difference between the function Φ and the elliptic integral becomes significant here, for the initial coordinate ξ_0 is large when $t \gg \tau$ and ξ is small. An analysis presented in the Appendix leads to the following asymptotic expression for the untrapped particles:

$$\Phi(\xi_0; 1/\lambda^2, \varepsilon) = F(\xi_0, \lambda)\tilde\lambda + \Psi(\lambda, \tilde\lambda, \varepsilon), \qquad (2.28)$$

where the "reduced modulus" $\tilde\lambda$ is given by

$$1/\tilde\lambda^2 = 1/\lambda^2 - \varepsilon\xi_0, \qquad (2.29)$$

$$\Psi = \frac{2}{\pi\varepsilon} K(\tilde\lambda)\tilde\lambda \left(\frac{1}{\tilde\lambda^2} - \frac{1}{\lambda^2}\right) - \frac{4}{\varepsilon\pi}\left[\frac{E(\tilde\lambda)}{\tilde\lambda} - \frac{E(\lambda)}{\lambda}\right]. \qquad (2.30)$$

($K(\tilde\lambda)$ and $E(\tilde\lambda)$ are complete elliptic integrals of the first and second kinds, respectively). Substituting (2.27) and (2.28) in (2.21) we obtain

$$\lambda F(\xi, \lambda) - \tilde\lambda F(\xi_0, \tilde\lambda) = t/\tau + \Psi(\lambda, \tilde\lambda, \varepsilon). \qquad (2.31)$$

Solving Eq. (2.31) formally with respect to ξ_0, we get

$$\xi_0 = \mathrm{am}\left\{\frac{\lambda}{\tilde\lambda} F(\xi, \lambda) - \frac{t}{\tilde\lambda\tau} - \frac{\Psi}{\tilde\lambda}, \, \tilde\lambda\right\}. \qquad (2.32)$$

Let now $|\tilde\lambda| < 1$. We can then write the well-known expansion (see, for example, [13])

$$\mathrm{am}(z, \tilde\lambda) = \frac{\pi z}{2K(\tilde\lambda)} - i\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n} \frac{q^n}{1 + q^{2n}} \exp\left[\frac{in\pi z}{K(\tilde\lambda)}\right],$$

$$q(\tilde\lambda) = \exp\left[-\frac{\pi K(\sqrt{1 - \tilde\lambda^2})}{K(\tilde\lambda)}\right]. \qquad (2.33)$$

With increasing t, the initial coordinate ξ_0 increases on the average linearly, and this average change is described by the first term of the expansion (2.33). We shall accordingly call this term the "principal" part of the initial coordinate and in the determination of $\tilde\lambda(\lambda, \xi_0)$ in accordance with (2.29) we shall replace ξ_0 by its principal part

$$1/\bar\lambda^2 = 1/\lambda^2 - \varepsilon\bar\xi_0, \qquad (2.34)$$

where

$$\bar\xi_0 = \frac{\pi}{2K(\tilde\lambda)\tilde\lambda}\left[\lambda F(\xi, \lambda) - \frac{t}{\tau} - \Psi\right]. \qquad (2.35)$$

Substituting the expression for Ψ from (2.30) into (2.35) and using (2.34), we obtain the following relation between λ and $\tilde\lambda$:

$$\frac{E(\tilde\lambda)}{\tilde\lambda} - \frac{E(\lambda)}{\lambda} = \frac{\varepsilon\pi t}{4\tau} - \frac{\varepsilon\pi\lambda}{4} F(\xi, \lambda). \qquad (2.36)$$

This equation determines the $\tilde\lambda(\xi, \lambda, t)$ dependence in implicit form. At sufficiently small t, namely

$$\varepsilon\frac{t}{\tau} = \frac{\tau}{T}\frac{t}{T} \ll 1, \qquad (2.37)$$

we can easily obtain from (2.36) an explicit expression for the function $\tilde\lambda(\xi, \lambda, t)$. Indeed, since

$$\bar\xi_0 \sim t/\tau, \qquad (2.38)$$

we see from (2.34) that under the condition (2.37) the difference between $\tilde\lambda$ and λ is small, so that we can write

$$\frac{E(\tilde\lambda)}{\tilde\lambda} - \frac{E(\lambda)}{\lambda} \approx \frac{\partial}{\partial\lambda}\left[\frac{E(\lambda)}{\lambda}\right](\tilde\lambda - \lambda) = -\frac{K(\lambda)}{\lambda^2}(\tilde\lambda - \lambda). \qquad (2.39)$$

Substituting (2.39) in (2.36) and neglecting the terms containing ε^2, we obtain

$$\tilde\lambda = \lambda + \frac{\varepsilon\pi\lambda^3}{K(\lambda)}\left[F(\xi, \lambda) - \frac{t}{\tau\lambda}\right]. \qquad (2.40)$$

We now proceed to an expression for the initial velocity $(\dot\xi)_0$. Putting in (2.13) $\xi = \xi_0$ and using (2.32) we get

$$(\dot\xi)_0 = \frac{1}{\tau\tilde\lambda} \mathrm{dn}\left\{\frac{\lambda}{\tilde\lambda} F(\xi, \lambda) - \frac{t}{\tau\tilde\lambda} - \frac{\Psi}{\tilde\lambda}, \, \tilde\lambda\right\}. \qquad (2.41)$$

Assuming here that $|\tilde\lambda| < 1$ and taking into account the expansion

$$\mathrm{dn}(z, \tilde\lambda) = \frac{\pi}{2K(\tilde\lambda)} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\pi}{2K(\tilde\lambda)} \frac{q^n}{1 + q^{2n}}$$

$$\times \exp\left[\frac{in\pi z}{K(\tilde\lambda)}\right], \qquad (2.42)$$

where $q(\tilde\lambda)$ have been determined in (2.33), we obtain for the average velocity $(\bar{\dot\xi})_0$ the expression

$$(\bar{\dot\xi})_0 = \pi/2K(\tilde\lambda)\tilde\lambda\tau \quad (|\tilde\lambda| < 1). \qquad (2.43)$$

Let us consider the limiting cases of formula (2.43). At $|\tilde{\lambda}| \ll 1$ we can write $(\dot{\xi})_0 \approx (\bar{\dot{\xi}})_0 \approx 1/\tilde{\lambda}\tau$ or, using formulas (2.40) and (2.13), $(\dot{\xi})_0 = \dot{\xi} + \epsilon\tau/2\tau^2$. Expressing ϵ in terms of the phase acceleration, in accordance with (2.6) and (2.3), we get

$$(\dot{\xi})_0 = \dot{\xi} + akt/2, \qquad (2.44)$$

which corresponds to motion of a free particle in a reference frame moving with acceleration a.

In the other limiting case $|\tilde{\lambda}| \to 1$ we obtain $(\bar{\dot{\xi}})_0 \to 0$. This case also has simple physical meaning. It corresponds to particles located at the initial instant of time on the envelope of the maxima of the potential shown in Fig. 1. (The equation of the latter is of the form $\epsilon\xi = 1/\lambda^2 - 1$.)

It is also seen from Fig. 1 that for the untrapped particles

$$\xi_0 < (1/\lambda^2 - 1)\epsilon^{-1}, \quad \epsilon > 0, \qquad (2.45)$$

$$\xi_0 > (1/\lambda^2 - 1)\epsilon^{-1}, \quad \epsilon < 0. \qquad (2.46)$$

Expressing here λ in terms of $\tilde{\lambda}$ with the aid of (2.34) we find that relations (2.45) and (2.46) are equivalent to the inequality

$$\tilde{\lambda}^2 < 1, \qquad (2.47)$$

which is thus valid for all the untrapped particles.

Let us stop further to discuss the signs of the quantities λ and $\tilde{\lambda}$. It is seen from (2.13) and (2.41) that it is convenient to define them in such a way that the signs of λ and $\tilde{\lambda}$ coincide with the signs of the final and initial velocities, respectively. We assume now, for concreteness, that $\epsilon > 0$, i.e., the plot of the "potential energy" has the form shown in Fig. 1. Then positive λ correspond to particles with positive $\tilde{\lambda}$ (they move in a positive direction all the time from the initial instant to the final one). As to the negative λ, they correspond to negative $\tilde{\lambda}$ only in the case when the particle arrives from the initial state at the final one without experiencing any reflection in the process. For particles that experience reflection, negative λ correspond to positive $\tilde{\lambda}$.

In order for a particle arriving at the point ξ at the instant t to have experienced a reflection at an earlier instant, it is necessary that it have a parameter λ in a certain interval

$$-1 < \lambda < \lambda_{cr}(\xi, t) < 0. \qquad (2.48)$$

On the other hand, if $\lambda = \lambda_{cr}(\xi, t) + \delta$, where δ is a small positive number, then the particle arrives at the point ξ at the instant t without reflection. Obviously, the corresponding value of $\tilde{\lambda}$ should in this case be sufficiently close to -1. The equation of the curve $\lambda = \lambda_{cr}(\xi, t)$ can be obtained approximately from (2.37). Neglecting terms of higher order of smallness than $\epsilon t/\tau$, we obtain from (2.36)

$$-\frac{E(\tilde{\lambda})}{\tilde{\lambda}} + \frac{E(\lambda)}{\lambda} + \frac{\epsilon\pi t}{4\tau} = 0. \qquad (2.49)$$

This equation defines the curve $\tilde{\lambda}(\lambda, t) = \text{const}$. Going here to the limit $\tilde{\lambda} = -1$, we obtain an equation for the limiting curve $\lambda = \lambda_{cr}(t)$ (discarding terms of order ϵ) in the form

$$1 + \frac{E(\lambda)}{\lambda} + \frac{\epsilon\pi t}{4\tau} = 0. \qquad (2.50)$$

It is further necessary to note that in the derivation of expressions (2.28) and (2.32), and consequently of (2.36), it was assumed that the particles arrive from the initial state at the final one without reflections (i.e., $\lambda > \lambda_{cr}$ and sign λ = sign λ). It is not particularly difficult to perform the corresponding analysis also for the region (2.48), with reflections taken into account. As a result we obtain for particles experiencing reflection the following relation in place of (2.36):

$$\frac{E(\tilde{\lambda})}{\tilde{\lambda}} - \frac{E(\lambda)}{\lambda} = 2 - \frac{\epsilon\pi\lambda}{4}F(\xi, \lambda) + \frac{\epsilon\pi t}{4\tau}$$
$$(\tilde{\lambda} > 0, \quad \lambda < 0). \qquad (2.51)$$

We proceed now to derive expressions for the distribution function. The latter are obtained by substituting (2.26) and (2.41) in (2.16). To write down the asymptotic expressions for the distribution functions that describe on the average the state established at $t \gg \tau$ (see [8,14]), we should substitute in place of the $(\dot{\xi})_0$ their mean values. Taking into account formula (2.43) for the untrapped particles and the relation $(\bar{\dot{\xi}})_0 = 0$, which holds for the trapped particles (this can easily be verified from (2.26)), we obtain

$$\overline{f_t(\xi, \lambda, t)} = f_0(\omega/k), \qquad (2.52)$$

$$\overline{f_{ut}(\xi, \lambda, t)} = f_0\left(\frac{\omega}{k}\right) + f_0'\left(\frac{\omega}{k}\right)\frac{\pi}{\tau k K(\tilde{\lambda})\tilde{\lambda}}. \qquad (2.53)$$

Here f_t is the distribution function of the trapped particles, and f_{ut} is the distribution function of the untrapped particles. It is necessary to bear in mind here that in formula (2.53) it is necessary to use the expression (2.40) for $\tilde{\lambda}$ at $\lambda > \lambda_{cr}$ and the expression (2.51) at $\lambda < \lambda_{cr}$.

2. CALCULATION OF THE DECREMENT

The decrement of the wave can be obtained from the formula

$$\gamma = -4\pi\mathscr{E}_0^{-2}\overline{(j\mathscr{E})} \qquad (3.1)$$

(the bar denotes averaging over the wavelength), where $\mathscr{E}(x, t)$ is the intensity of the wave field, defined by expression (2.2), and

$$j(x, t) = -e\int_{-\infty}^{\infty} \dot{x}f(x, \dot{x}, t)d\dot{x}. \qquad (3.2)$$

Substituting (3.2) and (2.2) in (3.1) and changing over to the variables ξ, λ, defined by formulas (2.7), we obtain (discarding small terms of order $v_r k/\omega$)

$$\gamma = \frac{2}{\pi}\omega_p^2\frac{\omega\tau}{k}\int\int d\xi d\lambda \frac{f(\xi, \lambda, t)\sin 2\xi}{\lambda^2\sqrt{1 - \lambda^2(\sin^2\xi + \epsilon\xi)}}. \qquad (3.3)$$

As is well known, the contribution made to the integral (3.3) by the time-oscillating terms in the expansion (2.42) vanishes when $t/\tau \gg 1$ (integration with respect to λ is equivalent to averaging over t). To determine the asymptotic expression for γ at $t \gg \tau$ it suffices therefore to calculate the contribution made to the decrement by the nonoscillating part of the distribution function represented by formulas (2.52) and (2.53). The expression for γ can then be reduced to the form

$$\gamma = \frac{4\gamma_L}{\pi}\iint_s d\xi d\varkappa \frac{\sin 2\xi}{\varkappa^2\sqrt{1 - \varkappa^2\sin^2\xi}}\frac{1}{\tilde{\lambda}K(\tilde{\lambda})}, \qquad (3.4)$$

where

$$\gamma_L = \frac{\pi}{2}\,\omega_p{}^2\,\frac{\omega}{k^2}\,f_0{}'\left(\frac{\omega}{k}\right) \qquad (3.5)$$

is the linear decrement of the Landau damping, and the new integration variable is expressed in terms of λ as follows:

$$\varkappa^{-2} = \lambda^{-2} - \varepsilon\xi. \qquad (3.6)$$

The region of integration S in (3.4) contains only those points $(\xi, \varkappa)$ which correspond to untrapped particles; they lie inside the curve ABCDEF on Fig. 2 (the contribution from the region outside this line, corresponding to the trapped particles, is equal to zero).

Expressing the quantity $\tilde{\lambda}$ in terms of $\varkappa$, ξ with the aid of relations (2.36), (2.51), and (3.6), we obtain the following equations for $\tilde{\lambda}(\xi, \varkappa, t)$ (accurate to terms of order ϵ inclusive):

$$\frac{E(\tilde{\lambda})}{\tilde{\lambda}} = \frac{E(\varkappa)}{\varkappa} + \frac{\varepsilon\pi t}{4\tau} + \varepsilon\left[\frac{\varkappa K(\varkappa)}{2}\,\xi - \frac{\pi\varkappa F(\xi, \varkappa)}{4}\right], \qquad (3.7)$$

$$\frac{E(\tilde{\lambda})}{\tilde{\lambda}} = \frac{E(\varkappa)}{\varkappa} + 2 + \frac{\varepsilon\pi t}{4\tau} + \varepsilon\left[\frac{\varkappa K(\varkappa)}{2}\,\xi - \frac{\pi\varkappa F(\xi, \varkappa)}{4}\right]. \qquad (3.8)$$

The first of these equations pertains to particles coming from the initial point of the phase plane to the final without experiencing reflection (for these we have sign $\tilde{\lambda}$ = sign $\varkappa$), and the second for particles experiencing reflection ($\tilde{\lambda} > 0$, $\varkappa < 0$).

The equation for the limiting curve on the $(\varkappa, \xi)$ plane separating the regions of the reflected particles and those experiencing no reflection can be obtained from (2.50) and (3.6). Discarding terms of order ϵ, we have for this curve

$$1 + \frac{E(\varkappa)}{\varkappa} + \frac{\varepsilon\pi t}{4\tau} = 0. \qquad (3.9)$$

(It is represented by the line ab in Fig. 2; the straight line cd is its mirror image on the region of positive $\varkappa$.)

Thus, in the region ABCba of Fig. 2 we should express $\tilde{\lambda}$ in terms of $\varkappa$ and ξ by using formula (3.8), and in the remaining part of the integration region we should use (3.7).

To eliminate $\tilde{\lambda}$ from the integral (3.4) with the aid of these equations, it is convenient to introduce a new function R(w), defined in parametric form as follows:

$$R = 1/\mu K(\mu), \quad w = E(\mu)/\mu. \qquad (3.10)$$

It is easy to verify that the function R(w) is odd, takes on real values in the interval $1 \le w < \infty$ (and complex ones at $-1 < w < 1$), and increases monotonically when w changes from unity to infinity. At $w - 1 \ll 1$
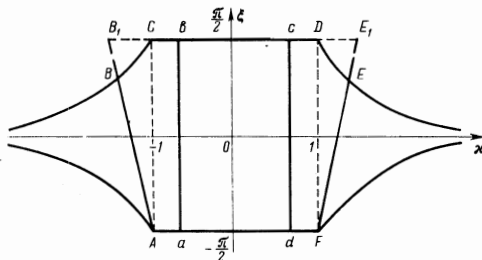

FIG. 2. Region of integration in (3.4). The curves BC and DE are described by the equation $\xi = \sin^{-1}(1/|\varkappa|)$.

and $w \gg 1$, we have the following asymptotic relations $(R(w) = -R(-w))$:

$$R(w) \approx \frac{2}{\ln 8 - \ln(w - 1)}, \quad w - 1 \ll 1; \qquad (3.11a)$$

$$R(w) \approx \frac{4}{\pi^2}\,w - \frac{\pi^2}{128w^3}, \quad w \gg 1. \qquad (3.11b)$$

Calculating now the integral in (3.4) accurate to terms of order ϵ, we obtain

$$\gamma = 2\pi\gamma_L|\varepsilon|\left\{\int_1^{1+g} dw\,[R(w - g - 2) - R(w + g)]\left[\frac{4}{\pi^2} - R'(w)\right]\right.$$
$$\left. + \int_{1+g}^{\infty} dw\,[R(w - g) - R(w + g)]\left[\frac{4}{\pi^2} - R'(\omega)\right]\right\}, \qquad (3.12)$$

where

$$R'(w) = dR/dw, \quad g(t) = |\varepsilon|\pi t/4\tau. \qquad (3.13)$$

Thus, in the approximation considered, the sign of the damping decrement (or of the growth increment) coincides with the sign of the corresponding decrement γ_L in the linear approximation, and is independent of the sign of ϵ, which characterizes the direction of the inhomogeneity.

At small values of g, i.e., at $t \ll 4\tau/\pi|\epsilon|$ (we must, however, have $t \gg \tau$, for in the derivation of (3.12) we used an averaged distribution function), we obtain from (3.12)

$$\gamma = \frac{40\pi}{3\ln^2 g}|\varepsilon|\gamma_L. \qquad (3.14)$$

When $g \sim 1$, i.e., at $t \sim 4\tau/\pi|\epsilon|$, the increment is of the order of

$$\gamma \sim |\varepsilon|\gamma_L. \qquad (3.15)$$

With further increase of t $(t \gg 4\tau/\pi|\epsilon|)$, formula (3.12) no longer holds, for at these values of t there arrive in the resonant region particles located at the initial instant sufficiently far from the resonant region, so that their distribution function can no longer be represented in the form of the expansion (2.16). We note in this connection that the quantity $t_0 = \tau/|\epsilon|$ is of the order of magnitude of the characteristic time of renovation of the resonant region, since the velocity changes within this time by an amount of the order of the resonant velocity (2.9); at small ϵ this time is, generally speaking, very large (it is certainly larger than T in (2.4)).

APPENDIX

ASYMPTOTIC EXPRESSION FOR THE FUNCTION $\Phi(x; \mu, \epsilon)$

We consider the function $\Phi(x; \mu, \epsilon)$, defined in (2.20) at $|x| \gg 1$ and $|\epsilon| < 1$. We assume, for concreteness, that $x > 0$ and $\epsilon > 0$. We put

$$x = N\pi + \varphi, \qquad (A.1)$$

where N is a large integer and $0 < \varphi < \pi$. Using the easily verified identity

$$\Phi(x; \mu, \varepsilon) = \Phi(x - \pi, \mu - \varepsilon\pi, \varepsilon) + \Phi(\pi, \mu, \varepsilon),$$

we get

$$\Phi(x;\mu,\varepsilon) \equiv \Phi(N\pi + \varphi;\mu,\varepsilon)$$

$$= \Phi(\varphi;\mu,-\varepsilon\pi N,\varepsilon) + \sum_{m=0}^{N-1}\Phi(\pi;\mu-\varepsilon m\pi,\varepsilon). \qquad (A.2)$$

(We shall henceforth assume that $\mu - \epsilon\pi N > 0$.)

Since $\varphi \lesssim \pi$, the first term in the right-hand side of (A.2) can be written in the form

$$\Phi(\varphi;\mu-\varepsilon\pi N,\varepsilon) = \Phi(\varphi;\mu-\varepsilon\pi N,0) + O(\varepsilon)$$

$$= \frac{1}{\sqrt{\mu-\varepsilon\pi N}}F[x-\pi N,(\mu-\varepsilon\pi N)^{-\frac12}] + O(\varepsilon). \qquad (A.3)$$

Using further the general relation (see, for example, [13])

$$F(x-\pi N, k) = -2K(k)N + F(x,k), \qquad (A.4)$$

we can write

$$\Phi(\varphi;\mu-\varepsilon\pi N,\varepsilon) = -2K\left(\frac{1}{\sqrt{\mu-\varepsilon\pi N}}\right)\frac{N}{\sqrt{\mu-\varepsilon\pi N}}$$

$$+ \frac{1}{\sqrt{\mu-\varepsilon\pi N}}F[x,(\mu-\varepsilon\pi N)^{-\frac12}]. \qquad (A.5)$$

We now proceed to calculate the sum in (A.2). Since it is assumed that $N \gg 1$, we can write the following asymptotic expression at $\epsilon \ll 1$:

$$\sum_{m=0}^{N-1}\Phi(\pi;\mu-\varepsilon\pi m,\varepsilon) \approx -\frac{1}{\varepsilon\pi}\int_{\mu}^{\mu-\varepsilon\pi N}\Phi(\pi;\sigma,\varepsilon)d\sigma. \qquad (A.6)$$

With the same accuracy with which we obtained the expression (A.6), we can put in it

$$\Phi(\pi;\sigma,\varepsilon) \approx \Phi(\pi;\sigma,0) = 2\sigma^{-\frac12}K(\sigma^{-\frac12}).$$

Recognizing that

$$\int \sigma^{-\frac12}K(\sigma^{-\frac12})d\sigma = 2\sigma^{\frac12}E(\sigma^{-\frac12}), \qquad (A.7)$$

we obtain

$$\int_{\mu}^{\mu-\varepsilon\pi N}\Phi(\pi;\sigma,\varepsilon) = 4\left\{(\mu-\varepsilon\pi N)^{\frac12}E\left(\frac{1}{\sqrt{\mu-\varepsilon\pi N}}\right) - \mu^{\frac12}E\left(\frac{1}{\sqrt{\mu}}\right)\right\}. \qquad (A.8)$$

Gathering together the formulas (A.2), (A.5), (A.6), and (A.8), we obtain

$$\Phi(x;\mu,\varepsilon) = -2K(\bar\mu^{-\frac12})N\bar\mu^{-\frac12} + \bar\mu^{-\frac12}F(x,\bar\mu^{-\frac12})$$

$$- (4/\varepsilon\pi)\{\bar\mu^{\frac12}E(\bar\mu^{-\frac12}) - \mu^{\frac12}E(\mu^{-\frac12})\}, \qquad (A.9)$$

where

$$\bar\mu = \mu - \varepsilon\pi N \approx \mu - \varepsilon x. \qquad (A.10)$$

It is easy to verify that the asymptotic formulas (A.9) and (A.10) are valid also at large negative x. Putting in (A.9) and (A.10)

$$\mu = 1/\lambda^2, \quad x = \xi_0, \qquad (A.11)$$

we obtain the expressions (2.28)–(2.30).

Note added in proof (21 January 1972). At arbitrary t, the increment is determined by the expression

$$\Gamma = \frac{8}{\pi}\omega_p^2\frac{\omega\tau}{k}\varepsilon\left[f_0\left(\frac{\omega}{k(x)}\right) - f_0\left(\frac{\omega}{k(x)} - at\right)\right] + \gamma - \frac{64g|\varepsilon|\gamma_L}{\pi^3},$$

where γ is determined by the formula (3.12) and a is the phase acceleration of the wave. At $t \lesssim \tau/|\varepsilon|$, this expression goes over into (3.12). When $t \gg \tau/|\varepsilon|$, the difference of the last two terms becomes equal to $64|\varepsilon|\gamma_L/\pi^3$.

In the case $\epsilon > 0$, the second term in the square brackets is equal, accurate to a numerical coefficient, to the rate of increase of the energy of the trapped particles dragged by the wave. The untrapped parts in this case are decelerated and give up energy to the wave. When $\epsilon < 0$ the picture is accordingly reversed. The result presented here is obtained by a simple generalization of the expansion (2.16) to the case $t > \epsilon/|\epsilon|$.

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