Theory of Interaction Between a Standing Wave Field and Gas

E. V. BAKLANOV and V. P. CHEBOTAEV

Siberian State Research Institute for Metrology; Institute of Semiconductors, Siberian Division, USSR Academy of Sciences

Submitted July 9, 1971

Zh. Eksp. Teor. Fiz. **62**, 541-550 (February, 1972)

The problem of interaction between the field of standing wave and a gas is solved. An analytic expression is obtained for polarization of the medium when the field strengths are such that $\gamma\chi/\Gamma \ll 1$ where γ is the level width, Γ the line width, and χ the saturation parameter. Since ordinarily $\gamma/\Gamma \ll 1$, the solution is valid for strong fields. For $\chi \ll 1$ the absorption coefficient equals that calculated by Lamb's theory with an accuracy to χ^2. Polarizability of the medium is represented by three terms. The first is identical to the solution obtained from the rate equations [4] in which coherence effects are neglected. The other two terms yield the contribution of coherence effects and are proportional to the parameter γ/Γ.

MANY questions in the theory of gas lasers require the solution of problems concerning the resonant interaction of moving atoms with fields of arbitrary intensity. It is well known that an exact analytic solution can be obtained only in the case of one traveling wave[1]. Even in the case of two waves the problem can be solved accurately only for the particular case of a standing wave with a frequency equal to the transition frequency, and with identical level relaxation constants[2]. For low field intensities, the method of successive approximations with respect to the saturation parameter χ was developed by Lamb in his well known paper[3]. To avoid difficulties connected with the exact solution of the equations for the density and matrix, approximate rate equations were used[4,5], in which coherent effects occurring when the atom interacts with the field and the spatial inhomogeneity of the medium were completely neglected. The problem of the interaction of the field of a standing wave of arbitrary intensity in a gas was considered by Stenholm and Lamb[6] and by Feldmann and Feld[7]. An analysis of their results calls for the use of computers. An approximate method of finding the dependence of the generation power on the frequency, as applied to the analysis of the experimental data, was developed by Holt[8].

A comparison of the aforementioned results[4-7] shows that particularly strong differences arise in the velocity distributions of the atoms. On the other hand, the shape of the Lamb dip does not differ qualitatively from the shape obtained when the rate equations are used. In spite of this, allowance for the coherent effects in problems of generation stability and radiation fluctuations is necessary. It is desirable to have a solution of the problem in analytic form.

In an earlier paper[9], in the analysis of the interaction of two opposing traveling waves, one of which is weak, we have shown that the relative contribution of the coherent effects and of the effects of population to the absorption of a weak signal is determined by the parameter γ/Γ, where Γ is the half-width of the line, $\gamma = 2\gamma_1\gamma_2/(\gamma_1 + \gamma_2)$, and γ_2 and γ_1 are the widths of the upper and lower levels[1)]. When several fields with frequencies $\omega_1, \omega_2, \omega_3, \ldots$, interact with an atom, combination frequencies $\omega_1 - \omega_2, \omega_1 - \omega_3, 2(\omega_1 - \omega_2), \ldots$ appear in the polarization. One can expect allowance for the polarization at these frequencies to lead to an additional contribution to the absorption, with respect to the parameter γ/Γ, in comparison with the absorption obtained from the rate equation. The bulk of the present paper is devoted to a determination of these corrections.

In strong fields, allowance for the coherent effects requires a determination of all the harmonics, something impossible to do in analytic form. In relatively weak fields, when $\gamma\chi/\Gamma \ll 1$, we can expect the contribution of the coherent effects to be determined only by the first harmonic of the combination frequencies. Rautian[4] consider the solution of stimulated emission of atoms moving in a field of a strong standing wave, within the framework of the transition probability and, under the conditions $\gamma_2/\gamma_1 \ll 1$ and $\gamma_2\chi/\gamma_1 \ll 1$; the coherent effects were disregarded. In this approximation, as expected, the solution of the problem is in essence the solution of the rate equations. The condition $\gamma\chi/\Gamma \ll 1$ can be rewritten in the form $(dE/\Gamma)^2 \ll 1$, which denotes that there are no oscillations for the transition probabilities. The contribution of the coherent effects is determined by the line width and turn out to be small, whereas the saturation effects can be appreciable, since they are determined to a greater degree by the long-lived level. For the resonant interaction of two opposing waves of equal frequency with a gas, it is more convenient to consider the spatial harmonics of the polarization of the medium. Obviously, taking the first harmonic into account is equivalent to taking into account the polarization of the atom at the first combination frequency in a coordinate system connected with the atom.

In the present paper we illustrate the general approach to the solution of the problem, using as an example the interaction of the standing-wave field with the gas. We note that the case of a standing wave is special. Unlike two opposing waves of unequal intensity, in a pure standing wave it is necessary to take into account all the spatial harmonics even if $\gamma\chi/\Gamma \ll 1$. This is due to the contribution of those atoms whose velocity projection is of the order of γ/k (k is the

1)In the case of waves traveling in the same direction, allowance for coherent effects gives qualitative differences even in the case when $\chi \ll 1$ [4,10].

wave vector)[2)]. The additional contribution of these atoms turns out to be of the order of γ/k and must be taken into account in our approximation, even when $\gamma\chi/\Gamma \ll 1$. However, as shown below, at low velocities the problem can be solved exactly, and at $\gamma\chi/\Gamma \ll 1$ the contribution of all the harmonics can be taken into account.

In the optical region, the relaxation constants are as a rule noticeably different. Therefore the parameter $\gamma/\Gamma \ll 1$, and consequently the condition $\gamma\chi/\Gamma \ll 1$ can be satisfied at large χ, which suffices for a rigorous analysis of the effects in most cases of known gas lasers. The presence of collisions, which lead to a loss of phase and do not change the lifetimes of the levels, also decreases the ratio γ/Γ.

1. PERTURBATION METHOD

To find the polarization of the medium we start from the equations for the elements $\rho_{21} = \rho_{22} = n_2$ and $\rho_{11} = n_1$ of the density matrix averaged over the moments of the excitations

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z} + \gamma_2\right) n_2 = idE(z)\,(\rho_{21}^*\,e^{-i\omega t} - \rho_{21}\,e^{i\omega t}) + \gamma_2 N_2(v),$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z} + \gamma_1\right) n_1 = -\,idE(z)\,(\rho_{21}^*\,e^{-i\omega t} - \rho_{21}\,e^{i\omega t}) + \gamma_1 N_1(v), \quad (1)$$

$$\left(\frac{\partial}{\partial t} + i\omega_0 + v\frac{\partial}{\partial z} + \Gamma\right) \rho_{21} = idE(z)\,e^{-i\omega t}\,n.$$

Here $E(z) = E(e^{ikz} + e^{-ikz})$ is the standing-wave field represented in the form of two opposing traveling waves of amplitude $2E$; ω is the field frequency ($k = \omega/c$), ω_0 is the transition frequency, γ_2 and γ_1 are the rates of decay of the upper and lower levels, Γ is the line width, d is the dipole matrix element of the transition in units of $\hbar$, and $n = n_1 - n_2$.

The quantities

$$N_2(v) = (N_2/\sqrt{\pi}v_0)\,\exp\,(-v^2/v_0^2),\quad N_1(v) = (N_1/\sqrt{\pi}v_0)\,\exp\,(-v^2/v_0^2)$$

are respectively the populations of the upper and lower levels in the absence of fields, v is the projection of the atom velocity on the z axis, and v_0 is the thermal velocity.

If we seek the solution of (1) in the form

$$n_1 = \sum_{m=-\infty}^{\infty} n_{1,m}\,e^{ikmz},\qquad n_2 = \sum_{m=-\infty}^{\infty} n_{2,m}\,e^{ikmz},$$

$$\rho_{21} = e^{-i\omega t}\sum_{m=-\infty}^{\infty} \rho_m\,e^{ikmz},$$

then we readily obtain

$$n_m = -2idE\Gamma^{-1}L_m\,(f_{m+1} + f_{m-1}) + N(y)\,\delta_{m,0},$$

$$c_m f_m = -2dE\,(n_{m+1} + n_{m-1}),$$

where

$$L_m = \frac{\Gamma(\gamma_{12} + imy)}{(\gamma_1 + imy)\,(\gamma_2 + imy)},\qquad c_m = my - i\Gamma - \frac{\Omega^2}{my - i\Gamma},$$

$$y = kv,\ \Omega = \omega - \omega_0,\ n_m = n_{1,m} - n_{2,m},\ f_m = \rho_{-m}^* - \rho_m,$$
$$\gamma_{12} = (\gamma_1 + \gamma_2)/2,\ N(y) = (N/\sqrt{\pi}v_0)\,\exp\,[-(y/kv_0)^2].$$

[2)]If the intensities of the opposing waves are unequal, the contribution of the atoms with low velocities to the higher harmonics decreases, and at $|\chi_1 - \chi_2| \ll 1$, apparently, only the first harmonic is important (χ_1 and χ_2 are the saturation parameters of the first and second waves).

If we introduce new unknowns

$$\psi_m = -f_m/2dEN(y), \qquad (2)$$

then we obtain for ψ_m the following recurrence relations[3)]:

$$a_m\psi_m + \eta b_{m+1}\psi_{m+2} + \eta b_{m-1}\psi_{m-2} = \delta_{m+1,0} + \delta_{m-1,0}, \qquad (3)$$

where

$$a_m = c_m + \eta b_{m+1} + \eta b_{m-1},\ b_m = -i\Gamma L_m,$$

$\eta = \gamma\chi/\Gamma$, $\gamma = \gamma_1\gamma_2/\gamma_{12}$, and $\chi = 4(dE)^2/\gamma\Gamma$ is the saturation parameter for one of the traveling waves.

When solving the system (3) we shall assume $\eta = \gamma\chi/\Gamma$ to be a small parameter. This enables us to use a perturbation method to solve (3). It is easy to see, however, that this is not always feasible. Indeed, in the region $y \sim \Gamma\eta$ we have $\eta L_m \sim 1$, i.e., the system (3) must be solved exactly. We shall return subsequently to the region $y \sim \Gamma\eta$. For the time being we assume that $y \gg \Gamma\eta$ making it possible to seek the solution of (3) in the form

$$\psi_1 = \psi_1^{(0)} + \psi_1^{(1)} + \ldots,$$

$$\psi_3 = \psi_3^{(1)} + \psi_3^{(2)} + \ldots,$$

with the zeroth approximation independent of η, $\psi^{(1)} \sim \eta\psi(0)$, etc.

To find the zeroth approximation we have from (3) the system

$$(c_1 - i\chi\Gamma)\,\psi_1^{(0)} + i\chi\Gamma\psi_1^{*(0)} = 1,$$

$$i\chi\Gamma\psi_1^{(0)} + (c_1^* + i\chi\Gamma)\,\psi_1^{*(0)} = 1$$

(we took into account the fact that $\psi_{-m} = -\psi_m^*$), whence

$$\psi_1^{(0)} = c_1^*\,(y^2 + \Gamma^2)/D, \qquad (4)$$

where

$$D = [(y - \Omega)^2 + \Gamma^2]\,[(y + \Omega)^2 + \Gamma^2] + \Gamma^2\chi[(y - \Omega)^2 + \Gamma^2] + \Gamma^2\chi[(y + \Omega)^2 + \Gamma^2]$$

or

$$D = [y^2 + (\Gamma a)^2]\,[y^2 + (\Gamma b)^2],$$

$$a^2 = 1 + \chi - \delta^2 + [\chi^2 - 4\delta^2]^{1/2},\qquad b^2 = 1 + \chi - \delta^2 - [\chi^2 - 4\delta^2]^{1/2},$$

$$\delta = \Omega/\Gamma\quad (\mathrm{Re}\,a > 0,\ \ \mathrm{Re}\,b > 0).$$

To find the first approximation we have

$$(c_1 - i\chi\Gamma)\,\psi_1^{(1)} + i\chi\Gamma\psi_1^{*(1)} = -\,\eta b_2\psi_1^{(0)},$$

$$i\chi\Gamma\psi_1^{(1)} + (c_1^* + i\chi\Gamma)\,\psi_1^{*(1)} = -\,\eta b_2^*\psi_1^{*(0)}.$$

Hence

$$\psi_1^{(1)} = -\eta c_1^{*2}b_2(y^2 + \Gamma^2)^2/D^2 + i\eta\Gamma\,(b_2c_1^* - b_2^*c_1)\,(y^2 + \Gamma^2)\,/\,D^2. \quad (5)$$

From (4) and (5) we have for the imaginary part of ψ_1, which we shall need in what follows

$$\mathrm{Im}\,\psi_1 = \Gamma(y^2 + \Gamma^2 + \Omega^2)\,/\,D + \eta\Gamma D^{-2}\,\mathrm{Re}\,\{L_2[(y\ +\ i\Gamma)^2 - \Omega^2]^2(y - i\Gamma)^2\}. \quad (6)$$

For $\gamma_1 = \gamma_2$ we have

$$\mathrm{Im}\psi_1 = \Gamma(y^2 + \Gamma^2 + \Omega^2)\,/\,D + \eta\Gamma^2[(y^2 + \Gamma^2)^2 - \Omega^2]\,/\,D^2. \qquad (7)$$

If $|y| \ll \Gamma$, then

$$\mathrm{Im}\,\psi_1 = \frac{\Gamma}{\Gamma^2 + \Omega^2 + 2\chi\Gamma^2} + \eta\,\frac{\Gamma^3(\Gamma^2 - \Omega^2)}{(\Omega^2 + \Gamma^2)\,(\Omega^2 + \Gamma^2 + 2\chi\Gamma^2)}. \qquad (8)$$

[3)]A system equivalent to (3) was considered in [6,7].

2. REGION OF LOW VELOCITIES

We now consider the region of low velocities, and take for simplicity the case $\gamma_1 = \gamma_2 = \gamma$.[4)] We turn to Eqs. (1). When $kv \ll \Gamma$, the derivatives with respect to z in the last equation can be left out. Eliminating ρ_{21} from the system (1), we have

$$\left(y\frac{\partial}{\partial\varphi}+\gamma\right)n=-2\gamma\nu n\cos^2\varphi+\gamma N(y), \tag{9}$$

where $\nu = 2\chi\Gamma^2/(\Gamma^2+\Omega^2)$, $\varphi = kz$, and f is expressed in terms of n with the aid of the relation

$$f=-\frac{4i\Gamma dEn}{\Omega^2+\Gamma^2}\cos\varphi.$$

The harmonic $\operatorname{Im}\psi_1$ of interest to us is equal to

$$\operatorname{Im}\tilde{\psi}_1=\operatorname{Re}\left\{\frac{\Gamma}{(\Gamma^2+\Omega^2)N(y)}(n_0+n_2)\right\},$$

where

$$n_0=\frac{1}{2\pi}\int_0^{2\pi}n\,d\varphi,\qquad n_2=\frac{1}{2\pi}\int_0^{2\pi}ne^{-2i\varphi}d\varphi$$

are the harmonics of the population difference.

In this region we have denoted ψ_1 by $\tilde{\psi}_1$, so as not to confuse it with (6). In view of the fact that the region of our preceding analysis was limited to velocities $|y| \gg \eta\Gamma$ ($\eta \ll 1$), and Eq. (9) is valid when $|y| \ll \Gamma$, we have a velocity region $\Gamma\eta \ll |y| \ll \Gamma$ in which both approximations are valid. Thus, by solving (9), we obtain a solution of (1) for all y.

The solution of (9) is

$$n=N(y)\frac{\gamma}{y}\int_0^\infty dt\exp\left\{-\frac{\gamma}{y}[\mu t+\nu\sin t\cos(2\varphi-t)]\right\}, \tag{10}$$

where $\mu = 1 + \nu$. For $\operatorname{Im}\tilde{\psi}_1$, using (20), we have

$$\operatorname{Im}\tilde{\psi}_1=\frac{\Gamma}{\Gamma^2+\Omega^2+2\chi\Gamma^2}-\frac{1}{2\chi\Gamma}\left(\frac{n_0}{N(v)}-\frac{1}{\mu}\right). \tag{11}$$

When $y \gg \gamma\chi = \eta\Gamma$ we have

$$\operatorname{Im}\tilde{\psi}_1=\frac{\Gamma}{\Gamma^2+\Omega^2+2\chi\Gamma^2}-\frac{\Gamma^4}{4(\Gamma^2+\Omega^2)^2}\frac{\eta\gamma}{y^2}. \tag{12}$$

3. CALCULATION OF POLARIZATIONS

The polarization induced in the medium by the field E(z) is the average dipole moment per unit volume

$$P=d\int_{-\infty}^{\infty}\rho_{21}dv+\text{c.c.}$$

It can be written in the form

$$P=\sum_{m=-\infty}^{\infty}\alpha_m Ee^{-i\omega t+ikmz}+\text{c.c.}$$

where

$$\alpha_m=\alpha\frac{1}{\pi}\int_{-\infty}^{\infty}\psi_m\left(1+\frac{\Omega}{my-i\Gamma}\right)\exp\left[-\frac{y^2}{(kv_0)^2}\right]dy,$$

$\alpha = \sqrt{\pi}Nd^2/kv_0$, and α_m is the polarizability at the m-th spatial harmonic in the standing-wave field.

We are interested in the imaginary part of the polarizability for $m = 1$. We introduce for it the special symbol:

$$\beta=\operatorname{Im}\frac{\alpha_1}{\alpha}=\frac{1}{\pi}\int_{-\infty}^{\infty}\operatorname{Im}\psi_1\exp\left[-\frac{y^2}{(kv_0)^2}\right]dy.$$

We shall continue the calculations for the case $kv_0 \gg \Gamma(1+\chi)^{1/2}$. This will enable us to take the exponential outside the integral at the point $y = \Omega$ in all cases, and simply omit the exponential in the case of deviations kv_0.

Let us calculate β in the case $\gamma_1 = \gamma_2 = \gamma$. To this end we represent β in the form

$$\beta=\frac{1}{\pi}\int_{-\infty}^{-y_0}\operatorname{Im}\psi_1\,dy+\frac{1}{\pi}\int_{y_0}^{\infty}\operatorname{Im}\psi_1\,dy+\frac{1}{\pi}\int_{-y_0}^{y_0}\operatorname{Im}\tilde{\psi}\,dy,$$

where y_0 is a certain point in the region $\Gamma\eta \ll y_0 \ll \Gamma$. Obviously,

$$\beta=\frac{1}{\pi}\int_{-\infty}^{-y_0}\operatorname{Im}\psi_1\,dy+\frac{1}{\pi}\int_{-y_0}^{y_0}\frac{\Gamma dy}{\Gamma^2+\Omega^2+2\chi\Gamma^2}$$
$$+\frac{1}{\pi}\int_{y_0}^{\infty}\operatorname{Im}\psi_1\,dy+\frac{1}{\pi}\int_{-y_0}^{y_0}\left(\operatorname{Im}\tilde{\psi}_1-\frac{\Gamma}{\Gamma^2+\Omega^2+2\chi\Gamma^2}\right)dy. \tag{13}$$

The second integral of (13) can be replaced by $\int_{-\infty}^{\infty}\operatorname{Im}\psi_1 dy$. According to (8), we neglect here in β a quantity on the order of $\eta y_0/\Gamma$. (We recall that we seek β accurate to η, and $y_0/\Gamma \ll 1$). The integration in the last term will extend from $-\infty$ to ∞. From (12) we can easily estimate the ensuing error. If $\gamma\mu \ll y_0$, then

$$\left|\int_{y_0}^{\infty}\left(\operatorname{Im}\tilde{\psi}_1-\frac{\Gamma}{\Gamma^2+\Omega^2+2\chi\Gamma^2}\right)dy\right|\sim\frac{\eta\gamma}{y_0}.$$

This quantity can be neglected if $\gamma/y_0 \ll 1$. This means in fact that we assume $\gamma/\Gamma \ll 1$ for all χ. The requirement $\gamma\mu \ll y_0$ in (12) for arbitrary χ also presupposed that $\gamma/\Gamma \ll 1$.

Taking the foregoing into account, we can rewrite (13) in the form

$$\beta=\frac{\Gamma}{\pi}\int_{-\infty}^{\infty}\frac{y^2+\Gamma^2+\Omega^2}{[y^2+(\Gamma a)^2][y^2+(\Gamma b)^2]}dy$$
$$+\eta\Gamma^3\operatorname{Re}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(y^2+\Gamma^2)^2-\Omega^4}{[y^2+(\Gamma a)^2]^2[y^2+(\Gamma b)^2]^2}dy$$
$$+\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\operatorname{Im}\tilde{\psi}_1-\frac{\Gamma}{\Gamma^2+\Omega^2+2\chi\Gamma^2}\right)dy. \tag{14}$$

The first two integrals in (14) can be calculated with the aid of the theory of residues. For the calculations connected with the third term see Appendix 2. As a result we obtain

$$\beta=\frac{1}{a+b}\left[1+\left(\frac{1+\delta^2}{1+\delta^2+2\chi}\right)^{1/2}\right]+\eta\frac{1}{2(a^2-b^2)^2}$$
$$\times\left[\frac{(5a^2-b^2)[(a^2-1)^2-\delta^4]}{(a^2-b^2)a^3}-\frac{4(a^2-1)}{a}\right.$$
$$\left.+\frac{(5b^2-a^2)[(b^2-1)-\delta^4]}{(b^2-a^2)b^3}-\frac{4(b^2-1)}{b}\right]-\frac{\gamma}{2\Gamma}\frac{\chi}{(1+\delta^2+2\chi)^2}A, \tag{15}$$

where

$$A=-\frac{8}{\pi p}\int_0^\infty\ln\left\{\frac{1}{2}\left[1+\left(1-p\frac{\sin^2 t}{t^2}\right)^{1/2}\right]\right\}dt,$$
$$p=4\chi^2/(1+\delta^2+2\chi)^2. \tag{16}$$

The value of A depends little on χ or Ω. As seen from (21) below, it can range from $A = 1$ to $A \approx 1.4$. At the center of the line ($\delta = 0$) we have

[4)] We note that the solution in this region can be obtained for $\gamma_2 \gg \gamma_1$ as well as for $\gamma_1 \gg \gamma_2$.

$$\beta=\frac{1}{(1+2\chi)^{1/2}}+\frac{\gamma\chi}{2\Gamma(1+2\chi)^{3/2}}-\frac{\gamma\chi}{2\Gamma(1+2\chi)^{2}}A_0, \tag{17}$$

where A_0 is the value of A at $\delta = 0$. For $\delta \gg (1 + 2\chi)^{1/2}$ we have $\beta = (1 + \chi)^{-1/2}$. For $\chi \ll 1$, making straightforward but rather cumbersome calculations, we obtain from (15), accurate to χ^2:

$$\beta=1-\frac{1}{2}\chi\left(1+\frac{1}{1+\delta^2}\right)+\frac{3}{8}\chi^2\left[1+\frac{1}{1+\delta^2}+\frac{2}{(1+\delta^2)^2}\right]$$
$$+\frac{\gamma}{8\Gamma}\chi^2\left[\frac{1}{1+\delta^2}+\frac{2(1-\delta^2)}{(1+\delta^2)^2}+\frac{1-3\delta^2}{(1+\delta^2)^3}\right]. \tag{18}$$

The expression (18) coincides with the absorption coefficient calculated by Lamb's theory, accurate to χ^2 at $\gamma/\Gamma \ll 1$[11].

4. ANALYSIS OF RESULTS AND PHYSICAL INTERPRETATION

Formula (15) describes the dependence of the absorption coefficient of the standing-wave field on the frequency deviation and on the saturation parameter. The first term of (15) corresponds to the solution obtained from the rate equations. The second and third terms are proportional to γ/Γ. At $\Omega \gg \Gamma(1 + \chi)^{1/2}$, the contributions of these effects tends to zero and, as expected, the absorption coefficient becomes equal to $(1 + \chi)^{-1/2}$, corresponding to the saturated absorption coefficient of each of the traveling waves. This can be easily understood physically. At deviations Ω $\Omega \gg \Gamma(1 + \chi)^{1/2}$ the opposing traveling waves interact with different atoms and their contribution to the absorption can be taken into account independently.

At $\Omega = 0$, the absorption coefficient is given by formula (17). Both opposing wave interact with the same atoms, and the contribution of the coherent effects is maximal. The first term of (17) determines the main contribution to the absorption coefficient. It differs from the case of large deviations in that the saturation parameter is doubled. Thus, the difference $(1 + \chi)^{-1/2} - (1 + 2\chi)^{-1/2}$ determines the depth of the Lamb dip. The last two terms are due to the spatial modulation of the medium, i.e., they are connected with the appearance of spatial harmonics in the population difference and in the polarization. The first term is connected with the contribution of the atoms whose velocities are $kv \sim \Gamma(1 + \chi)^{1/2}$. In this velocity region, the amplitudes of the spatial harmonics decrease in terms of the parameter $\gamma\chi/\Gamma$, and to obtain the solution with the required accuracy it is necessary to take into account the zeroth and second spatial harmonics of the population difference. From the point of view of the action of the field on the atom, we take into account the coherence between the states of the atom at the levels 2 and 1, due to the field. Physically this corresponds to additional modulation of the dipole moment of the atom by frequencies of the order of dE. In view of the fact that the time of the coherent interaction of the atom with field is $1/\Gamma$, the indicated modulation is significant when $dE/\Gamma \sim 1$. Thus, the additional contribution to the polarization in terms of the parameter $\gamma\chi/\Gamma = 4(dE/\Gamma)^2 \ll 1$, corresponds to allowance for the coherent corrections to the dipole moment of the atom in terms of the same parameter. We note once more that the saturation effects can be appreciable, since they are determined by the saturation parameter.

The third term in (17), due to the velocities $v \sim \gamma/\Gamma$, makes a negative contribution to the absorption. Its appearance is due to the specific features of the interaction of the standing-wave field with the atoms.

The field vanishes at the nodes, and this leads to a sharp increase of the population difference, and consequently to a decrease of the strong-field absorption. From (9) we obtain for the distribution of the population difference at $v = 0$

$$n=\frac{N}{1+2\chi+2\chi\cos 2kz}.$$

At $z_n = \pi k^{-1}(n + 1/2)$ we have $n = N$, i.e., the population difference is equal to the unsaturated value. The values of the spatial harmonics at $v = 0$ are given in Appendix 1. At $\chi \ll 1$, the amplitudes of the harmonics decrease like $\chi^{m/2}$ ($m = 0, 2, 4, \dots$), i.e., only the zeroth harmonic is significant. It is easy to understand why the region of slow atoms is determined by the condition $v \sim \gamma/\Gamma$. At $v \gg \gamma/k$, any of the excited atoms spends the greater part of its time in the region of the averaging field. To estimate the additional contribution of this region to the absorption coefficient at $\chi \sim 1$, we can put $\Delta n \sim 1$ and therefore the contribution of this region to the gain is of the order of $\alpha_0\gamma/\Gamma$, which agrees in order of magnitude with (17) at $\chi \sim 1$. When $\chi \gg 1$, the contribution from the atoms with low velocities can be neglected, since the third term in (17) decreases like $1/\chi$, whereas the second is proportional to $\chi^{-1/2}$.

It is interesting to note the property of the expansion (18) at $\chi \ll 1$. In each of the regions $v \sim \gamma/k$ and $v \sim \Gamma/k$ there arise terms that are linear in χ. Their contributions, however, are equal and opposite, so that coherent corrections appear only as a result of the χ^2 terms.

The authors are indebted to the participants of the seminar at the Institute of Physics Problems of the Siberian Division of the USSR Academy of Sciences (July 1970), particularly to V. S. Smirnov and B. L. Zhelnov, for a discussion of the region of applicability of the obtained solution. The authors thank A. V. Chaplik for a discussion of the work.

APPENDIX I

From (10), the harmonics of the population difference at $y \ll \Gamma$ are given by

$$n_k=N(y)\frac{\gamma}{y}\int_0^\infty dt\exp\left[-\frac{\gamma\mu}{y}t\right]\frac{1}{2\pi}$$
$$\times\int_0^{2\pi}d\varphi\exp\left[-ik\varphi-\frac{\gamma\nu}{y}\sin t\cos(2\varphi-t)\right]$$
$$(k=0,\ \pm2,\ \pm4\dots).$$

At $y = 0$ we have

$$n_k=(-1)^{k/2}\frac{N(0)}{(1+4\chi)^{1/2}}\left[\frac{4\chi}{[1+(1+4\chi)^{1/2}]^2}\right]^{k/2}. \tag{19}$$

It should be noted that (19) is valid at $y = 0$ for arbitrary relaxation constants γ and Γ.

From (9) we can obtain for the harmonics the following recurrence relations

$$n_{k+2}=-n_{k-2}-\frac{2n_k}{\nu}\left(\mu+\frac{iky}{\gamma}\right)+\frac{2}{\nu}\delta_{k,0}N(y).$$

In particular, for $k = 0$ we have, recognizing that $n_k = n^*_{-k}$,

$$\mathrm{Re}\, n_2 = -\frac{\mu}{\nu}\left(n_0 - \frac{N(y)}{\mu}\right). \qquad (20)$$

APPENDIX II

Let us calculate the integral

$$J = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\mathrm{Im}\,\bar{\psi}_1 - \frac{1}{\Gamma(1+\delta^2+2\chi)}\right]dy.$$

According to (15), it can be written in the form

$$J = -\frac{\gamma\chi}{2\Gamma(1+\delta^2+2\chi)^2}A,$$

$$A = \frac{8}{\pi p}\int_0^\infty \frac{dy}{y}\int_0^\infty dt \exp\left(-\frac{\gamma\mu}{y}t\right)\frac{1}{\pi}\int_0^\pi\left[\exp\left(-\frac{\nu\gamma}{y}\sin t\cos\varphi\right) - 1\right]d\varphi,$$

where we have put

$$\frac{1}{1+\delta^2+2\chi} = \frac{\gamma}{1+\delta^2}\frac{1}{y}\int_0^\infty dt\, e^{-\gamma\mu t/y}\frac{1}{\pi}\int_0^\pi d\varphi.$$

We proceed to calculate A. After introducing the new variable $z = \gamma/\chi$ and changing the order of integration, we get

$$A = \frac{8}{\pi p}\int_0^\infty dt\frac{1}{2\pi}\int_0^{2\pi} d\varphi\int_0^\infty \frac{dz}{z}\left[e^{-z(\mu t+\nu \sin t\cos\varphi)} - e^{-z\mu t}\right].$$

Since

$$\int_0^\infty z^{\alpha-1}e^{-Qz}\,dz = \frac{\Gamma(\alpha+1)}{\alpha Q^\alpha}$$

(see, for example,[12], p. 331), the integration with respect to z in the expression for A can be carried out by making the substitution $1/z \to 1/z^{-\alpha+1}$ and letting α tend to zero. This yields

$$A = \frac{8}{\pi p}\int_0^\infty dt\frac{1}{\pi}\int_0^\pi d\varphi \ln\frac{\mu t}{\mu t + \nu\sin t\cos\varphi}$$

Recognizing that (see[12], p. 541)

$$\int_0^\pi \ln(c + q\cos x)\,dx = \pi\ln\frac{c+(c^2-q^2)^{1/2}}{2},$$

we obtain finally the formula (16) of the text.

To calculate A it is convenient to expand the integrand in (16) in a series, assuming p to be a small quantity; the required integrals can be obtained in[12], pp. 460 and 464. Ultimately we get

$$A = 1 + \frac{1}{4}p + \frac{11}{96}p^2 + \ldots$$

Since p varies in the range $0 < p < 1$, the obtained expression can be used in practice for all values of p.

[1]W. R. Bennett, Jr., Phys. Rev. **126**, 580 (1962).

[2]S. G. Rautina and I. I. Sobel'man, Zh. Eksp. Teor. Fiz. **44**, 834 (1963) [Sov. Phys.-JETP **17**, 564 (1963)].

[3]W. E. Lamb, Phys. Rev. **134**, 1428 (1964).

[4]S. G. Rautina, Trudy FINA **43**, Nauka, 1968, p. 3.

[5]H. Greenstein, Phys. Rev. **175**, 438 (1968).

[6]S. Stenholm, and W. E. Lamp, Phys. Rev. **181**, 618 (1969).

[7]B. J. Feldmann and M. S. Feld, Phys. Rev. **A1**, 1375 (1970).

[8]H. K. Holt, Phys. Rev. **A2**, 233 (1970).

[9]E. V. Baklanov and V. P. Chebotaev, Zh. Eksp. Teor. Fiz. **60**, 552 (1971). [Sov. Phys.-JETP **33**, 300 (1971)].

[10]E. V. Baklanov and V. P. Chebotaev, ibid. **61**, 922 (1971) [**34**, 490 (1972)].

[11]K. Uehara and K. Shimoda, Japan J. Appl. Phys. **4**, 921 (1965).

[12]I. S. Gradshtein and I. M. Ryzhik, Tablitsy Integralov, Summ, Ryadov i Proizvendenii (Tables of Integrals, Sums, Series, and Products), Fizmatgiz (1963).

Translated by J. G. Adashko
67