Shock Wave Structure in the Radiation Spectrum During Bose Condensation of Photons

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The spectrum of photons interacting with electrons via the induced Compton effect is considered. Assuming weak energy transfer per collision, we previously predicted that a discontinuity may arise in the dependence of intensity on frequency ("shock wave" in phase space). In the present paper it is shown that if the finite temperature of the electrons is taken into account, the slope of the spectrum increases only up to a certain limit, after which an oscillatory dependence of intensity on frequency arises on the shortwave side of the spectrum. The structure of the shock wave is thus found to be more complex than previously assumed. It is similar to a collisionless wave in a plasma and not to a viscous wave in a neutral gas.

1. INTRODUCTION

THE discovery of powerful compact sources of low-frequency radiation in astronomical objects has attracted attention to the interaction between intense electromagnetic waves having a brightness temperature $kT_b \gg m_ec^2$ and free electrons.

Let us consider a spatially-homogeneous and isotropic situation, wherein the radiation spectrum is specified by the spectral energy density $\mathscr{E}_\nu$ of the radiation or by the occupation number $n(\nu) = (c^3/8\pi h\nu^3)\mathscr{E}_\nu$. The space is filled with electrons with density N_e and temperature T_e, which we assume specified and constant, although the results remain unchanged for a variable, and particularly self-consistent temperature (which the electrons acquire in the given radiation field). The scattering of the radiation by the free electrons leads to a redistribution of the energy and of the intensity over the spectrum; thus, the object of the investigation is the function $\mathscr{E}_\nu(t, \nu)$, where t is the time. Instead of using $\mathscr{E}_\nu$, we can characterize the radiation by the brightness temperature T_b, which is connected with $\mathscr{E}_\nu$ (we are considering long-wave unpolarized radiation) by the Rayleigh-Jeans relation

$$\mathscr{E}_\nu = 8\pi kT_b / c\lambda^2 = 8\pi kT_b\nu^2 / c^3.$$

Since $\mathscr{E}_\nu$ is in the general case not in equilibrium, $T_b(\nu, t)$ is also a function of the frequency. We are investigating the case of long-wave radiation of high-intensity, so that $T_b \gg T_e$ in a wide frequency range. As is well known[1,2], the interaction of the radiation having the higher brightness temperature with the colder electrons is then accompanied by drawing of energy from the radiation[1)], as a result of which the radiation spectrum is altered in the low-frequency region[3][2)]. If, furthermore, $kT_b \gg m_ec^2 \gg kT_e$, then (i) the induced scattering is stronger than the spontaneous scattering and (ii) the integral equation for the realignment of the spectrum can be transformed into a differential equation at a spectrum width $\Delta\nu \gg \Delta\nu_D = \nu\sqrt{2kT_e/m_ec^2}$. Here $\Delta\nu_D$ is the Doppler width of the spectrum and corresponds to the thermal velocities of the electrons.

Such a transformation was first performed by Kompaneets[5], who obtained the equation

$$\frac{\partial n}{\partial t} = \frac{\sigma_T N_e h}{m_e c}\frac{1}{\nu^2}\frac{\partial}{\partial\nu}\nu^4\left(n^2 + n + \frac{kT_e}{h}\frac{\partial n}{\partial\nu}\right). \quad (1)$$

As applied to our problem, this equation can be written in simpler form

$$\partial g / \partial\tau = g\partial g / \partial\nu, \quad (2)$$

where

$$g = \nu^2 n = \frac{c^3}{8\pi h\nu}\mathscr{E}_\nu; \quad d\tau = 2\frac{\sigma_T N_e h}{m_e c}dt; \quad \sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2.$$

This nonlinear equation was studied by Levich and one of the authors[3]; its characteristics are the lines

$$d\nu / d\tau = -g, \quad (3)$$

corresponding to a decrease of frequency at a rate proportional to the spectral density of the radiation energy (more accurately, proportional to the quantity g, which is connected with this density).

Under definite initial conditions, the spectrum evolution in accordance with Eq. (2) leads in the course of time to the formation of an infinite derivative $\partial g/\partial\nu$. For this purpose it is necessary and sufficient that there exist a point of inflection on the low-frequency side of the $g(\nu, 0)$ curve, i.e., the derivative $dg(\nu, 0)/d\nu$ should have a maximum at definite values $\nu = \nu_0$ and $g = g_0$:

$$\frac{dg}{d\nu}(\nu_0, 0) > 0, \quad \frac{d^2g}{d\nu^2}(\nu_0, 0) = 0, \quad \frac{d^3g}{d\nu^3}(\nu_0, 0) < 0.$$

The situation is mathematically similar to nonlinear propagation of an acoustic wave in a gas, wherein the dependence of the wave velocity on the amplitude gives rise first to an infinite derivative $\partial\rho/\partial x \to \infty$ and $\partial p/\partial x \to \infty$, and then to a shock wave. In analogy, the formation of a "shock wave" was predicted in[3] also for the spectrum of the electromagnetic radiation under the conditions described above.

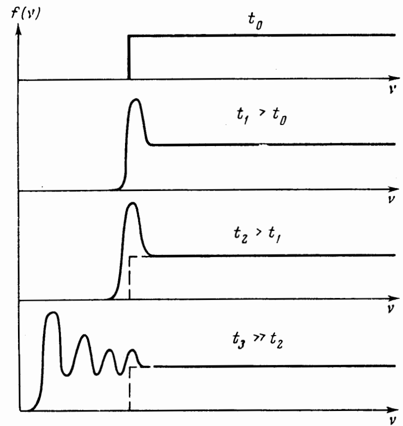
It should be noted that such situations were considered much earlier as applied to plasma oscillations. A nonlinear equation similar to (2) was derived in[6] for longitudinal plasma waves and it was noted that the evolution leads to a narrowing of the wave front. In its idea, this reference (see also[7]) anticipates the results of [3].

[1)]To realize this case it is necessary that the electrons lose energy in some manner that does not depend on the considered interaction with the low frequency radiation.

[2)]This effect was apparently observed in experiment[4].

We note also that Kompaneets[5] and the workers that followed him[1-3] have used the quantum language in that they have considered Compton scattering of photons and the corresponding transfer of the momentum $h\nu/c$ and of the energy, with account taken of both spontaneous and induced processes, as is indicated in Eq. (1) by the factor $(1 + n)$, which is characteristic of Bose particles (photons). Planck's constant h, however, is cancelled out everywhere, so that actually the problem in question is classical, and the quantum language only makes the description more convenient[3)].

Let us turn from the history to the gist of the problem. The occurrence of an infinite derivative $\partial g/\partial \nu$ during the course of the evolution means violation of the condition necessary for changing over from the integral scattering equation to Kompaneets' differential equation (1), namely that g be smooth. An analysis of the integral equation shows that instead of a shock wave moving as proposed in[3] along the energy axis towards lower frequencies, there occurs an oscillatory frequency dependence of the radiation intensity. The resultant spectrum is represented by a set of narrow spectral lines of width $\Delta\nu \sim \Delta\nu_D$, spaced a distance $\Delta\nu \sim \Delta\nu_D$ apart and moving towards the lower frequencies (see the figure).



Evolution of the radiation spectrum $g(\nu)$ as a result of Compton interaction with thermal electrons. The initial spectrum is chosen in the form of a step function. The motion of the quanta along the frequency axis to the right of the discontinuity is determined by the induced Compton effect, and the motion near the discontinuity by the spontaneous effect.

2. FORM OF INTEGRAL EQUATION

The present paper is devoted to an analysis of the situation arising when the spectrum abruptly becomes steep. For such an analysis it is necessary to return to an integral equation of the form

$$\frac{\partial g(\nu,t)}{\partial t} = Ag(\nu,t)\int K(\nu,\mu)g(\mu,t)\,d\mu. \quad (4)$$

Here $A = 2\sigma_T N_e h/m_e c = \tau/t$. The effective width of the kernel $K(\nu, \mu)$ of Eq. (4) corresponds to the average change of the frequency following a single scattering by the moving electrons:

$$K \neq 0 \text{ for } \left|\ln\frac{\mu}{\nu}\right| = \frac{|\mu-\nu|}{\nu} \leq \frac{v}{c} \sim \sqrt{\frac{kT_e}{m_e c^2}}.$$

The kernel K is the difference between the probability of the photons moving from all μ to ν and the inverse process of scattering from ν to all μ. It is therefore natural that the kernel K alternates in sign and moreover is antisymmetrical, $K(\nu, \mu) \equiv -K(\mu, \nu)$.

The integral equation (4) should correspond to a number of conditions. In Compton scattering, the total number of quanta in the system is conserved. Since g is proportional to the number of photons per unit frequency, we have

$$0 = \frac{dN_\nu}{dt} = \frac{\partial}{\partial t}\int_0^\infty g(\nu)\,d\nu = \iint K(\nu,\mu)g(\nu)g(\mu)\,d\nu\,d\mu. \quad (5)$$

The antisymmetry of K ensures identical satisfaction of this condition. Additional information concerning K can be obtained by using the fact that when the number of quanta in the system is sufficient the joint action of the induced and spontaneous Compton processes should lead to a spectrum of the Rayleigh-Jeans type at low frequencies. The determination of the exact form of the kernel K is not part of our problem, however.

In the limit as the electron temperature $T_e \to 0$ and K is correspondingly narrow, the kernel can be replaced by the singular derivative $\delta'(\nu, \mu)$ of the Dirac function. In this approximation, the integral equation is transformed into the differential equation (2) written out above. Our task is to analyze the integral equation at low but finite T_e. We note that as $T_e \to 0$ the differential equation leads to a discontinuity, so that the analysis of the structure of the front requires allowance for $T_e \neq 0$ at any T_e.

The first seemingly natural assumption is that the solution contains instead of the discontinuity $g(\nu_0 + 0) \neq g(\nu_0 - 0)$ a transition region of width $\Delta\nu \sim \nu_0 v/c \sim \nu_0\sqrt{kT_e/m_e c^2}$, which moves downward along the frequency axis as a viscosity-smeared shock wave with velocity proportional to the average value $\frac{1}{2}[g(\nu_0 + 0) + g(\nu_0 - 0)]$.

However, even without a detailed mathematical analysis it can be seen that the solution is more complicated. In fact, the use of a simple "step" ($g = g_1$ when $\nu < \nu_0$ and $g = g_2$ when $\nu > \nu_0$ with $g_1 < g_2$) as the initial condition causes the growth to be slower to the left of the discontinuity than to the right (since the integral in (4) is continuous, it follows that $\partial g/\partial t \sim g$). Thus, the discontinuity tends not to be smoothed out but to increase, and a sharp maximum $g > g_2$ is produced to the right of the discontinuity. The situation is clearly illustrated by the limiting case when $g_1 = 0$, i.e., when the initial spectrum is described by a step function. Then at $\nu > \nu_0$ we have predominant induced Compton scattering and motion of the quanta towards lower frequencies. In the region $\nu < \nu_0$ there are no quanta, and therefore there is no induced process and the quanta can enter this region only with the aid of spontaneous Compton scattering. As a result, the quanta should accumulate and a narrow spectral region with width $\Delta\nu \sim \Delta\nu_D$ should be produced near the discontinuity. The influence of the spontaneous Compton effect on the parameters of this line will be considered in Sec. 4.

[3)]Compare the ideas of Tsytovich[8] (and also Zel'dovich[9]) concerning the number of quanta as an invariant of the classical field, and also those of Paradoksov [10] concerning the usefulness of quantum language. We note that spontaneous scattering by free electrons is also classical.

In the general case $g_2 > g_1 \neq 0$ it turns out that at $\nu \sim \nu_0$ the single smooth discontinuity gives way to a complicated periodic dependence of g on ν. Simultaneously with the general displacement of the singularity towards lower frequencies we have a growth of the amplitude of the oscillations, whose period in the frequency scale is of the order of $\nu v/c$.

The question of possible realization and observation of such a characteristic spectrum under astrophysical conditions is beyond the scope of the present paper. In Sec. 5 below we shall give only an example of a situation in which the described picture can be realized. To answer this question it is first necessary to solve the problem in a realistic geometry and with allowance for the reaction of the radiation on the electrons[4)].

We confine ourselves here to an idealized formulation of the problem, for the purpose of disclosing under very simple assumptions the structure of the wave in pure form. In numerical calculations, on the other hand, we can use the following form of the kernel:

$$K(\nu,\mu)=\frac{d}{d\mu}[a\exp\{-a^2(\nu-\mu)^2\}],\quad a\sim\sqrt{\frac{m_e c^2}{kT_e}}\frac{1}{\nu}. \tag{6}$$

In Sec. 3 below we shall determine the analytic properties of (4).

3. ANALYTIC PROPERTIES OF EQ. (4) AND LIMITING CASES

We choose a scale, changing over to $x = \nu/a$, $y = \mu/a$, $t = b\tau$, such that the kernel of the equation

$$\frac{\partial g(x,t)}{\partial t}=g(x,t)\int K(x-y)\,g(y,t)\,dy \tag{4$'$}$$

has the following properties ($z = x - y$):

$$\text{a) } \int K(z)\,dz=0,\ \text{b) } \int zK(z)\,dz=1,\ \text{c) } \int z^3K(z)\,dz=1,$$

$$\text{d) } K(z)=\frac{d\Phi(z)}{dz}\quad \text{e) } \Phi(+z)=\Phi(-z),\quad \text{f) } \int\Phi(z)\,dz=1. \tag{7}$$

Conditions (a) and (e) reflect the antisymmetry of the kernel K(z), condition (d) is a definition of the symmetrical function $\varphi(z)$ (which can be considered to be Gaussian), and (b), (c), and (f) are normalization conditions.

Equation (4′) can be rewritten in the form

$$\frac{\partial g}{\partial t}=g\int\Phi\frac{\partial g}{\partial y}\,dy, \tag{8}$$

which yields for a slowly-varying g

$$\partial g/\partial t=g\,\partial g/\partial x. \tag{2$'$}$$

According to (2), one can visualize the quanta as "moving" with velocity g/2. We note that the flow velocity is not equal to the perturbation-propagation velocity, but is one-half the latter. The differential equation (2) has two obvious "conservation laws" that are essential in what follows:

$$\frac{d}{dt}\int_{x_1}^{x_2} g\,dx=-\frac{g^2(x_1)}{2}+\frac{g^2(x_2)}{2}, \tag{9}$$

$$\frac{d}{dt}\int_{x_1}^{x_2}\ln g\,dx=-g(x_1)+g(x_2). \tag{10}$$

It is easy to verify that both laws also obtain for the integral equation, if the function g in the vicinities of x_1 and x_2 is constant and equal respectively to g_1 and g_2 over several units of the chosen scale (we recall that the region of influence of the kernel $K(x-y)$ is of the order of unity).

Let us attempt to construct a solution using a transition region of arbitrary shape moving in stationary fashion to the left with velocity u:

$$g(x,t)=\varphi(x+ut);\ \varphi(-\infty)=g_1,\ \varphi(+\infty)=g_2. \tag{11}$$

For such a solution, regardless of the form of φ, the derivatives of the integrals in (9) and (10) have perfectly defined values:

$$\frac{d}{dt}\int_{-\infty}^{+\infty} g\,dx=u(g_2-g_1)=\frac{g_2^2-g_1^2}{2}, \tag{9$'$}$$

$$\frac{d}{dt}\int_{-\infty}^{+\infty}\ln g\,dx=u\ln\frac{g_2}{g_1}=g_2-g_1, \tag{10$'$}$$

from which we obtain two different values of u:

$$u'=(g_2+g_1)/2, \tag{12}$$

$$u''=(g_2-g_1)/(\ln g_2-\ln g_1), \tag{13}$$

which coincide only in the limit as

$$g_2\to g_1\to g_0,\qquad u'\to u''\to g_0.$$

This means that when $g_2 \neq g_1$ no stationary solution is possible!

The difference between u′ and u″ is small: putting $g_2 = g_0 + \alpha$ and $g_1 = g_0 - \alpha$, we obtain $u' = g_0$ and $u'' = g_0 - \alpha^2/3g_0 + \ldots$

Let us attempt to determine the possible quasistationary solution when $\alpha \ll g_0$, neglecting the difference between u′ and u″. To this end we seek the asymptotic forms of a solution of the type $\varphi(x + g_0t)$ to the left and to the right of the discontinuity

$$\varphi=g_1+\beta(x+g_0t), \tag{14}$$

$$\varphi=g_2+\gamma(x+g_0t). \tag{15}$$

Assuming β to be small in (14) and γ to be small in (15), we seek a solution in exponential form

$$\beta(\eta)=Be^{p\eta};\ \gamma(\eta)=Ce^{-q\eta};\ \eta=x+g_0t. \tag{16}$$

Substituting such β and γ in (2), we get

$$g_0=g_1\exp(p^2/4)=g_2\exp(q^2/4). \tag{17}$$

Since $g_1 < g_0 < g_2$, it follows from (17) that p is real but q is imaginary. This means that in the quasistationary solution the front of the wave clings exponentially to the low-frequency region of x_1, g_1, but the high-frequency asymptotic form $g \to g_2$ is reached in an oscillatory manner.

The completely linearized case $\delta = g_2 - g_1 \ll g_0$ does not contain the important property of the nonlinear problem—the steepening of g(x, t), i.e., the growth of $|\partial g/\partial x|_{max}$ when the initial function g(x, 0) is slowly varying. Nonetheless, the linear case is not devoid of interest. The equation

$$\frac{\partial\delta(x,t)}{\partial t}=\int K\delta(y,t)\,dy \tag{18}$$

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4)It is curious that when $\mathcal{E}_\nu$ has a periodic dependence on ν one can expect a particularly strong increase of the induced radiation pressure on the electrons[11].

for the Fourier components

$$\delta = \int \delta_k e^{i\omega t + ikx} dk \quad (19)$$

yields the dispersion equation

$$\omega(k) = ke^{-k^2/4}. \quad (20)$$

At small k we have

$$\omega = k - 0{,}25k^3. \quad (21)$$

At large k

$$\omega = ke^{-k^2/4}. \quad (22)$$

We see therefore that the slowly varying δ moves (as $k \to 0$) as a unit with unity velocity, so that

$$i\omega t + ikx = ik(t + x). \quad (23)$$

To the contrary, the discontinuity, or any other singularity characterizing the asymptotic form as $k \to \infty$, stands still, since at large k the dependence on t disappears in the limit as $\omega \to 0$.

4. DISCUSSION

A detailed clarification of the picture calls for a numerical calculation. However, even the presented considerations show that in induced Compton interaction between a high-intensity radiation whose spectrum has an inflection point on the low-frequency side and thermal electrons one should expect a unique resultant spectrum with several minima and maxima. Spontaneous scattering may smooth them out on the high-frequency side.

The very appearance of intensity maxima at frequencies not corresponding to any resonances in the system is so curious that it is worthwhile to discuss the situation even before the entire picture is quantitatively explained.

The structure and evolution of the radiation spectrum can easily be explained qualitatively in the simplest example in which the initial spectrum is chosen in the form of a step function: $g(\nu > \nu_0) = \text{const}$, $g(\nu < \nu_0) = 0$. Then at $\nu > \nu_0$ the motion of the quanta along the frequency axis is determined by the induced Compton effect and has a velocity

$$\left.\frac{d\nu}{dt}\right|_{ind} = -Ag = -\frac{\sigma_T N_e h\nu^2}{m_e c} n. \quad (3')$$

When $\nu < \nu_0$ there are no quanta, the induced process does not take place, and the quanta can enter in this region only with the aid of the spontaneous Compton scattering. The probability of the spontaneous scattering is $w = \sigma_T N_e c$ and the average change of frequency in one act is $|\Delta\nu|_{sp} \sim v\nu/c$. Comparing $|\Delta\nu_{sp}|$ with the induced frequency shift

$$|\Delta\nu|_{ind} = \frac{1}{w_{sp}} \left|\frac{d\nu}{dt}\right|_{ind}$$

during the same time $1/w_{sp}$ we see that at

$$kT_b = nh\nu > m_e cv = \sqrt{m_e c^2 kT_e} \quad (24)$$

the motion of the quanta along the frequency axis as a result of the induced processes is faster than that due to the spontaneous processes, and that in the region $\nu \sim \nu_0$ there should occur an accumulation of the quanta. Obviously in the zone with $\Delta\nu/\nu_0 \sim v/c$ the number of accumulated quanta should be such that their outflow towards lower frequencies as a result of spontaneous scattering should equal the influx from the higher frequencies, due to the induced processes. As a result a narrow line is produced near ν_0 (see the figure). Subsequently, since the flow of quanta towards the lower frequencies is conserved, this line should move with velocity (3′), but the spectrum can already acquire a complex oscillatory structure. It is easy to make a rough estimate (which is patently exaggerated in view of the neglect of the role of the induced processes in the formation of the line) of the stationary height of the line:

$$\frac{T_b(L)}{T_b} = \frac{n_L}{n} = \frac{g_L}{g} \sim \frac{|d\nu/dt|_{ind}(g)}{|\Delta\nu|_{sp}(L)\, w_{sp}} \sim \frac{h\nu_0 n}{m_e cv} = \frac{kT_b}{m_e c^2}\sqrt{\frac{m_e c^2}{kT_e}}. \quad (25)$$

5. POSSIBILITY OF ASTROPHYSICAL APPLICATIONS

It was noted in[12] that induced Compton interaction of low-frequency radiation with a thermal plasma can greatly distort the spectra of quasars, galactic nuclei, and pulsars in the frequency region where the brightness temperature of the radiation is $kT_b > m_e c^2/\tau_T$. Here $\tau_T = \sigma_T N_e l$ is the optical thickness with respect to the Thomson scattering of the radiation region with characteristic dimension l. Since the brightness temperatures of the radiation of compact radio sources in quasars and galactic nuclei reach $T_b \sim 10^{13}$ °K $\sim 10^3 - 10^4\, m_e c^2$, and in pulsars even $T_b \sim 10^{25}$ °K $\sim 10^{15}\, m_e c^2$, it follows that one can hope to observe a narrow intense line in the spectra of sources whose initial spectra have inflection points on the low-frequency side. We note that in this case we have a stationary problem wherein the quanta emerge from a spatially limited region in which the quanta are produced and scattered.

In order for the inhomogeneity of the source (and the differences between the emission spectra in spatially separated regions of the sources) to cause no smearing of the lines in question, it would be of particular interest to consider the following geometrical arrangement of the scattering sources. An electron cloud of optical thickness τ_T and linear dimension l is located between a source of radius R and the observer, the distance r between the cloud and the source being much larger than either R or l. Then, if $kT_b > (m_e c^2/\tau_T)(r/R)^3$[12], then a narrow intense emission line having a brightness temperature larger than that in the continuous spectrum should be produced in the spectrum.

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