Curvature of Physical Space and the Difference between Gravitational and Inertial Fields

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The curvature tensor of three-dimensional physical space is determined with the aid of the Schouten-Wagner theory of the curvature of nonholonomic manifolds. The concept of the characteristics of the nonholonomy of time and space is introduced. The influence of a gravitational field on these characteristics is considered. A classification of frames of reference is given.

1. INTRODUCTION

THE goal of the present article is a formulation, within the framework of the general theory of relativity (GTR), of a method of distinguishing between true and fictitious gravitational fields with the aid of the Wagner curvature tensor[1] of three-dimensional physical space. In the final analysis this distinction turns out to be possible because, in the first place, not only space but in general time also participates in the formation of the geometry and, in the second place, because gravitational and inertial fields are now uniquely distinguished at the level of four-dimensional space-time with the aid of the Riemann world curvature tensor, namely, at this level no inertial field is present; it appears only when the single manifold of events (the Lorentz manifold) is split into time and space, and is determined by the method of such a split; at the four-dimensional level a gravitational field is identical to the world tensor of curvature $R_{\alpha\beta\gamma\delta}$.[2]

The basic definitions are cited and discussed in Sec. 2, namely, the definition of the frame of reference, and the definition of its space and time. The first two definitions are essentially contained in the articles by Zel'manov,[3,4] which describe his theory of chronometric invariants. The idea of this theory is to turn, when considering physical fields, from the four-dimensional geometrical picture to a three-plus-one-dimensional dynamically-geometric picture, but to remain within the framework of the GTR, that is, to turn to a point of view in which the difference between space and time is explicitly and consistently taken into consideration, in particular by the differences between the standards of length and duration, clocks and yardsticks. The four-dimensional geometrical world quantities and differential operators are then split into spatial and time parts, having usually physical (dynamical) meaning. At the same time the theory of chronometric invariants intrinsically introduces the concept of nonholonomy into the GTR. In the present article (Secs. 3 and 4) this concept is universalized, and the expression for the curvature of space as the curvature of a nonholonomic manifold (distribution) is defined more precisely. Without this more precise definition it would be impossible, in particular, to rigorously define closed, purely spatial, finite geometrical figures such as triangles, circles, cubes, etc. in a rotating (that is, not having simultaneity) or in a deformed-space frame of reference. However, the inclusion of the Schouten-Wagner theory[1,5] of the curvature of nonholonomic manifolds into the GTR permits us to choose these spatial figures to be closed in the sense of four-dimensional geometry. In Sec. 5 we estimate the influence of a gravitational field on the spectrum (series, list) of possible combinations of the characteristics of the nonholonomy of space and time. The result is that a gravitational field shifts this spectrum, excluding certain combinations and admitting others.

The signature $(+---)$ is assumed for the metric $g_{\mu\nu}$. Greek indices take everywhere the values 0, 1, 2, 3; Latin indices take only the values 1, 2, 3. In the symbol V_4^k for a distribution (nonholonomic manifold), the subscript corresponds to the dimension of the basis (point manifold) and the superscript corresponds to the dimension of the tangent fiber; TV_4 denotes the tangent manifold (tangent fiber bundle) V_4^4; the duality of the distributions is indicated by a line above the appropriate symbol. Brackets around the indices denote alternation or cyclic permutation:

$$f_{[ij]} = {}^1\!/_2 (f_{ij} - f_{ji}), \quad f_{[ijk]} = f_{ijk} + f_{jki} + f_{kij}.$$

Chronometrically invariant derivatives are denoted by asterisks; $\partial_\alpha \equiv \partial/\partial x^\alpha$; $b^{\mu\nu}_{\alpha\beta} \equiv b^\mu_\alpha b^\nu_\beta$. The direct sum $E_1 + E_3$ of the Euclidean straight line E_1 and the plane E_3, having a single point in common, denotes the four-dimensional plane E_4, each vector of which is resolved into components along E_1 and E_3; for distributions the direct sum is taken point by point. The velocity of light is set equal to unity.

2. BASIC DEFINITIONS

Three-dimensional physical space is always the space of a certain reference body. We shall assume the reference body to be an idealized (test, continuous) single deformable medium with monotonically and continuously running clocks at each of its points. Let there be two of these clocks at each point, an arbitrarily running clock and a standard clock. The first set of clocks arbitrarily parametrize the world line of an element of the medium, giving the co-time (the coordinate time y^0); the second set of clocks are canonical, specifying the invariant time (the arc length τ of the world line). We note that geometrization essentially penetrates into the GTR just at this point, where quantities of a different nature are become identical, viz., the time interval (a physical concept) and the arc length of a four-dimensional curve (a geometrical concept). The test nature of the reference body means that its mass must be infinitesimal, so that without the expenditure of energy one can impart any arbitrary state of motion to the medium and regard it as motionless relative to any real reference body specified beforehand, whose behavior it is necessary to describe.

Since the trajectories of the elements of the continuous reference body, moving in laminar fashion, do not intersect with each other, their world lines form a congruence, that is, a family of curves such that just one curve of the family passes through each point (elementary event). This congruence is geometrically equivalent to the physical reference body.

We shall define a reference frame as an oriented congruence Γ of time-like curves γ in the Lorentz manifold V_4. We note that some congruence always exists globally in the Lorentz manifold with its structure of a continuous and oriented field of isotropic cones, but we must not require from it the property of being geodesic everywhere and so forth; in particular, a synchronous frame of reference may not exist everywhere.

If each particle of the test medium has some kind of fixed characteristic spatial directions, one can also include them in the definition of the frame of reference (thereby modifying it) with the aid of additional spatial congruences. The complete set of such congruences is equivalent to the field of a tetrad, and it is convenient to take it to be orthonormalized. In the frame of reference determined by the method indicated above, the spatial directions are determined more readily with the aid of a pair of infinitesimally close curves. The time direction is given by the direction of the curve γ from the congruence Γ, that is, by the unit tangent vector of the four-dimensional velocity of an element of the medium or by the tangent itself with its orientation.

Let us construct the concept of three-dimensional physical space. The instantaneous local space of the reference frame is orthogonal to the corresponding time direction, since the former is an infinitesimal region consists, roughly speaking, of simultaneous events for this system. Strictly speaking, in curved space-time the time and space directions and the light cone (all vectors) passing through a given point do not lie in space-time itself, but in a tangential centro-Euclidean plane E_4 having one common point with space-time. This does not mean, of course, that the above-mentioned directions take us outside the limits of space-time: for a fixed center of the tangential Minkowski world E_4, all of its other points designate not other events but only to directions, and the tangent plane itself is a space consisting only of directions going out from the given elementary event. We cannot subdivide an elementary event itself (a point) into spatial and temporal parts, but we can separate directions and define local time and local space as a time-like straight line E_1 orthogonal to a three-dimensional small area (hyperplane) E_3, and the direct sum of the two gives the local E_4. The time of the reference frame then corresponds to the set $T\Gamma$ of lines E_1 which are tangent to the congruence Γ, and the space of the reference frame corresponds to the set of three-dimensional areas (triplanes) E_3 which are orthogonal to it. In other words, time and space correspond to dual (orthogonal and mutually rigged, nonisotropic) distributions in the tangent fiber bundle TV_4 of the Lorentz manifold V_4.

Let us introduce the following definitions. The time Θ of the reference frame Γ is its tangent fiber bundle $T\Gamma$. The space Σ of the reference frame Γ is the dual of its time distribution $\overline{T\Gamma}$.

It is obvious that

$$TV_4 = \Theta + \Sigma,$$
$$(\Theta = T\Gamma = V_4^1, \quad \Sigma = \overline{\Theta} = V_4^3, \quad TV_4 = V_4^4),$$

that is, the direct sum of space and time forms space-time together with its directions. In the present definition space is taken not only as the space of directions, but also as a collection of "positions": in V_4^3 one can easily define the enveloping, dual (orthogonal) distribution V_4^1 of the congruence Γ, each curve γ of which is formed by a sequence of events and which fixes the "position," the spatial point, taken for all instants of time.

It might be possible to define space simply as a collection of "positions," i.e., as a Riemannian three-dimensional manifold V_3, each point of which is a time-like world line—this is the fiber bundle of the manifold of events V_4 itself with one-dimensional time fibers γ. In this case space is formed by the "contraction" of space-time with respect to time. But in this connection it remains outside the field of view, or else the nonholonomy of physical space[3,4] and time[6] is inadequately specified, and a satisfactory theory of the curvature of three-dimensional physical space becomes impossible. In fact, the curvature of the holonomic Riemannian space V_3 is described by its Riemann tensor, while the curvature of the nonholonomic distribution V_4^3 is described by a generalization of its Wagner tensor.[1] Their difference arises as a consequence of the fact that when a vector taken around a closed spatial contour, in the first case (a V_3 type space) the possible time shift of the origin and of the end of the contour is not taken into consideration. This is equivalent, for example when summing the angles of a triangle that is being deformed together with space (it is clear that this sum attests to the curvature of space) to ignoring the fact that the vertices γ_1, γ_2, and γ_3 of a finite, geodesic triangle are taken at different instants of time and in general cannot be taken simultaneously. In other words, the absence of (the impossibility of defining) finite spatial figures is ignored. This happens because time in such a rather extended conception of space (a V_3 type space) is essentially excluded as a factor of the spatial geometry, and at the same time it is retained in the requirement of simultaneity, in synchronization along the contour of a geometrical figure. On account of this, we lose, for example, the concepts of the state of a physical system, the mass of an extended object (or the energy contained in a volume), since this state or this object are assumed to be taken at a certain instant of time, which on the whole does not exist. Speaking differently, the nonholonomicity of physical space is ignored here.

By taking space as the nonholonomic manifold V_4^3, we obtain the right to regard finite spatial figures as closed in the sense of four-dimensional geometry, foregoing the requirement of simultaneity which is impossible to satisfy in general, but introducing the requirement that there be no time gap due to circuiting the figure. This is made possible by the inclusion of time as a factor of the spatial geometry (see Sec. 4), with a consistent development of the idea of the intrinsic geometry of only one (nonholonomic) space. We also note that the inclusion of distributions permits us to subdivide space time into space and time which are everywhere mutually orthogonal for any arbitrary reference frame, and not just for a semigeodesic (synchronous) frame of reference.

3. THE NONHOLONOMY OF TIME AND SPACE

In the narrow sense of the word, the nonholonomy of three-dimensional space is to be understood as the absence of hypersurfaces enveloping the local spaces.[4] In the broad sense of the word, nonholonomy refers to the noncommutativity of differential operators. Following E. Cartan, we shall distinguish between four kinds of nonholonomy of a single manifold, namely, its external and interior curvatures and its exterior and interior torsions. For a one-dimensional manifold (the time), only the exterior curvature is possible.

The interior curvature corresponds to the noncommutativity of covariant derivatives, that is, to a parallel displacement along two directions in the manifold. To observe the exterior curvature it is necessary to choose one of these directions orthogonal to the given manifold, embedded in the exterior manifold that envelopes the given manifold (for space and time this exterior manifold is space-time). Interior torsion appears in connection with a Cartan circuit a point along a contour. Its model can be the rolling without slipping of the tangent plane on the surface. The interior torsion of the surface differs from zero if the contour on the plane is open for a closed contour in this surface. The Riemannian manifold V_4 with the Lorentz signature, which is the generally-accepted model of space-time at the present-day level of physical knowledge, does not have interior torsion. One can show that this implies its absence for both space and time separately. The situation is different with regard to the exterior torsion, which arises when a point is transported over small areas of the nonintegrable distribution. In particular, by synchronizing the clocks along a closed spatial contour, we are displaced each time with respect to the small area of simultaneity $E_3 \in V_4^3$ and we return to another small area of the previous world line; the transported point comprises events that are simultaneous along the contour. The gap which appears lies in a direction that is external to the space (is timelike).

Let us consider the possible forms or the characteristics of nonholonomy of two nonholonomic manifolds—space and time—and their physical interpretation. In this connection we note that space-time has only a single nonholonomy characteristic, namely, the interior curvature, and space does not have more than three (unique) variants of nonholonomy (interior torsion is excluded).

Let us define the reference frame Γ by the field of the unit time-like vector $u^\alpha = dx^\alpha/ds$, $u_\alpha u^\alpha = 1$. Its expansion in terms of covariant derivatives is of the form:

$$\nabla_\mu u_\nu = u_\mu f_\nu + d_{\mu\nu} + \omega_{\mu\nu},$$
$$f_\nu = u^\alpha \nabla_\alpha u_\nu, \quad d_{\mu\nu} = d_{\nu\mu}, \quad \omega_{\mu\nu} = -\omega_{\nu\mu}.$$

The vector f^λ is the first curvature vector of Γ and describes the deviation of each world line $\gamma \in \Gamma$ from a geodesic; it is interpreted as the acceleration of free fall, taken with the opposite sign, in the given frame of reference. Within the framework of Zel'manov's theory of chronometric invariants, f^λ is the vector of the gravitational-inertial force, $d_{\mu\nu}$ is the tensor characterizing the rates of deformation, and $\omega_{\mu\nu}$ is the tensor characterizing the rotation of the reference frame. One can show that the Pfaffian form $\omega = u_\alpha dx^\alpha$ is integrable if and only if $\omega_{\mu\nu} = 0$. In this case the congruence Γ is normal, it admits everywhere holonomic (in the narrow sense of the word) spaces with global simultaneity which are orthogonal to its hypersurface. If a small two-dimensional spatial contour is specified by a simple bivector $f^{\mu\nu}$, then the time gap associated with the synchronization of the colcks along the contour is proportional to $\omega_{\mu\nu} f^{\mu\nu} u^\lambda$. We shall therefore call the nonholonomy tensor[7]

$$M_{\mu\nu}{}^\lambda = \omega_{\mu\nu} u^\lambda$$

of the distribution Σ, the tensor of exterior torsion of the space.

The tensor

$$N_{\mu\nu}{}^\lambda = d_{\mu\nu} u^\lambda$$

is the tensor of the exterior curvature of the space. In fact, if for example the vector joining the ends of the normals $u^\alpha d\tau$ to the hypersurface is displaced parallel to the hypersurface by the amount $d\tau$, where the vector dx^λ joins the bottoms of the normals, then the displaced vector will differ from the vector dx^λ by the amount $d_\mu^\lambda dx^\mu d\tau$. This means noncommutativity of displacements along the directions which are tangential and normal to Σ. We shall call the tensor

$$H_{\mu\nu}{}^\lambda = M_{\mu\nu}{}^\lambda + N_{\mu\nu}{}^\lambda = (\omega_{\mu\nu} + d_{\mu\nu}) u^\lambda = b_{\mu\nu}^{\alpha\beta} \nabla_\alpha b_\beta^\lambda$$

the tensor of the total exterior nonholonomy of the space (compare with[1,7]).

Whereas it is necessary to take two normals to the space in order to observe the exterior curvature of the space Σ, in the case of the time Θ it is necessary to take two normals to Γ in order to obtain the vector f^λ characterizing the nonholonomy of the time. The reasons for its appearance are the nongravitational forces, deflecting the world lines of the elements of the medium from geodesics and comparatively variable in terms of influence on the rate of time's tempo. We determine the interior curvature of space in Section 4.

It is useful to note that the class k of the exterior form $\omega = u_\alpha dx^\alpha$ gives a natural classification of reference frames (there are four classes). This class is equal to the number of unknown functions in terms of which the form ω can be expressed. The physical meaning of the classification is evident from the following list:

$k=1$	$\omega = d\tau,$	$f_\lambda = 0,$	$\omega_{\mu\nu} = 0;$
$k=2$	$\omega = \varphi d\tau,$	$f_\lambda = -b_\lambda^\rho \partial_\rho \ln \varphi \neq 0,$	$\omega_{\mu\nu} = 0;$
$k=3$	$\omega = \varphi d\tau + d\rho,$	$f_\lambda = 0,\ f_\lambda \neq 0,$	$\omega_{\mu\nu} \neq 0;$
$k=4$	$\varphi = \varphi d\tau + \psi d\rho,$	$f_\lambda \neq 0,$	$\omega_{\mu\nu} \neq 0.$

In particular, the semi-geodesic (synchronous) frame of reference belongs to the first class: The congruence Γ is geodesic ($f^\lambda = 0$) and normal ($\omega_{\mu\nu} = 0$). Time flows at the same rate everywhere and there is simultaneity (the spatial hypersurfaces are geodesically parallel). For k = 2, as a consequence of the nongravitational forces, the tempo of time is different in different parts of the body, but simultaneity (the possibility of synchronization) remains because of the absence of rotation. Simultaneity is not present for k = 3 and 4, and as a consequence of rotation space decomposes into disconnected local elements. It is obvious that the relativistic nonholonomy of space (rotation) and time (acceleration) may create a rather exotic picture of the extent and history of the reference body (the universe).

From the foregoing meaning of the quantities f^{λ} and $\omega_{\mu\nu}$ it follows that they must appear in connection with the commutation of chronometrically invariant derivatives,[3] i.e., derivatives in the directions Θ and Σ. If the vector field $u^{\lambda}(x)$ defines the time Θ, then the quantity $a_{\alpha}^{\beta} = u_{\alpha}u^{\beta}$ is the projector[8] on Θ, and $b_{\alpha}^{\beta} = g_{\alpha}^{\beta} - u_{\alpha}u^{\beta}$ is the projector on Σ (it is obvious that $g_{\mu\nu} = a_{\mu\nu} + b_{\mu\nu}$). Therefore, the chronometrically invariant derivatives have the form $a_{\alpha}^{\beta}\partial_{\beta}$, $b_{\alpha}^{\beta}\partial_{\beta}$ (compare with[5,7]). If we choose the time coordinate y^0 along Γ and the spatial coordinates y^i co-moving with the reference body, coinciding with Lagrangian variables, and numbering $\gamma \in \Gamma$ (we call these the internal coordinates for the reference frame), we obtain (the notation is obvious):[3]

$$u^{\alpha} = \{g_{00}{}^{-1/2}, 0, 0, 0\}, \quad u_{\alpha} = \{g_{00}{}^{1/2}, 0, 0, 0\},$$
$$X_0 = {}^*\partial_0 = d/d\tau = g_{00}{}^{-1/2}\partial_0,$$
$$X_i = {}^*\partial_i = \partial_i - g_{0i}g_{00}{}^{-1}\partial_0,$$
$$[X_0X_i] = f_iX_0, \quad [X_iX_k] = 2\omega_{ik}X_0,$$
$${}^*\partial_0\omega_{ik} + {}^*\partial_{[i}f_{k]} = 0,$$
$${}^*\partial_{[i}\omega_{jk]} + f_{[i}\omega_{jk]} = 0.$$

We use here the Poisson brackets and the Jacobi identities. We also note the following equations:[3]

$$d_{ik} = -{}^1/{}_2{}^*\partial_0 b_{ik}, \quad d^{ik} = {}^1/{}_2{}^*\partial_0 b^{ik},$$
$$b^{ik} = g^{ik}, \quad b_{ik} = g_{ik} - g_{0i}g_{0k}g_{00}{}^{-1}.$$

It is obvious that the spatial metric coincides with the one given in[9].

4. THE CURVATURE OF THREE-DIMENSIONAL PHYSICAL SPACE

A parallel displacement in the space Σ (covariant differentiation) is determined by the inducement of a parallel displacement[5] in space-time. Thus, if a vector undergoes a parallel displacement in the sense of the external metric $g_{\mu\nu}$, then its projection on Σ is regarded as a parallel displacement in Σ in the sense of the spatial metric $b_{\mu\nu}$. If we systematically develop the concept of an intrinsic spatial geometry, then we take the starting point to be a spatial metric and construct from it, with the aid of extension of the manifold[1], the metric of the nearest holonomic manifold containing the given nonholonomic manifold, i.e., in the present case the metric $g_{\mu\nu}$ (here holonomicity is to be understood in the narrow sense of integrability of the corresponding exterior form, that is, in the sense of exterior torsion), which is induced independently of the displacement defined in Σ. The following equations hold in the internal coordinates:[1,3,5,7]

$${}^*\nabla_iQ_j = {}^*\partial_iQ_j - {}^*\Delta_{ij}{}^kQ_k, \quad Q_j \in \Sigma,$$
$${}^*\Delta_{ij}{}^k = {}^*\Delta_{ji}{}^k = {}^1/{}_2 b^{kl}({}^*\partial_i b_{jl} + {}^*\partial_j b_{il} - {}^*\partial_l b_{ij}).$$

The curvature tensor of the three-dimensional physical space Σ should be defined so that its vanishing will give an integrable connectivity, i.e., the result of a parallel displacement is independent of the path. Here it is understood that from a given local space we always return to it, and not simply to one and the same world line.

The Schouten[1,5,7] tensor for the distribution V_4^3

$$K_{ijk}{}^l = 2{}^*\partial_{[i}{}^*\Delta_{j]k}^{l} + 2{}^*\Delta_{m[i}^{l}{}^*\Delta_{j]k}^{m} - 2M_{ij}{}^{\rho}\Lambda_{\rho k}^{l},$$
$$(\Lambda_{\rho k}^{l} = 2b_k^{\mu}a_{\rho}^{\nu}\partial_{[\mu}b_{\nu]}^{l}, \quad b_k^{\mu} = \partial x^{\mu}/\partial y^k, \quad b_k^{\mu}b_{\mu}^{l} = \delta_k),$$

which appears in the commutator

$$2{}^*\nabla_{[i}{}^*\nabla_{j]}v^l = 2M_{ij}{}^{\rho}D_{\rho}'v^l + K_{ijk}{}^lv^k$$

(here $D_{\rho}'v^l = a_{\rho}^{\sigma}\partial_{\sigma}v^l + \Lambda_{\rho b}^{l}v^b$ is the covariant derivative with respect to time), is the curvature tensor of Σ only in the trivial case of its holonomy ($M_{\mu\nu}^{\lambda} = 0$)—only then do the induced (chronometrically invariant) derivatives commute when the Schouten tensor vanishes.

The nonholonomic curvature tensor Σ will be taken to mean the following tensor, which differs from the Wagner curvature tensor[1] for nonholonomic distributions of the type V_n^{n-1} only in details (we obtain the Wagner tensor for $\omega^2 = 1$):

$$L_{\alpha\beta\gamma}{}^{\delta} = b_{\gamma\nu}^{\mu\delta}R_{\alpha\beta\mu}{}^{\nu} = \partial_{[\alpha}\widetilde{\Gamma}_{\beta]\gamma}^{\delta} + 2\widetilde{\Gamma}_{\rho[\alpha}^{\sigma}\widetilde{\Gamma}_{\beta]\gamma}^{\rho},$$
$$\widetilde{\Gamma}_{\alpha\beta}^{\gamma} = b_{\alpha\beta k}^{ij\gamma}{}^*\Delta_{ij}^{k} + a_{\alpha}^{\rho}b_{\beta k}^{i\gamma}\Lambda_{\rho i}^{k} + b_{\beta k}^{i\gamma}K_{\alpha i}^{k},$$
$$K_{\alpha j}{}^k = {}^1/{}_4 M^{il}{}_{\alpha}K_{ilj}{}^k\omega^{-2}, \quad 2\omega^2 = \omega_{\mu\nu}\omega^{\mu\nu}.$$

The curvature tensor L depends on the third derivatives of the spatial three-dimensional metric—the Schouten tensor for Σ enters in the Christoffel symbols $\widetilde{\Gamma}$. This complication was necessary in order to develop the concept of the intrinsic geometry of a single space and to build it up in the case of nonholonomy to the holonomic enveloping manifold with the aid of the holonomy tensor of the space itself. The time here is included as a factor in the formation of the purely spatial geometry—as the possibility of space having an exterior torsion. If we go, on the other hand, from space-time to space and take the initial four-dimensional metric $g_{\mu\nu}$, then we obtain in the tensor L only its second derivatives. The increase in the order of the derivatives occurs when the manifold is extended to the outside and its dimension is increased, but not when it is decreased, nor by going over to imbedded manifolds which, in the presence of exterior torsion, serve as certain projections of the curvature tensor of the exterior manifold.

Thus, the Wagner curvature tensor of a nonholonomic manifold (space) is essentially always determined in terms of the Riemannian tensor of a holonomic manifold, which coincides with the given nonholonomic manifold or encloses it. Here we did not obtain the Riemann tensor itself for space-time because only the displacement of purely spatial vectors was taken.

The intrinsic curvature of space manifests itself, for example, in the fact that the sum of the angles of a spatial triangle which is closed in space-time is not equal to π. In order to obtain the integral volume characteristics (total mass, energy, etc.) after going from space to space-time, we may take as the region of integration a set of local spaces on a spacelike hypersurface, i.e., a quasi-hypersurface. The integral characteristics depend here on the choice of the hypersurface and on the reference frame, i.e., they are functionals, and they have the unique meaning of conserved quantities only when independent of the choice of the hypersurface.

Usually the four-dimensional point of view is assumed at once, thereby avoiding those difficulties which arise in connection with a consistent development of the three-plus-one-dimensional point of view.

5. COMPARISON OF GRAVITATIONAL AND INERTIAL FIELDS

In connection with the transition from space-time to space and time, we obtain instead of the intrinsic curvature (R) of space-time: 1) the interior curvature (L) of space, 2) the exterior torsion (M) of space, 3) the exterior curvature (N) of space, and 4) the exterior curvature (f) of time. In this connection the three forms of exterior nonholonomy (M, N, f) have a dynamical interpretation: At the three-dimensional one-dimensional level, geometry appears as a physical process. One can partly determine the dynamical meaning of the interior curvature in terms of its relation to the three remaining quantities: in vacuum the latter determine the curvature (and the entire picture) completely. We will now be interested in the combinations in which the nonholonomy characteristics may be encountered.

In the flat Minkowski world R_4 the gravitational field, identical to the world tensor of Riemann, is not present, and the decomposition

$$TR_{\iota} = \Theta + \Sigma$$

determines the purely inertial field of a given reference frame, which is thus a property of the reference frame, and it simultaneously defines the effect of the nonholonomy of its space and time. Here the characteristics of nonholonomy counterbalance each other and vanish at the four-dimensional level. The four-dimensional curvature violates the cited equilibrium and transforms a purely inertial field into a gravitational-inertial field. To the extent that a gravitational field is observed at the three-plus-one-dimensional level in terms of the same nonholonomy characteristics of space and time as in the case of a purely inertial field, a certain qualitative (not local) equivalence of gravitation and inertia appears—even though the field of gravitation, in contrast to the field of inertia, is not at all a property of the reference frame, and even though these fields are different in principle. For the electromagnetic field such an equivalence, for example, is not present: charge is a less universal characteristic than mass, and it is still impossible to describe the behavior of charges in the universal language of the geometrical properties of space and time.

From the expression for the curvature tensor of the space Σ it follows that in the Minkowski world and in the presence of rotation of the reference frame, its space has zero curvature—such as, for example, the space of a rotating disk. Here the deviation of the geometry of space from Euclidean is caused by its exterior torsion (exterior curvature is absent for steady-state rotation). In the absence of rotation and in the presence of deformation of the reference body, the interior curvature of space does not vanish, being represented in this case by a family of hypersurfaces (with tangential hyperplanes). Thus, in the Minkowski world the combination LM of interior curvature and exterior torsion of space is excluded (i.e., also the combinations LMN, LMf, and LMNf). The gravitational field includes this combination among the admissible ones, but on the other hand it excludes inertial reference frames. Consequently the following assertion is valid (the principle of three-plus-one-dimensional nonholonomy): a gravitational field shifts the spectrum (list) of the possible combinations of the nonholonomy characteristics of the space and time of the reference frame—namely, the exterior torsion and the interior curvature of space, the exterior curvature of time and space. Namely, a gravitational field excludes their simultaneous total absence and, by removing the prohibition to combine exterior torsion with the interior curvature of space, it admits the simultaneous total presence of all four characteristics.

Simply speaking, in an inertial field all the characteristics of nonholonomy may be absent, and in a gravitational field—they may be present, but not vice versa. The presence of a gravitational field manifests itself in this shift ($12 \to 15$) of the beginning and end of the list, which was compiled in a natural manner, of the admissible (out of 16) combinations of the four kinds of nonholonomy. At the four-dimensional level, the shift is trivial, namely the appearance of a single characteristic ($0 \to 1$).

In conclusion let us present the useful formulas of Gauss, Codazzi, and Ricci for Σ,[7,10] enabling us in the spatial metric to find the projections of the world curvature tensor which vanish for all of its essential components, and thereby to judge the curvature of space-time. We write down these formulas in terms of the internal coordinates (y^0, y^i) in that form which was derived by Zel'manov (private communication):

$$^{*}\partial_0 d_{ij} - (d_i^{\;k} + \omega_i^{\;k})(d_{jk} + \omega_{jk}) + {}^1/_2({}^{*}\nabla_i f_j + {}^{*}\nabla_j f_i) - f_i f_j = X_{ij} = b_{ij}^{\alpha\gamma} u^{\beta} u^{\delta} R_{\alpha\beta\gamma\delta},$$

$$^{*}\nabla_j(d_{ik} + \omega_{ik}) - {}^{*}\nabla_i(d_{jk} + \omega_{jk}) - 2f_k\omega_{ij} = Y_{kij} = b_{kij}^{\gamma\alpha\beta} u^{\delta} R_{\alpha\beta\gamma\delta},$$

$$-H_{ijkl} + 2\omega_{kl}(d_{ij} + \omega_{ij}) + (d_{ki} + \omega_{ki})(d_{lj} + \omega_{lj})$$

$$-(d_{kj} + \omega_{kj})(d_{li} + \omega_{li}) = Z_{ijkl} = b_{ijkl}^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$$

$$(H_{ijk}^{\;\;\;l} \equiv 2^{*}\partial_{[i}{}^{*}\Delta_{j\,]k}^{l} + 2^{*}\Delta_{m[i}^{l}{}^{*}\Delta_{j\,]k}^{m}).$$

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