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Spin Waves in Complex Magnetic Structures

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The elementary excitation spectra of a system containing p magnetic sublattices are systematized. The possibility of existence of such substances was discussed in [1]. The effect of a magnetic field and also of usual Heisenberg interaction on the spin-wave spectrum is considered.

1. MAGNETIC STRUCTURES CONNECTED WITH THE ORBITAL ANGULAR MOMENTUM

In our preceding paper[1] we have shown that allowance for the orbital magnetism leads to complicated magnetic structures with several sublattices. A simple example of substances with non-quenched orbital angular momentum are compounds of rare-earth elements. In view of the strong spin-orbit coupling, the f-shell electrons in rare-earth element atoms are described by the total angular momentum. One should expect at least fourfold degeneracy of the d-shell electrons to remain also in transition-metal compounds, where the crystal fields are large compared with the spin-orbit coupling (see [1]).

In [1] we considered the ground state of a crystal consisting of atoms with multiply-degenerate electronic levels. It was assumed that such a system is satisfactorily described by the Hubbard Hamiltonian[2] and that there is one electron for each atom. It was also assumed that the characteristic "kinetic" energy of the electron in the band is small compared with the Coulomb energy of repulsion of two electrons by one atom. The latter assumption leads to a strong localization of the electronic states and to the appearance of a gap in the electron spectrum (dielectric).

Under these assumptions, the problem of the ground and weakly-excited states reduces to an investigation of the effective Hamiltonian[1], where

$$H_0 = g\sum_{\mathbf{r},\mathbf{a}} P_{\mathbf{r},\mathbf{r}+\mathbf{a}}, \tag{1}$$

$P_{\mathbf{r},\mathbf{r}'}$ is an operator representing the states at the sites $\mathbf{r}$ and $\mathbf{r}'$. Summation with respect to $\mathbf{r}$ is over the crystal sites and $\mathbf{a}$ are the basis vectors of the lattice.

The ground state of the system is constructed as a superposition of the vectors $\Lambda(\lambda_1, \lambda_2, \ldots, \lambda_N)$, where λ_i denotes the number of the electronic state at the i-th site. The vectors Λ form a vector space Λ with dimensionality p^N, where p is the number of degenerate electronic states on the atom ($\lambda_i = 1, 2, \ldots, p$) and N is the total number of atoms.

The problem is similar in many respects to the investigation of the ground state of an antiferromagnet (for $p = 2$, the Hamiltonian (1) coincides with the ordinary Hamiltonian of an antiferromagnet). However, the simplest Hartree approximation used in the theory of antiferromagnetism is not suitable in the case $p > 2$. In the Hartree approximation, the ground state of the crystal with Hamiltonian (1) is any state in which the electronic states of the neighboring states are orthogonal. Thus, in this approximation the ground state is degenerate with a degeneracy degree of the order of p^N.

In [1] we proposed a certain generalization of the Hartree method. The crystal was subdivided into elementary cells containing several atoms each (for example, into squares made up of four atoms in the case of a two-dimensional quadratic cell, into cubes made up of eight atoms in a primitive cubic cell, etc.).

The wave function of the crystal was represented in the form of a product of wave functions of all such cells. It was shown that in the ground state there are produced p identical sublattices corresponding to different values of λ. The sublattices are so arranged that identical states are at maximal distances. The ground state defined in this manner is still degenerate, but its entropy does not depend on the volume (the degree of degeneracy is p!).

2. SPIN WAVES AS ROTATION IN Λ SPACE

The Hamiltonian (1) is invariant against uniform rotation, in all the sites, of the p-dimensional complex vectors of the electronic states φ_λ. Therefore, if such an operation is performed on the ground state, its energy remains unchanged. Obviously, when the rotation parameters change slowly from site to site, the energy of the state differs little from that of the ground state. For the stationary state it is natural to expect the rotation of the variation parameters to have the character of a plane wave whose energy tends to zero together with the wave vector. Such oscillations will be called arbitrarily acoustic spin waves (ASW) in analogy with an ordinary antiferromagnet.

We shall now classify the ASW.

The electronic states at the sites are transformed under an infinitesimally small rotation in accordance with the law

$$\varphi_\lambda \to \varphi_\lambda + \varepsilon_{\lambda,\lambda'}\varphi_{\lambda'}, \tag{2}$$

where $\|\epsilon_{\lambda,\lambda'}\|$ is a complex antisymmetrical matrix. Any rotation in p-dimensional space can be represented in the form of rotations in $p(p-1)/2$ mutually orthogonal planes. Since each complex $\epsilon_{\lambda,\lambda'}$ is determined by two parameters, we have a total of $p(p-1)$ independent parameters determining the rotation. Consequently, there exist $p(p-1)$ independent ASW for each wave vector **q**. These p(p − 1) ASW branches differ from one another in their "polarizations." It is natural to classify the spin waves in accordance with the planes in which the rotation takes place. We shall thus speak of ASW(1, 2), (1, 3), etc. A plane does not define the ASW uniquely: there exist two spin waves degenerate in energy and corresponding, for example, to real and imaginary ϵ. Any other infinitesimally small rotation $\hat{\epsilon}$ can be represented as a superposition of rotations in the coordinate planes:

$$\hat{\varepsilon}\varphi = (1 + \varepsilon_{ik} I_{ik})\varphi, \quad ((i, k) = (1, 2), (1, 3), \ldots), \tag{3}$$

where I_{ik} are the generators of the rotation in p-dimensional space.

One more principle of excitation classification is connected with the integrals of motion of the Hamiltonian (1), namely the total numbers N_λ of the electrons in the state λ. In the ground state all the N_λ are equal to each other. The lowest excited state, say in the (1, 2) plane, is characterized by the smallest possible (but not zero) value, $|N_1 - N_2| = 2$. A state with a large degree of excitation is characterized either by $|N_1 - N_2| > 2$, or by simultaneous inequality of several N_λ to one another.

In the present paper we confine ourselves to elementary ASW, which we take to mean the following: if an ASW is excited in the (m, m′) plane then $|N_m - N_{m'}| = 2$, and all other $N_\lambda = N/p$.

3. ASW DISPERSION

We shall show that, just as in an antiferromagnet, the frequency of the ASW in the absence of a magnetic field depends linearly on the wave vector. The difference from an antiferromagnet consists in the fact that anisotropy of the spectrum with respect to the directions **q** sets in. In an antiferromagnet p = 2 and only ASW of one type (1, 2) are possible. In complex magnets, differently polarized ASW have different properties with respect to the crystallographic axes, and this gives rise to the anisotropy.

To calculate the excitation spectrum it is necessary to know the wave function Φ_0 of the ground state. In [1] we found the ground state by a variational method. But to elucidate the general properties of the ASW spectra we need not a concrete knowledge of Φ_0, but only some general information concerning the ground state. We shall assume that sublattices exist. This means, in particular, that the probability $n_{\gamma\mathbf{r}}$ of finding the state γ at the site **r** of the sublattice numbered γ is larger than the probability $n_{\gamma'\mathbf{r}}$, with $\gamma' \neq \gamma$. It is natural to write Φ_0 in the representation of the vectors Λ:

$$\Phi_0 = \sum C(\Lambda)\Lambda. \tag{4}$$

We seek the excited states of the system in the form

$$\overline{\Phi} = \hat{S}\Phi_0, \tag{5}$$

$$\hat{S} = \prod_{\mathbf{r}} \hat{s}(\varepsilon_{\mathbf{r}}). \tag{6}$$

The product is taken over the lattice sites. Here $\hat{s}(\epsilon)$ is the operator of rotation through an angle ϵ, say, in the (m, m′) plane. Accurate to second order in ϵ we have

$$\hat{s}(\varepsilon) = (1 - {}^1\!/_2|\varepsilon|^2) + (\varepsilon\sigma), \tag{7}$$

where $(\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}) = \epsilon_1\sigma_x + \epsilon_2\sigma_y$, $\epsilon = \epsilon_1 + i\epsilon_2$, and σ_x and σ_y are Pauli matrices. To simplify matters, we shall henceforth assume ϵ to be real.

The problem of determining the ASW spectrum is equivalent to minimization of the following form, which is quadratic in ϵ,

$$\langle\Phi_0\hat{S}^{-1}H_0\hat{S}\Phi_0\rangle - S\langle\Phi_0\hat{S}^{-1}(N_m - N_{m'})\hat{S}\Phi_0\rangle. \tag{8}$$

The Lagrangian multiplier E has the meaning of excitation energy.

Let us consider the first term of (8). Confining ourselves to terms not higher than of second order in ϵ, we obtain

$$\langle\Phi_0\hat{S}^{-1}H_0\hat{S}\Phi_0\rangle = E_0 + \sum_{\mathbf{r},\mathbf{r}'} K_{\mathbf{r},\mathbf{r}'}(\varepsilon_{\mathbf{r}}^2 + \varepsilon_{\mathbf{r}'}^2 - 2\varepsilon_{\mathbf{r}}\varepsilon_{\mathbf{r}'}), \tag{9}$$

where

$$K_{\mathbf{r},\mathbf{r}'} = -2\mathscr{H}_{\mathbf{r},\mathbf{r}'} + \sum_{\mathbf{r}''}\mathscr{H}_{\mathbf{r},\mathbf{r}''}\delta_{\mathbf{r},\mathbf{r}'} + \sum_{\mathbf{r}''}\mathscr{H}_{\mathbf{r}'',\mathbf{r}'}\delta_{\mathbf{r},\mathbf{r}'} \tag{10}$$

$$\mathscr{H}_{\mathbf{r},\mathbf{r}'} = g\{-{}^1\!/_2\langle\Phi_0 P_{\mathbf{r},\mathbf{r}'}(\hat{n}_{m\mathbf{r}} + \hat{n}_{m\mathbf{r}'} + \hat{n}_{m'\mathbf{r}'} + \hat{n}_{m'\mathbf{r}})\Phi_0\rangle + \langle\Phi_0(\hat{n}_{m\mathbf{r}}\hat{n}_{m'\mathbf{r}'} + \hat{n}_{m'\mathbf{r}}\hat{n}_{m\mathbf{r}'})\Phi_0\rangle\},$$

$\hat{n}_{m\mathbf{r}}$ and $\hat{n}_{m'\mathbf{r}}$ are the operators of the occupation numbers of the states m and m′ on the site **r**.

From the invariance of the Hamiltonian to a state rotation that is uniform over the entire crystal, it follows that

$$\sum_{\mathbf{r}'} K_{\mathbf{r},\mathbf{r}'} = 0. \tag{11}$$

It will be convenient in what follows to specify the notation in greater detail. We divide the crystallographic lattice into identical cells of volume p (see, for example, Fig. 1 for p = 4). We introduce the vector index R of the cell and the number γ of the sublattice ($\gamma = 1, 2, \ldots, p$). These two quantities defines the lattice vector **r** completely. We shall therefore use instead of $K_{\mathbf{r},\mathbf{r}'}$ the symbol $K_{\mathbf{R},\mathbf{R}'}^{\gamma,\gamma'}$. By virtue of the homogeneity, it is obvious that $K_{\mathbf{R},\mathbf{R}'}^{\gamma,\gamma'}$ depends only on the difference R − R′. Relation (11) takes the form

$$\sum_{\gamma',\mathbf{R}'} K_{\mathbf{R},\mathbf{R}'}^{\gamma,\gamma'} = 0. \tag{12}$$

Let us consider now the second term of the quadratic form (8). Using (6), we get

$$\langle\Phi_0\hat{S}^{-1}(N_m - N_{m'})\hat{S}\Phi_0\rangle = \sum_{\mathbf{r}}\langle\Phi_0\hat{s}_{\mathbf{r}}^{-1}(\hat{n}_{m\mathbf{r}} - \hat{n}_{m'\mathbf{r}})\hat{s}_{\mathbf{r}}\Phi_0\rangle. \tag{13}$$

In the representation of the operators of the occupation

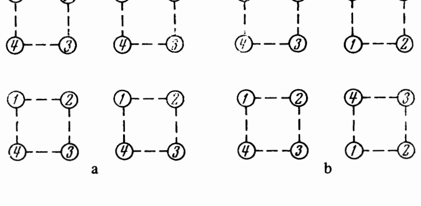

FIG. 1. Basis vectors of sublattices: a–(2, 0), (0, 2); b–(2, −1), (2, 1).

numbers of the states m and m′, formula (13) becomes

$$\langle\Phi_0\hat{S}^{-1}(N_m-N_{m'})\hat{S}\Phi_0\rangle=\sum_{\mathbf{r}}(1-2\varepsilon_{\mathbf{r}}^2)\langle\Phi_0(\hat{n}_{m\mathbf{r}}-\hat{n}_{m'\mathbf{r}})\Phi_0\rangle. \qquad (14)$$

In the derivation of (13) we used the fact that Φ_0 is an eigenfunction of the operators $\hat{N}_\lambda$ ($\hat{N}_\lambda\Phi_0 = N\Phi_0/p$), and therefore terms proportional to ϵ do not appear in (14).

It is obvious the coefficients of ϵ_r^2 in (14) have different signs for the sites of the sublattices m and m′, since $\langle\Phi_0\hat{n}_{mr}\Phi_0\rangle$ have the meaning of the probability of the state m at the site r in the ground state. The remaining sublattices can be broken up into pairs, such that the first component of the pair is located relative to sublattice m in the same manner as the second in the sublattice m′. Such pairs will be called conjugate. For these pairs, the coefficients of ϵ_r^2 also have opposite signs. Finally, we can encounter sublattices that are symmetrical with respect to m and m′. For such sublattices, the coefficient of ϵ_r^2 is equal to zero. We shall call such sublattices self-conjugate.

Thus, we can break up the entire aggregate of sublattices into three groups, the second of which contains sublattices conjugate to those of the first group, and the third self-conjugate sublattices. The indices of the conjugate sublattices will be the same as those of the first group (α, β), but primed. The self-conjugate sublattices will be assigned other indices (μ, ν). The index γ will be used to number all the sublattices.

In the assumed notation, the conditions for the minimum of the quadratic form (8) take the form

$$\sum_{\gamma,\mathbf{R}_2}K_{\mathbf{R}_1\mathbf{R}_2}^{\alpha\gamma}\varepsilon_{\mathbf{R}_2}{}^{\gamma}+EL_\alpha\varepsilon_{\mathbf{R}_1}{}^{\alpha}=0,$$

$$\sum_{\gamma,\mathbf{R}_2}K_{\mathbf{R}_1\mathbf{R}_2}^{\alpha'\gamma}\varepsilon_{\mathbf{R}_2}{}^{\gamma}-EL_\alpha\varepsilon_{\mathbf{R}_1}{}^{\alpha'}=0,$$

$$\sum_{\gamma,\mathbf{R}_2}K_{\mathbf{R}_1\mathbf{R}_2}^{\mu\gamma}\varepsilon_{\mathbf{R}_2}{}^{\gamma}=0. \qquad (15)$$

After changing over to the Fourier representation with respect to R, we obtain the following dispersion equation:

$$\begin{vmatrix} K^{\alpha\beta}(\mathbf{q})+EL_\alpha\delta_{\alpha\beta} & K^{\alpha\beta'}(\mathbf{q}) & K^{\alpha\nu}(\mathbf{q}) \\ K^{\alpha'\beta}(\mathbf{q}) & K^{\alpha'\beta'}(\mathbf{q})-EL_\alpha\delta_{\alpha'\beta'} & K^{\alpha'\gamma}(\mathbf{q}) \\ K^{\mu\beta}(\mathbf{q}) & K^{\mu\beta'}(\mathbf{q}) & K^{\mu\nu}(\mathbf{q}) \end{vmatrix}=0, \qquad (16)$$

where each of the elements denotes a quadratic block consisting of the Fourier components of the matrix K (with the exception of obvious changes in the diagonal blocks).

We note that at **q** = 0 and E = 0 the determinant (16) vanishes identically by virtue of the condition (12).

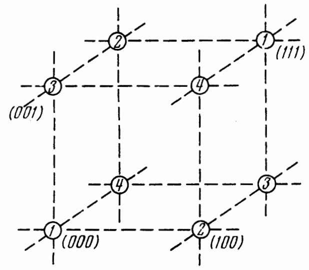

FIG. 2. Basis vectors of sublattices: (1, 1, 1), (-1, 1, 1) and (-1, -1, 1).

Let us show that (16) is a polynomial in E^2. Indeed, let us interchange the first and second rows of the determinant (2) and then the first and second columns. If we recognize that $K^{\alpha'\beta'} = K^{\alpha\beta}$, $K^{\alpha\beta'} = K^{\alpha'\beta}$, and $K^{\alpha\mu} = K^{\alpha'\mu}$, then the resultant determinant differs from (16) only in that E is replaced by −E, thus proving our statement.

Near **q** = 0, all the matrix elements K(**q**) can be expanded in powers of q_i. The expansion begins with the quadratic term, since the matrix K is an invariant to the inversion of **q**. However, the quadratic form for each matrix element is generally speaking not isotropic. Thus, at small **q** we obtain from (16)

$$E^2=a_{ij}q_iq_j. \qquad (17)$$

As an illustration of the general relations, we present the results of calculation of the spectrum for the configurations shown in Figs. 1 and 2. We have used here the approximate ground-state function

$$\Psi_0=\prod_j\psi_j,$$

where the ψ_j stand for the wave functions of the elementary square or cube, which were calculated in [1].

We present the results for the configuration in Fig. 1a:

$$E_{(1,2)}(\mathbf{q})=E_{3,4}(\mathbf{q})\sim\sqrt{q_x^2+0.19q_y^2},$$
$$E_{(1,4)}(\mathbf{q})=E_{(2,3)}(\mathbf{q})\sim\sqrt{q_y^2+0.19q_x^2},$$
$$E_{(1,3)}(\mathbf{q})=E_{(2,4)}(\mathbf{q})\sim\sqrt{q_x^2+q_y^2}.$$

For the configuration of Fig. 1b we have

$$E_{(1,2)}(\mathbf{q})=E_{(3,4)}(\mathbf{q})=E_{(1,3)}(\mathbf{q})=E_{(2,4)}(\mathbf{q})\sim\sqrt{q_x^2+0.16q_y^2},$$
$$E_{(1,4)}(\mathbf{q})=E_{(2,3)}(\mathbf{q})\sim\sqrt{q_y^2+0.59q_x^2}.$$

Finally, for a primitive cubic lattice we have

$$E_{(1,2)}(\mathbf{q})=E_{(3,4)}(\mathbf{q})\sim\sqrt{q_x^2+0.25(q_y^2+q_z^2)},$$
$$E_{(1,4)}(\mathbf{q})=E_{(2,3)}(\mathbf{q})\sim\sqrt{q_y^2+0.25(q_z^2+q_x^2)},$$
$$E_{(1,3)}(\mathbf{q})=E_{(2,4)}(\mathbf{q})\sim\sqrt{q_z^2+0.25(q_x^2+q_y^2)}.$$

4. OPTICAL BRANCHES OF THE SPECTRUM

The local rotations described by (5) and (6) are the most general ones in Λ space. Therefore the dispersion equation (16), which follows only from the assumption concerning the form of the wave function (5), the variational principle (8), and the existence of sublattices, determines not only the ASW but also the entire aggregate of "spin" motions. It is natural to call the spectrum branches separated from the ground state by a gap "optical spin waves," in obvious analogy with the optical branches of crystal vibrations. The magnitude of this gap is of the order of g.

Let us find the number of optical spin branches. To this end we note that each value of **q**, for a given rotation plane (m, m′), corresponds to p − $\bar{p}$ (m, m′) solutions of Eq. (16). The symbol $\bar{p}$ (m, m′) has been introduced for the number of self-conjugate sublattices corresponding to rotation in the plane (m, m′). Two branches, as already shown, are acoustic. Summing over all possible rotation planes, we obtain

$$\sum_{(m,m')} (p - \bar{p}(m, m') - 2) = {}^1\!/_2 p(p-1)(p-2) - \sum_{(m,m')} \bar{p}(m, m'). \quad (18)$$

In particular, for an ordinary antiferromagnet p = 2, $\bar{p}$ = 0, and there are no optical branches.

The simplest quadratic lattice shown in Fig. 1a has the property that for rotation in the (1, 3) plane the sublattices 2 and 4 are self-conjugate. Of ourse, the same pertains also to sublattices 1 and 3 for rotation in the (2, 4) plane. Then, in accord with (18), there are eight optical branches.

For the configuration shown in Fig. 1b, there are no self-conjugate sublattices and the number of optical branches is 12.

Finally, for the cubic lattice (Fig. 2), for rotation in any plane, two self-conjugate sublattices arise. The number of optical branches in this case is equal to zero.

5. GROUND AND WEAKLY-EXCITED STATES IN A MAGNETIC FIELD

We shall describe the magnetic field by an increment to the Hamiltonian (1) of the operator

$$H_1 = h \sum_{\mathbf{r},\lambda} \lambda \hat{n}_{\lambda \mathbf{r}}. \quad (19)$$

It is easy to see that both H_0 and the total Hamiltonian $H_0 + H_1$ commute with the occupation numbers N_λ, which thus remain conserved in the magnetic field.

We denote by $E_0(N_1, \ldots, N_p)$ the minimum eigenvalue of the Hamiltonian H_0 for specified numbers N_λ. The absolute minimum of E_0 coincides with the energy of the ground state of the Hamiltonian H_0. We assume that the absolute minimum of E_0 corresponds to $N_1 = N_2 = \ldots = N_p = N/p$. This assumption is confirmed by a variational calculation (see [1]).

In a weak magnetic field, the equilibrium value of N_λ differs somewhat from N/p. We put

$$N_\lambda(h) = N/p + \delta N_\lambda. \quad (20)$$

We shall show that δN_λ depends linearly on the magnetic field. To this end, we expand E_0 near the absolute minimum in powers of δN_λ. We confine ourselves only to quadratic terms. In view of the complete symmetry of E_0 with respect to all N_λ, the increment δE_0 is of the form

$$\delta E_0 = \frac{a}{2} \sum_{\lambda=1}^{p} (\delta N_\lambda)^2 + b \sum_{\lambda \neq \lambda'} \delta N_\lambda \delta N_{\lambda'}. \quad (21)$$

The total energy E(h) of the system in a magnetic field is obviously

$$E(h) = E_0 + \delta E_0 + h \sum_{\lambda=1}^{p} \delta N_\lambda. \quad (22)$$

Minimizing this expression with respect to δN_λ, we find

$$\delta N_\lambda = -\lambda h/(a-b). \quad (23)$$

In the derivation of (23) we have changed from numbering the states from 1 to p to numbering from $-(p-1)/2$ to $(p-1)/2$.

We see from our result that the values of δN_λ averaged over the ground state in a magnetic field are proportional to h.

We consider now the dispersion of ASW in a weak magnetic field. Homogeneous rotation now no longer commutes with the total Hamiltonian $H_0 + H_1$. This means that the spectrum of the excitations of the magnetic field is separated from the ground state by a gap. Nonetheless, the wave function of the excited state will be determined by formula (5) and (6), if Φ_0 is taken to mean the wave function of the ground state in the magnetic field.

Let us see how Eq. (16) for the determination of the spectrum is altered. Just as before, we shall consider for concreteness rotation in the plane (m, m′). The elementary ASW will be defined as before by the condition $|N_m - N_{m'}| = 2$.

The dispersion equation from which we obtain the ASW spectrum has, with a slight exception, the form (16). Thus, we add to the first diagonal element of the determinant the term

$$\tilde{E}(L_\alpha + \delta n_{m\alpha} - \delta n_{m'\alpha}),$$

to the second

$$\tilde{E}(-L_\alpha + \delta n_{m\alpha'} - \delta n_{m'\alpha'})$$

and to the third

$$\tilde{E}(\delta n_{m\mu} - \delta n_{m'\mu}).$$

Here $\tilde{E} = E + {}^1\!/_2 (m - m')h$.

It is very important that the coefficients $K^{\gamma_1\gamma_2}(\mathbf{q})$, which are determined only by averages of the Hamiltonian H_0, satisfy the relation (12) as before, although each of the coefficients depends little on the magnetic field.

In fact, in the derivation of (12) we have used only two circumstances: first, the invariance of H_0 to homogeneous rotations in p-dimensional space, and second, the fact that Φ_0 is an eigenvector of the Hamiltonian H_0. Both assumptions remain valid also in a magnetic field.

It is now easy to obtain the form of the ASW spectrum in a magnetic field. The two branches, which were previously degenerate, now split and take the forms

$$E_1(\mathbf{q}) = \sqrt{a_{ij} q_i q_j + k^2 h^2} - kh + |{}^1\!/_2 (m - m')|h, \quad (24)$$

$$E_2(\mathbf{q}) = \sqrt{a_{ij} q_i q_j + k^2 h^2} + kh - |{}^1\!/_2 (m - m')|h. \quad (25)$$

Here k is a constant that depends on the redistribution of the occupation numbers over the sublattices and is bounded by the following inequality:

$$2k \geqslant |{}^1\!/_2 (m - m')|.$$

From (24) and (25) we see that

$$\omega_1 = h|{}^1\!/_2 (m - m')| \quad (26)$$

and

$$\omega_2 = h(2k - |{}^1\!/_2 (m - m')|) \quad (27)$$

are the antiferromagnetic-resonance frequencies.

6. ASW SPECTRUM WITH ALLOWANCE FOR EXCHANGE INTERACTION

In real compounds of rare-earth and transition elements there exists, in addition to the interaction described by the Hamiltonian (1), also the usual exchange interaction.

If the intraatomic spin-orbit coupling is large, the electrons are described by a total angular momentum **J**. Therefore the exchange interaction, which must be added to H_0, is of the form

$$H_j = -\sum_{\mathbf{r},\mathbf{r}'} I_j(\mathbf{r}-\mathbf{r}')\mathbf{J}_\mathbf{r}\mathbf{J}_{\mathbf{r}'}. \tag{28}$$

If the interaction (28) is small compared with H_0 (we shall consider only this case), then the ground state of the system differs little from the ground state of the Hamiltonian (1). As before, p sublattices will remain.

We have noted earlier that the operators of the total occupation numbers N_λ commute with H_0 and are therefore integrals of motion. These operators do not commute with the Hamiltonian H_j. Therefore instead of p independent integrals of motion there remains only one conserved quantity, connected with the operators of the total occupation numbers in the state λ:

$$J^z = \sum_{\lambda=-(p-1)/2}^{(p-1)/2} \lambda \hat{N}_\lambda. \tag{29}$$

In formula (29), the sum over λ means summation over all the projections of the total angular momentum J on the z axis.

The existence of the integral of motion (29) causes the elementary ASW to be characterized now by the smallest possible value $|\delta J^z| = 1$.

We have already shown that there exist $p(p-1)/2$ independent homogeneous rotations that leave the Hamiltonian (1) invariant. The consequence of this was the existence of $p(p-1)$ ASW branches. Not all these rotations, however, commute with H_j. Only one superposition of rotations in the coordinate planes leaves the Hamiltonian $H_0 + H_j$ invariant. Accurate to terms of second order in ϵ, this rotation is written in the form

$$\hat{s}(\varepsilon_\mathbf{r}) = 1 + (\varepsilon_\mathbf{r} J_\mathbf{r}^- - \varepsilon_\mathbf{r}^* \mathbf{J}_\mathbf{r}^+) + {}^1/_2(\varepsilon_\mathbf{r} J_\mathbf{r}^- - \varepsilon_\mathbf{r}^* \mathbf{J}_\mathbf{r}^+)^2. \tag{30}$$

The rotation $\hat{S} = \prod_\mathbf{r} \hat{s}(\epsilon_\mathbf{r})$, which is inhomogeneous over the crystal and corresponds to $|\delta J^z| = 1$, defines, together with $\epsilon_{\gamma\mathbf{r}} \sim e^{i\mathbf{q}\cdot\mathbf{r}}$, two branches of the spectrum. There remain, however, $p(p-1)/2 - 1$ independent rotations. Since these rotations no longer commute with the Hamiltonian $H_0 + H_j$, a gap will appear in the spectrum of the spin waves that correspond to these rotations. The value of the gap Δ is obviously proportional to $|\sum_\mathbf{r} I_j(\mathbf{r})|$ Such branches of the spectrum (their number is $p(p-1) - 2$) can be called "weakly optical" since the energy gap in the spectrum of the ordinary spin waves is of the order of g, and is much larger than Δ in our case.

In transition-metal compounds, the spin-orbit interaction is small and the exchange is due to the interaction of the spins of the magnetic atoms. The total Hamiltonian of the system is made up of H_0 and H_s:

$$H_s = -\sum_{\mathbf{r},\mathbf{r}'} I_s(\mathbf{r}-\mathbf{r}')\mathbf{S}_\mathbf{r}\mathbf{S}_{\mathbf{r}'}. \tag{31}$$

It was shown in [1] that the exchange constants in (1) and (31) are of the same order. Therefore, if $I_s > 0$, the spins have ferromagnetic ordering, and the orbital angular momenta form several sublattices. The number is equal to $2L + 1$ if the crystal fields are small, and at least to two in the opposite case. In such compounds, the systems of orbital and spin in momenta separate. The spectrum of the elementary excitations is therefore as follows: there is one spin-connected branch proportional to q^2, and $p(p-1)$ branches connected with the orbital angular momenta. At small wavelengths, the latter are linear in q.

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268