Description of Lepton-Hadron Reactions in Quantum Field Theory

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Deeply inelastic lepton-hadron reactions are studied within the framework of renormalized quantum field theory. Theories with zero charges are considered in the first part of the paper. Calculations of the reaction amplitudes are carried out completely in the major logarithmic approximation. It is also shown that the determination of the amplitudes reduces to a determination of the dependence of the vertex parts of a certain set of operators on the cutoff radius. Renorm-group equations are derived for such vertex parts and the possible form of the equation solutions is discussed.

1. INTRODUCTION

IN recent experiments[1] there have been extensive studies of lepton-hadron interactions at high energies; these studies have made it possible to establish the distributions of the charges and currents inside the nucleus at distances much shorter than its Compton radius.

There is no consistent theory of the phenomena occurring at such distances at present, and all the theoretical work in this direction consisted mainly of constructing phenomenological models[2].

In the present paper we attempt to predict definite properties of lepton-hadron reactions on the basis of the renormalized quantum field theory. Although the scattering amplitudes cannot be calculated completely in the theory with strong coupling, it is possible to obtain an experimentally verifiable infinite set of sum rules. Experiment will show whether the field theory in the existing form is suitable for the description of strong interactions.

The present article, which is the first part of our work, contains mainly the methodological results. We consider the theory of spin-1/2 particles interacting in the usual manner with pseudoscalar particles. At large nonrenormalized coupling constants, a null-charge exists in the theory, and the effective coupling is weak[3]. We consider this situation since, first, in this case the current distributions of interest to us can be calculated completely and, second, certain properties of amplitudes with weak coupling can be extended to the strong-coupling theory.

In Sec. 2 we calculate the amplitude of the strongly virtual Compton effect at zero angle in the principal logarithmic approximation, and show that in this case there is no Bjorken scaling[4].

In Sec. 3 we analyze the result and show that to calculate the amplitude of interest to us it suffices to ascertain how the vertex parts of certain operators, which are irreducible tensors of rank j, depend on the cutoff radius. This problem is solved for the general case in Sec. 4, where the renorm-group equations are formulated for vertex parts and it is shown that in the case of strong coupling the dependence on the cutoff radius is given by formula (4.13). The obtained formulas, besides being applicable directly to our problem, make it possible to generalize and correct the Johnson-Bjorken-Low theorem[5].

The present paper is a continuation and generalization of an article[6] in which e^+e^- annihilation into hadrons is discussed.

2. INTERACTION OF REAL AND VIRTUAL PARTICLES IN A THEORY WITH WEAK COUPLING (THE NULL-CHARGE CASE)

In renormalized field theories, the perturbation-theory series for the amplitudes contain logarithmic divergences. These divergences satisfy a renormalization condition, consisting in the fact that when the cutoff radius is changed the sum of the series is multiplied by a certain factor. As noted by Gell-Mann and Low[4], this condition, depending on the magnitude of the bare charge, can lead to two possible types of interaction. First, interaction with zero charge is possible, as discussed by Landau and Pomeranchuk[8], and leads to the relation

$$g^2 \propto \ln^{-1}(\Lambda/m) \ll 1,$$

where Λ is the cutoff radius and m is the particle mass. The second possibility is a situation with a finite charge, which was recently discussed again in[8-10] and leads to scale invariance at small distances.

The structure of the theory with weak coupling is much simpler than in the case of a finite charge. We shall therefore investigate in this section the processes of interest to us just in this theory. Such an analysis is of methodological significance, since it paves the way for the analysis of strong coupling in the succeeding sections.

Assume that there is one sort of charged baryons ψ with mass m and one sort of neutral pseudoscalar mesons φ with mass μ, interacting in accordance with the law

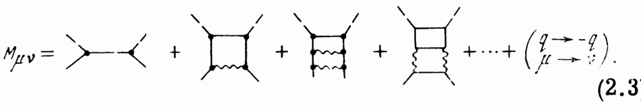
$$\mathscr{L}_{int} = ig_0\bar{\psi}\gamma_5\psi\varphi. \qquad (2.1)$$

The renormalized propagation function and the vertex part of the fields φ and ψ, D(p), S(p), and $\Gamma(p, q)$, and also the electromagnetic vertex $\Gamma_\mu(p, q)$, are given by

$$S(p) = \frac{1}{\hat{p}} s(\xi), \quad D(p) = \frac{1}{p^2} d(\xi),$$
$$\Gamma(q, p) = \gamma(\xi), \quad \Gamma_\mu(p, q) = \alpha(\xi)\gamma_\mu \qquad (2.2)$$

with $p^2 \gg m^2$ and $q \sim p$, where $\xi = \ln(q^2/m^2)$, and the functions s, d, γ, and α are determined in the Appendix in the principal-logarithm approximation[1], i.e., with allowance for terms of order $(g^2\xi)^n$ and neglecting terms of the type $g^2(g^2\xi)^n$.

Let us consider the problem of calculating the amplitude of the virtual Compton effect through zero angle at $q^2 \gg m^2$ (q^2 is the mass of the virtual quantum) in the logarithmic approximation. In this approximation, as will be shown below, we can confine ourselves to summation of ladder diagrams of the type



(2.3)

where the solid lines correspond to S, the wavy lines to D, and the points to the corresponding vertex parts of (2.2).

Let us find the rules for calculating the skeleton ladder diagrams. (Allowance for the vertex parts and the Green's functions is a relatively simple problem, which we shall consider later in the summation of the series (2.3).) The first diagram of (2.3) makes the contribution

$$M_{\mu\nu}{}^0 = \frac{1}{q^2+\nu}\alpha^2(\xi)s(\xi)f_{\mu\nu}(q,p),$$

($f_{\mu\nu}(q, p) = \mathrm{Sp}\,\gamma_\mu(\hat{q}+\hat{p})\gamma_\nu\hat{p}$, $\nu = 2pq$; averaging was carried out over the baryon spin).

It is convenient to consider next the skeleton diagrams.

The second diagram of (2.3) is given by the integral

$$M_{\mu\nu}^{(1)} = \frac{g_0{}^2}{(2\pi)^4}\int\frac{d^4k}{((p-k)^2+i0)^2(k^2+i0)((q+p-k)^2+i0)} \cdot \mathrm{Sp}(\gamma_5(\hat{p}-\hat{k})\gamma_\mu(\hat{p}+\hat{q}-\hat{k})\gamma_\nu(\hat{p}-\hat{k})\gamma_5\hat{p}). \tag{2.4}$$

We took into account the fact that the logarithmic contribution to (2.4) is due to $m^2 \sim \mu^2 \ll (p-k)^2 \sim k^2 \ll q^2$.

To calculate the integral (2.4) it is convenient to use the Sudakov variable[12] [1)]:

$$\begin{gathered} k = \alpha p' + \beta q' + k_\perp, \\ p' = p - m^2 q/\nu, \qquad q' = q - m^2 p/\nu, \\ d^4k = \nu d\alpha d\beta d^2k_\perp, \end{gathered} \tag{2.5}$$

$$\begin{gathered} (p-k)^2 + i0 = (\alpha-1)\beta\nu + k_\perp{}^2 + i0 + O(m^2), \\ k^2 + i0 = \alpha\beta\nu + k_\perp{}^2 + O(m^2) + i0, \\ (q+p-k)^2 + i0 = q^2 + \nu(1-\alpha) + i0 + O(k^2). \end{gathered}$$

In order for the singularities of the integral with respect to β to be on opposite sides of the real axis, it is necessary to satisfy the condition

$$0 < \alpha < 1. \tag{2.6}$$

The integral with respect to β then reduces to the residue at the pole of the denominator k^{-1}, i.e., at the point

$$\beta = k_\perp{}^2/\alpha\nu. \tag{2.7}$$

As a result we have

$$M_{\mu\nu}^{(1)} = \frac{g_0{}^2}{16\pi^2}\int\frac{d^2k_\perp}{k_\perp{}^4}\int\frac{\alpha\, d\alpha\, \mathrm{Sp}\{\ldots\}}{q^2+\nu(1-\alpha)}. \tag{2.8}$$

In calculating the trace it must be taken into account that in the essential region it turns out that $\alpha \sim 1$, $\beta \ll 1$, $m^2 \ll k_\perp^2 \ll q^2$. We can therefore write

$$\mathrm{Sp}\{\ldots\} \approx \{\mathrm{Sp}\gamma_5(\hat{p}(1-\alpha) - \hat{k}_\perp)\gamma_\mu(\hat{q}+\hat{p}(1-\alpha))\gamma_\nu \times(\hat{p}(1-\alpha)-\hat{k}_\perp)\gamma_5\hat{p}\} \approx k_\perp{}^2\,\mathrm{Sp}\,\gamma_\mu(\hat{q}+\hat{p}(1-\alpha))\gamma_\nu. \tag{2.9}$$

In the derivation of (2.9) we used the equality $\hat{p}\hat{p} = p^- \approx 0$, and the fact that μ and ν can be assumed, without loss of generality, to be longitudinal. Further

$$\int\frac{d^2k_\perp}{k_\perp{}^2} = \ln\frac{q^2}{m^2} \equiv \xi. \tag{2.10}$$

Therefore the final expression is

$$M_{\mu\nu}^{(1)} = \frac{g_0{}^2\xi}{16\pi^2}\int_0^1\frac{\alpha d\alpha\,\mathrm{Sp}\,\gamma_\mu(\hat{q}+\hat{p}(1-\alpha))\gamma_\nu\hat{p}}{q^2+\nu(1-\alpha)+i0} = \frac{g_0{}^2\xi}{16\pi^2}\int_0^1\alpha d\alpha M_{\mu\nu}^{(0)}(q,p(1-\alpha)), \tag{2.11}$$

where

$$M_{\mu\nu}^{(0)}(q,p(1-\alpha)) = \mathrm{Sp}\,\gamma_\mu\frac{1}{\hat{q}+\hat{p}(1-\alpha)}\gamma_\nu\hat{p}. \tag{2.12}$$

In calculating the higher-order ladder it is convenient to introduce integration with respect to the momenta of the steps k_i. The integrals with respect to β_i are then determined by the poles of the denominator k_i^{-2} under the condition that $\Sigma\alpha_i \leq 1$, $\alpha_i \geq 0$, $\beta_i = -k_{i\perp}^2/\alpha_i\nu$. At other values of α_i, the poles in β_i lie alongside of the contour and the integral is equal to zero. The logarithmic contribution arises in the region

$$\begin{gathered} m^2 \ll k_{1\perp}{}^2 \ll k_{2\perp}{}^2 \ll \ldots \ll k_{n\perp}{}^2 \ll q^2, \\ m^2/\nu \ll \beta_1 \ll \beta_2 \ll \ldots \ll \beta_n \ll 1, \\ \alpha_i \sim 1 \end{gathered} \tag{2.13}$$

(the momenta are reckoned from the bottom of the ladder). The integral can be written in the form

$$M_{\mu\nu}^{(n)} = \left(\frac{g_0{}^2}{16\pi^2}\right)^n\int\prod_i\frac{d^2k_{i\perp}}{k_{i\perp}{}^4} \times\int\prod_{s=1}^{n-1}\frac{\alpha_s d\alpha_s}{\left(1-\sum_1^s\alpha_i\right)^2}\alpha_n\, d\alpha_n\frac{N_{\mu\nu}}{q^2+\nu\left(1-\sum_1^n\alpha_i\right)}, \tag{2.14}$$

where $N_{\mu\nu}$ is the product of the traces and depends on the number and locations of the nucleon squares which enter in the ladder.

Let us calculate first $N_{\mu\nu}$ for the case when there are no nucleon squares at all, i.e., all the steps of the ladder except the uppermost are bosons. For this case we have

$$N_{\mu\nu} = \mathrm{Sp}\,\gamma_5(\hat{p}-\hat{k}_1)\gamma_5(\hat{p}-\hat{k}_1-\hat{k}_2)\ldots\gamma_\mu \times\left(\hat{p}+\hat{q}-\sum\hat{k}_i\right)\gamma_\nu\left(\hat{p}-\sum\hat{k}_i\right)\gamma_5\ldots(\hat{p}-\hat{k}_1)\gamma_5\hat{p}. \tag{2.15}$$

We note that in the terms corresponding to the vertical fermions we can write

$$\hat{p} - \sum\hat{k}_i \approx -\sum\hat{k}_{i\perp} \approx -\hat{k}_{s\perp}, \tag{2.16}$$

since $\beta_i \ll 1$, and the contribution proportional to $\hat{p}$ vanishes because $\hat{p}\hat{p} \approx 0$.

Consequently

$$N_{\mu\nu} \approx \prod_{i=1} k_{i\perp}{}^2\,\mathrm{Sp}\,\gamma_\mu\left(\hat{q}+\hat{p}\left(1-\sum\alpha_i\right)\right)\gamma_\nu\hat{p} \tag{2.17}$$

[1)] In the initial version of the article the integrals were calculated with the aid of the Feynman parametrization. The convenience of the Sudakov variables in the present problem was pointed out to the author by V. N. Gribov.

and finally for a ladder with n boson steps and one fermion step we have

$$M_{\mu\nu}^{(n)}=\left(\frac{g_0^2\xi}{16\pi^2}\right)^n\frac{1}{n!}\int\prod_{s=1}^{n-1}\alpha_s\,d\alpha_s\,\alpha_n\,d\alpha_n\left(1-\sum_1^s\alpha_i\right)^{-2}$$
$$\times M_{\mu\nu}^{(0)}\left(q,p\left(1-\sum_1^n\alpha_i\right)\right) \qquad (2.18)$$

For the case when there are many fermion steps, it is also easy to transform the traces in analogy with the foregoing. We present only the final result: in formula (2.18) replacement of the k-th boson step by a fermion step gives rise to a factor $1/\alpha_k$, and replacement of the k-th vertical fermion line by a boson line produces the factor

$$\left(1-\sum_1^k\alpha_i\right)^2,$$

The result must be multiplied by 4^m, where m is the number of nucleon squares. In all other respects, (2.18) remains unchanged.

We proceed now to sum the obtained expressions. This is conveniently carried out directly for the imaginary part of $M_{\mu\nu}$, which is of physical interest. We note that

$$\operatorname{Im}M_{\mu\nu}^{(0)}=\operatorname{Im}\frac{\alpha^2(\xi)s(\xi)}{q^2+\nu+i0}f_{\mu\nu}(q,p)$$
$$=f_{\mu\nu}(q,p)\pi\left\{\alpha^2(\xi)\delta(q^2+\nu)\right.$$
$$\left.+\alpha^2(\xi)\frac{ds}{d\xi}\frac{1}{q^2+\nu}\right\}\approx\pi f_{\mu\nu}(q,p)\alpha^2(\xi)s(\xi)\delta(q^2+\nu) \qquad (2.19)$$

Since at our accuracy the second term needs to be taken into account only when $q^2\approx-\nu$. The arbitrary term of the series can be written in the form

$$\operatorname{Im}M_{\mu\nu}^{(n)}=\frac{\pi\alpha^2(\xi)s(\xi)}{q^2}\operatorname{Sp}\gamma_\mu\left(\hat q+\frac{\hat p}{\omega}\right)\gamma_\nu\hat p$$
$$\times\left(\frac{g_0^2\xi}{16\pi^2}\right)^n\frac{1}{n!}\Phi^{(n)}(\omega),\quad \omega=-2pq/q^2, \qquad (2.20)$$

where the function $\Phi^{(n)}(\omega)$ is given by an n-fold integral with respect to α, which is constructed in accordance with the rules given above. To calculate these integrals, we consider the Mellin transform of $\Phi^{(n)}(\omega)$:

$$\Phi_j^{(n)}\equiv\int_1^\infty\frac{d\omega}{\omega^j}\Phi^{(n)}(\omega). \qquad (2.21)$$

The quantity $\Phi_j^{(n)}$, the physical meaning of which will be discussed later, is extremely easy to calculate for each individual diagram. Let us consider, for example, the third diagram of (2.3). According to our rules, for this diagram we have

$$\Phi^{(2)}(\omega)=\int\frac{\alpha_1 d\alpha_1\alpha_2 d\alpha_2}{(1-\alpha_1)^2}\delta(1-\omega(1-\alpha_1-\alpha_2)). \qquad (2.22)$$

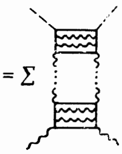
Calculating the Mellin transform of (2.22), we obtain

$$\Phi_j^{(2)}=[1/j(j+1)]^2. \qquad (2.23)$$

It is easy to see if a general diagram such as the last one in (2.3) is assumed to consist of m nucleon squares, and if the number of boson lines is n_0 in its base, n_1 in the first square, etc., then the Mellin transform becomes

$$\Phi_j=4^m\left[\frac{1}{j(j+1)}\right]^{\mathfrak{m}},\quad \mathfrak{m}=\sum_{k=0}^m n_k+m. \qquad (2.24)$$

We shall need later the amplitude $F_{\mu\nu}(q, p)$ of the virtual Compton effect on the meson. It is of the form

$F_{\mu\nu}=\Sigma$  (2.25)

The rules for calculating the Im $F_{\mu\nu}$ skeleton are analogous to the rules for the Im $M_{\mu\nu}$ skeleton. Namely:

$$\operatorname{Im}F_{\mu\nu}=\frac{\pi\alpha^2(\xi)s(\xi)}{q^2}\operatorname{Sp}\left(\gamma_\mu\left(\hat q+\frac{p}{\omega}\right)\gamma_\nu\hat p\right)g(\omega). \qquad (2.26)$$

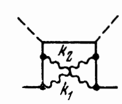
The function

$$g_j=\int_1^\infty\frac{d\omega}{\omega^j}g(\omega) \qquad (2.26')$$

is calculated here for a diagram in which the lower fermion square has n_0 bosons, the next square has n_1, etc., and the total number of fermion squares is m + 1, in accordance with the rule

$$g_j=\frac{2}{j}4^m\left[\frac{1}{j(j+1)}\right]^{\mathfrak{m}}. \qquad (2.27)$$

Before we proceed to the summation of (2.3), let us show that the diagrams which we do not plan to take into account, for example those of the type



(2.28)

make a small contribution.

Indeed, we recall that the logarithmic contribution to the third diagram of (2.3) was the result of small denominators connected with the vertical fermion lines. In the case of (2.28) we can obtain small denominators from the left-hand fermion line, by stipulating $m^2\ll k_{1\perp}^2\ll k_{2\perp}^2\ll q^2$, but for the right-hand line the analogous condition is

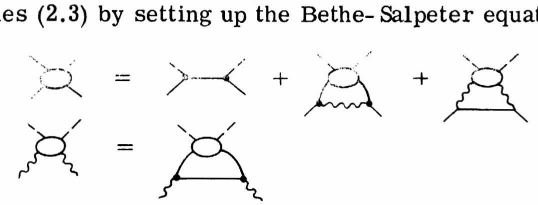
$$m^2\ll k_{2\perp}^2\ll k_{1\perp}^2\ll q^2.$$

Consequently, the region of integration where both denominators are large is determined by the condition

$$m^2\ll k_{1\perp}^2\sim k_{2\perp}^2\ll q^2,$$

and therefore yields only one logarithmic integration.

The physical meaning of separating the diagram (2.3) consists in the fact that the particles are emitted with $k_{i\perp}\ll k_{i+1,\perp}$ and therefore the interference between them has little probability, since they are separated in space and their wave functions overlap little (see the analogous reasoning in[6]). It is convenient to sum the series (2.3) by setting up the Bethe-Salpeter equations


(2.29)

Using the technique developed above for calculating the individual diagrams, we can easily write down the system (2.29) explicitly for the functions

$$f_j(\xi, \eta) = \int_1^\infty \frac{d\omega}{\omega^j} f(\omega, \xi, \eta),$$

$$g_j(\xi, \eta) = \int_1^\infty \frac{d\omega}{\omega^j} g(\omega, \xi, \eta), \qquad (2.30)$$

where $\xi = \ln(q^2/m^2)$, $\eta = \ln(p^2/m^2)$, and the functions (2.30) are generalizations of the functions (2.21) and (2.26′) for $\eta \neq 0$.

For these quantities, we rewrite (2.29) in the form

$$f_j(\xi,\eta) = 1 + \frac{1}{j(j+1)} \int_\eta^\xi dx\, \lambda(x) f_j(\xi, x) + \frac{2}{j+1} \int_\eta^\xi dx\, \rho(x) g_j(\xi, x),$$

$$g_j(\xi, \eta) = \frac{2}{j} \int_\eta^\xi dx\, \sigma(x) f_j(\xi, x), \qquad (2.31)$$

where

$$\lambda(x) = \frac{g_0^2}{16\pi^2} \gamma^2(x) s^2(x) d(x),$$
$$\rho(x) = \frac{g_0^2}{16\pi^2} \gamma^2(x) s(x) d^2(x) = \lambda(x) \frac{d(x)}{s(x)},$$
$$\sigma(x) = \frac{g_0^2}{16\pi^2} \gamma^2(x) s^3(x) = \lambda(x) \frac{s(x)}{d(x)}. \qquad (2.32)$$

Formula (2.31) is easiest to verify by putting $\lambda = \sigma = 1$ and $\eta = 0$, expanding in powers of ξ, and comparing with (2.24) and (2.27).

Equations (2.31) can be solved by differentiating them with respect to η, introducing the independent variable

$$z = \int_0^x dy\, \lambda(y)$$

and replacing the function g_j by the function

$$h_j(\xi, \eta) \equiv \frac{d(\eta)}{s(\eta)} g_j(\xi, \eta). \qquad (2.33)$$

If we then use the relation indicated in the Appendix

$$\frac{\partial}{\partial\eta} \ln \frac{d(\eta)}{s(\eta)} = \frac{3}{2}, \qquad (2.34)$$

then the system (2.31) reduces to the equations

$$-\frac{\partial f_j}{\partial z} = \frac{1}{j(j+1)} f_j + \frac{2}{j+1} h_j,$$
$$-\frac{\partial h_j}{\partial z} = \frac{2}{j} f_j - \frac{3}{2} h_j \qquad (2.35)$$

with initial conditions

$$f_j = 1,\ h_j = 0 \text{ for } z = \zeta \equiv \int_0^\xi \lambda(x)\, dx. \qquad (2.36)$$

From this we can easily find the function $f_j(\xi)$ of interest to us. The result is of the form

$$f_j(\xi) = \frac{(p_2 - \gamma) e^{p_1 \zeta} - (p_1 - \gamma) e^{p_2 \zeta}}{p_2 - p_1}, \qquad (2.37)$$

where

$$\gamma = 1 / j(j+1),$$
$$p_{1,2} = {}^1/_2 [\gamma - {}^3/_2 \pm \sqrt{\gamma^2 + 19\gamma + {}^9/_4}]. \qquad (2.38)$$

Thus, the amplitude of the virtual Compton effect in the principal logarithmic approximation is

$$f(\omega, \xi) = \alpha^2(\xi) s(\xi) \int_{a - i\infty}^{a + i\infty} dj\, \omega^{j-1} f_j(\xi), \qquad (2.39)$$

where the contour passes to the right of the singularities of the function f_j specified by (2.37). We shall investigate and generalize the results in the next section.

3. CONNECTION OF THE AMPLITUDE WITH RENORMALIZATION OF THE VERTICES OF DIFFERENT OPERATORS

To understand and generalize the obtained formulas, let us consider a somewhat different approach to the problem solved in Sec. 2. There we calculated the amplitude for the scattering of a quantum with momentum q by a fermion with momentum p in the region $q^2 \sim pq \gg m^2$. It is obvious that this configuration of momenta is essentially non-Euclidean, since

$$z = \frac{pq}{\sqrt{p^2 q^2}} = \frac{1}{2} \omega \sqrt{\frac{q^2}{p^2}} \gg 1. \qquad (3.1)$$

Euclidean configurations correspond to

$$q^2 \gg p^2, \qquad z \sim 1, \qquad \omega \sim \sqrt{p^2 / q^2} \ll 1. \qquad (3.2)$$

Nonetheless, knowledge of the asymptotic behavior in the region (3.2) makes it possible to determine the asymptotic behavior in the region (3.1), by using the relation

$$\left.\frac{d^n F}{d\omega^n}\right|_{\omega=0} = \frac{1}{\pi} \int_1^\infty \frac{\operatorname{Im} F(\omega') d\omega'}{\omega'^{n+1}} \equiv f_n, \qquad (3.3)$$

where n are even, $F(\omega, q^2, p^2)$ are the invariant amplitudes for the scattering of the quantum (F_1 will be investigated below, and F_2 has analogous properties), and formula (3.3) is a trivial consequence of the dispersion relations.

Owing to (3.3), the problem reduces to finding the coefficients of the expansion of the quantity F in z (or in ω) and to analytic continuation of these coefficients, i.e., actually of the partial waves as functions of the number. It is seen from the foregoing that we obtained for these coefficients (in the case of weak coupling) a simple system of equations (2.35). We shall explain below the cause of this simplicity and generalize the system (2.35) to the case of strong coupling.

Let us consider again one of the summed diagrams, namely a small square. We write its contribution, altering the notation for the momenta, in the form

$$F_1 \infty F_{\mu\mu}(q, p) = \int \frac{d^4 k}{(p-k)^2} \frac{1}{\hat{k}} \gamma_\mu \frac{1}{\hat{q} + \hat{k}} \gamma_\mu \frac{1}{\hat{k}}$$
$$= \text{const} \int \frac{d^4 k}{(p-k)^2} \frac{1}{\hat{k}} \frac{1}{\hat{q} + \hat{k}} \frac{1}{\hat{k}}. \qquad (3.4)$$

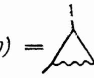
In the Euclidean region of the external variables we can assume the momenta of the virtual particles to be also Euclidean. From this we can easily conclude that when $q \gg p$ the values $k \ll q$ play an important role in (3.1), and the asymptotic form of (3.1) when $z \sim 1$ and $q^2 \gg p^2$ is

$$F_1 \infty \frac{q_\alpha}{q^2} \int \frac{d^4 k}{(p-k)^2} \frac{1}{\hat{k}} \gamma_\alpha \frac{1}{\hat{k}}, \qquad (3.5)$$

where we have made the substitution

$$1 / (\hat{q} + \hat{k}) \approx q_\alpha \gamma_\alpha / q^2. \qquad (3.6)$$

The integral T_α in (3.5) corresponds to the diagram

$\Gamma_\alpha^{(1)}(p) =$  $\sim \gamma_\alpha \ln(q/p)$ (3.7)

where the dashed line corresponds to γ_α, and the cutoff radius is replaced by the momentum q. The contribution of (3.5) to F_1 actually gives the asymptotic form of the small square at $q \gg p$, but it cancels out in the complete amplitude, since it is symmetrical under the substitution $q \to -q$. We therefore expand (3.6) up to the next term. The corresponding contribution to the asymptotic form is

$$F_1 \infty \frac{q_\alpha q_\beta}{q^4} \int \frac{d^4k}{(p-k)^2} \frac{1}{\hat{k}} \gamma_\alpha \hat{k} \gamma_\beta \frac{1}{\hat{k}}. \quad (3.8)$$

In (3.8) we are interested in the term proportional to $(pq)^2$, and therefore we can subtract from the tensor specified by the integral in (3.5) its trace multiplied by $\delta_{\alpha\beta}/4$. (This term makes a contribution $\propto p^2q^2$.) After this subtraction, (3.8) takes the form

$$F_1 \approx \frac{q_\alpha q_\beta}{q^4} \int \frac{d^4k}{(p-k)^2} \frac{1}{\hat{k}} \left(\gamma_\alpha k_\beta + \gamma_\beta k_\alpha - \frac{1}{2} \hat{k} \delta_{\alpha\beta}\right) \frac{1}{\hat{k}}. \quad (3.9)$$

After averaging over the spin, the integral (3.9) must be proportional to an irreducible tensor of the form $p_\alpha p_\beta - p^2\delta_{\alpha\beta}/4$. Therefore the linear divergence of (3.9) is illusory and the integral diverges logarithmically. Thus,

$$F_1 \approx \text{const} + \left(\frac{pq}{q^2}\right)^2 \ln(q^2/p^2). \quad (3.10)$$

The integral (3.9) corresponds to a correction to the vertex of the operator:

$$\bar{\psi}\gamma_\alpha \frac{\partial \psi}{\partial x_\beta} + \bar{\psi}\gamma_\beta \frac{\partial \psi}{\partial x_\alpha} - \delta_{\alpha\beta} \frac{1}{2} \bar{\psi}\gamma_\lambda \frac{\partial \psi}{\partial x_\lambda}. \quad (3.11)$$

Analogously, the term numbered n in the expansion with respect to ψ reduces to the vertex of an operator that is bilinear in ψ, contains $n - 1$ derivatives, and is an irreducible tensor of rank n from the point of view of the Lorentz group. The logarithmic divergence of the vertex part is cut off by the momentum q. Therefore the contribution of the square diagram has the property

$$\int_1^\infty \frac{d\omega}{\omega^{n+1}} \operatorname{Im} F_1 = \gamma_n \ln\left(\frac{q^2}{p^2}\right).$$

A similar result was obtained by the non-Euclidean approach (see (2.8)) which gave, in addition, the formula

$$\gamma_n \infty 1 / n(n+1).$$

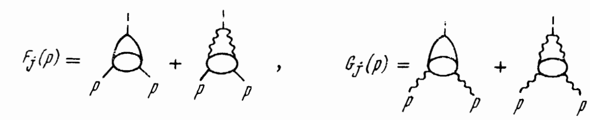
Let us consider now the entire ladder sequence of diagrams (2.3) (remaining within the framework of the weak coupling). According to the foregoing, the problem of calculating the quantities (3.3) reduces to finding the renormalizations of the vertices of operators that are bilinear in ψ, and the cutoff radius is assumed equal to q.

In the logarithmic approximation, these quantities satisfy equations of the type given in[11], which indeed lead to the system (2.35). As will be shown in our next paper, in the case of strong coupling the quantities $f_n(q^2, p^2)$ are also the vertex parts of certain operators, and the divergences are assumed to be cut off at the momentum q.

4. RENORMALIZATION OF THE VERTICES IN THE CASE OF STRONG COUPLING

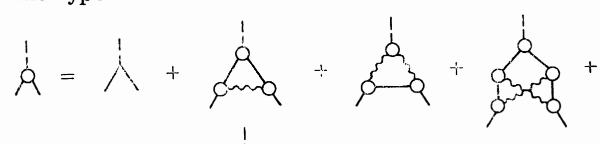
The problem of the character of renormalization of such vertex parts has not been solved as yet, although it was raised in connection with attempts to generalize and verify the Bjorken-Johnson-Low theorem[13]. We shall consider this problem below and obtain a general relation for the renormalization, valid for both strong and weak coupling.

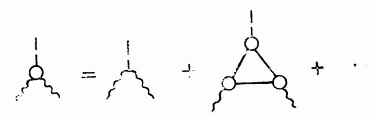
We shall investigate an aggregate of diagrams of the type


(4.1)

where the point of connection of the dashed line is set in correspondence with an irreducible tensor of rank j, made up of γ matrices and the vector k. (For example, at j = 2 this tensor takes the form $\gamma_\alpha k_\beta + \gamma_\beta k_\alpha - \hat{k}\delta_{\alpha\beta}/2$.)

The quantity (3.9) satisfies a system of equations of the type


(4.2)


(4.2′)

Iteration of the system (4.2) gives rise to logarithmically diverging integrals. We shall show that if two constants are specified, namely the amplitudes (4.2) and (4.2′) at a certain fixed momentum $p^2 = \lambda^2$, then the divergences turn out to be hidden in these two constants, i.e., the expressions for F(p) and G(p) in terms of $F(\lambda)$ and $G(\lambda)$ do not contain divergences in all orders of perturbation theory. To prove this, we note that all the integrals diverge logarithmically. For a triangular diagram this has been explained in connection with formula (3.9), and in the general case this is obvious from dimensionality considerations: after averaging over the spin, the coefficient preceding the irreducible tensor ($p_{\alpha_1} \dots p_\alpha^j$ — traces) is a dimensionless quantity and is given by logarithmic integrals. Consequently, we can write

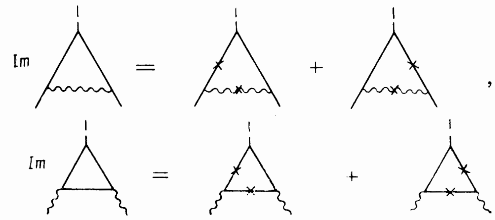
$$F(p^2) = F(\lambda^2) + \frac{p^2 - \lambda^2}{\pi} \int_{s_0}^{\infty} \frac{\operatorname{Im} F(s)\, ds}{(s-\lambda^2)(s-p^2)},$$

$$G(p^2) = G(\lambda^2) + \frac{p^2 - \lambda^2}{\pi} \int_{s_0}^{\infty} \frac{\operatorname{Im} G(s)\, ds}{(s-\lambda^2)(s-p^2)} \quad (4.3)$$

(Since the imaginary part of logarithm raised to a power is again a logarithm raised to a power and the integrals in (4.3) converge).

The problem reduces to proving that Im F and Im G, when calculated as functions of $F(\lambda^2)$ and $G(\lambda^2)$ and of the physical charges and masses, contain no divergences. It suffices to prove this for skeleton diagrams, for in accordance with ordinary renormalization theory the building up of the skeleton by replacing bare charges with physical ones does not lead to divergences.

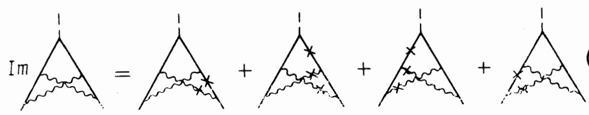
The imaginary parts of the simplest skeleton diagrams take the form



(4.4)

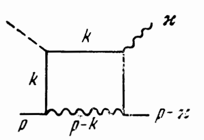
where, as usual, the crosses correspond to imaginary parts of the propagators. The energy conservation law imposes an upper limit here on the integration, and the integrals converge.

So far our proof did not differ in any way from the standard proof of renormalizability—the specific feature of the dashed line did not come into play in any way. In more complicated diagrams, this specific feature calls for caution. Let us examine the diagram



(4.4′)

In all terms except the first and the last, the integrals are bounded from above by energy conservation, and all the particles in them are real. The first term contains the amplitude



(4.5)

which involves virtual particles, and it is therefore necessary to study its convergence. In ordinary renormalization theory this is a trivial problem, for when the dashed line is replaced by a wavy line, the integral obviously converges like $\int d^4k/k^5$.

The dashed line, however, introduces additional powers of the momentum, and the problem becomes more complicated. The integral (3.14) then takes the form (for concreteness, at j = 2)

$$\int \frac{d^4k}{(p-k)^2}\frac{1}{\hat{k}-\hat{\varkappa}}\frac{1}{\hat{k}}\left(\gamma_\alpha k_\beta+\gamma_\beta k_\alpha-\frac{1}{2}\hat{k}\delta_{\alpha\beta}\right)\frac{1}{\hat{k}}.$$

This integral diverges logarithmically if p and κ are neglected in the denominators when $k \gg p, \kappa$. The integral, however, should be a traceless symmetrical tensor, and such a quantity cannot be constructed without using p or κ. Thus, in the considered field theory, the integrals (3.14) converge. Were we to consider a theory with vector particles, we would have at our disposal also the polarization vector, and (3.14) would yield a logarithmic divergence. Thus, our arguments do not pertain to the vector theory, in which the diagram (4.4) has a diverging imaginary part.

In the scalar theory, we can carry out for any diagram an analysis similar to that above and prove that the many-point graphs inside the diagrams do not result in divergences. This proves the foregoing statement.

We can now write a renorm-group equation for the quantities $F_j(p^2)$ and $G_j(p^2)$. It is easy to verify that by using subtractions at the point λ^2 and expanding Eqs. (4.2) and (4.2′) in the invariant charges $\alpha(\lambda)$ and $\beta(\lambda)$, which are connected with the meson-nucleon and meson-meson interactions (they are defined in[14]), we can obtain the following relations:

$$\begin{aligned} F_j(p^2) &= F_j(\lambda^2)\psi_1^j\left(\frac{p^2}{\lambda^2},\frac{m^2}{\lambda^2},\alpha(\lambda),\beta(\lambda)\right) \\ &+ G_j(\lambda^2)\frac{s(\lambda)}{d(\lambda)}\psi_2^j\left(\frac{p^2}{\lambda^2},\frac{m^2}{\lambda^2},\alpha(\lambda),\beta(\lambda)\right), \\ G_j(p^2) &= \frac{d(\lambda^2)}{s(\lambda^2)}F_j(\lambda^2)\psi_3^j\left(\frac{p^2}{\lambda^2},\frac{m^2}{\lambda^2},\alpha(\lambda),\beta(\lambda)\right) \\ &+ \psi_4^j\left(\frac{p^2}{\lambda^2},\frac{m^2}{\lambda^2},\alpha(\lambda),\beta(\lambda)\right)G_j(\lambda^2) \end{aligned} \tag{4.6}$$

(for example, the contribution of the triangular diagram subtracted at the point $\lambda^2 \gg m^2$ at $\lambda^2 \gg p^2 \gg m^2$ is

$$\begin{aligned} &\text{const}\, F_j(\lambda^2)\gamma_\lambda^2 d_\lambda s_\lambda^2 \ln(p^2/\lambda^2) \\ &= \text{const}\, F_j(\lambda^2)\alpha(\lambda)\ln(p^2/\lambda^2), \end{aligned} \tag{4.7}$$

which agrees with (4.6)).

Differentiating (4.6) with respect to p^2 and putting $\lambda^2 = p^2$, we obtain

$$\begin{aligned} p^2\frac{dF_j}{dp^2} &= \psi_1^j\left(\frac{m^2}{p^2},\alpha(p^2),\beta(p^2)\right)F_j(p^2)+\psi_2^j\frac{s}{d}G_j, \\ p^2\frac{dG_j}{dp^2} &= \psi_3^j\frac{d}{s}F_j+\psi_4^jF_j. \end{aligned} \tag{4.7′}$$

We introduce the functions

$$F_1^j(p^2) \equiv F^j(p^2),$$
$$F_2^j(p^2) = sG_j(p^2)/d.$$

Then the system (4.7′) takes the form

$$p^2\frac{dF_i^j}{dp^2} = \sum_{k=1}^{2}\psi_{ik}^j\left(\frac{m^2}{p^2},\alpha(p^2),\beta(p^2)\right)F_k^j, \tag{4.8}$$

where

$$\begin{aligned} &\psi_{11}=\psi_1,\quad \psi_{12}=\psi_2,\quad \psi_{21}=\psi_3, \\ &\psi_{22}=\psi_4-p^2\frac{\partial}{\partial p^2}\ln\left(\frac{s}{d}\right). \end{aligned} \tag{4.9}$$

According to the known renorm-group equations for s and d[14], ψ_{22} depends here on the same argument as the remaining ψ_{ik}.

The system (4.8) generalizes (2.35) to include the case of strong interaction. An important property of ψ_{ik}^j is that when $p^2 \gg m^2$ they do not depend on their first argument. This follows from the fact that none of the perturbation-theory diagrams contain infrared divergences in the m = 0 limit.

The system (3.18) makes it possible to determine the dependence, in which we are interested, of the quantities F_i^j on the cutoff radius Λ. For this purpose we note that the initial conditions for the system (3.18) are

$$F_i^j(p^2=\Lambda^2) = C_i^j \sim 1, \tag{4.10}$$

since there is no logarithmic region in the integrals of perturbation theory when $p \sim \Lambda$, and they are quantities of the order of unity. The exact value of C_i^j depends on the cutoff method and is immaterial to us.[1] From the formal solution of (4.8) in the form

$$\begin{pmatrix}F_1^j\\F_2^j\end{pmatrix} = T\exp\left\{-\int_{\ln(p^2/m^2)}^{\ln(\Lambda^2/m^2)}\|\psi^j(x)\|dx\right\}\begin{pmatrix}C_1^j\\C_2^j\end{pmatrix}, \tag{4.11}$$

where T is the Dyson operator, we obtain the system of equations

$$\frac{\partial F_i^j}{\partial \ln(\Lambda^2/m^2)} = -\sum_k \psi_{ik}^j\left(\frac{m^2}{\Lambda^2}, \alpha(\Lambda^2), \beta(\Lambda^2)\right) F_k^j. \quad (4.12)$$

When $\Lambda \gg m$ we have in the case of strong coupling $\alpha(\Lambda^2) \to \alpha_1$ and $\beta(\Lambda^2) \to \beta_1$, where α_1 and β_1 are certain constants. Consequently, the functions ψ_{ik}^j tend to the constants

$$\psi_{ik}^j(m^2/\Lambda^2, \alpha(\Lambda^2), \beta(\Lambda^2)) \to \psi_{ik}^j(0, \alpha_1, \beta_1) \equiv C_{ik}^j.$$

We thus obtain ultimately

$$F_i^j(p^2, \Lambda^2) = \sum_{k=1}^{2}\left(\frac{\Lambda^2}{m^2}\right)^{\gamma_k(j)} A_{ik}^j\left(\frac{p^2}{m^2}\right) \quad (4.13)$$

for strong coupling.

For weak coupling, the functions ψ can be expanded in powers of $\alpha(\lambda)$ and $\beta(\lambda)$, and we obtain Eq. (2.37), which is quite similar to (4.13)).

It is interesting to note that if the investigated operator is a conserved quantity, such as the operator of the current or of the energy-momentum tensor, then its vertex part (multiplied by the renormalization of the end points) should not depend on Λ (this follows formally from the Ward identities, and actually from the observability of the corresponding quantities). Therefore the corresponding $\gamma_k(j)$ should vanish. In the case of weak coupling, a demonstration of this general rule is the fact that in (2.38) we have

$$p_1(j=2) = {}^1/_2, \qquad p_2(j=2) < 0$$

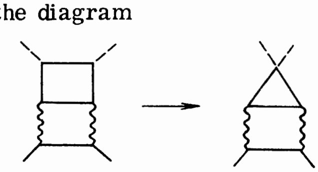
and consequently, after multiplying by

$$\alpha_c^2(\xi)s_c(\xi) = s_c^{-1}(\xi) = Z(\xi) = e^{-\frac{1}{2}\xi}$$

(see[15]) we obtain a quantity that does not depend on ξ.

This is connected with the fact that at j = 2 the calculated vertex is proportional to the vertex of the energy-momentum tensor. This question will be discussed in greater detail in our next paper.

We also wish to note here the following. The integrals (2.30) are connected with the vertices only at even j, so that at j = 1 the right-hand side of (2.30) coincides, according to (3.4), with the electromagnetic-current operator in the quadratic diagram but not in more complicated diagrams. For example, at j = 1 we have a contribution of the diagram


, (4.14)

which is forbidden for the electromagnetic current. Therefore $p_{1,2} \neq 1/2$ at j = 1 in (2.38).

A difference of this kind between even and odd j is connected with the exchange interaction and is analogous to the appearance of the signature in the Regge theory.

The author is grateful to V. N. Gribov and A. A. Migdal for useful discussions.

APPENDIX

In first order of perturbation theory we have

$$s(\xi) = 1 + g_0^2\xi/32\pi^2,$$
$$d(\xi) = 1 + g_0^2\xi/8\pi^2,$$
$$\gamma(\xi) = 1 + g_0^2\xi/16\pi^2.$$

Consequently the invariant charge is

$$q(\xi) = \gamma^2(\xi)s(\xi)d(\xi) = 1 + 5g_0^2\xi/16\pi^2 + O(\xi^2).$$

Hence, using the renorm-group[14], we obtain in the logarithmic approximation

$$q(\xi) = 1/[1-(5g^2/16\pi^2)\xi], \qquad s(\xi) = q^{-1/10}(\xi),$$
$$d(\xi) = q^{-2/5}(\xi), \qquad \gamma(\xi) = q^{-1/5}(\xi).$$

For the electromagnetic vertex $\alpha(\xi)$ we get from the Ward identity the relation

$$\alpha(\xi) = s^{-1}(\xi) = q^{1/10}(\xi).$$

[1]E. D. Bloom, W. B. Amblard, and R. I. Taylor et al., Preprint SLAC-PUB, 796, 1970.
[2]J. D. Bjorken and J. Paschos, Phys. Rev. **185**, 1975, 1969.
[3]L. D. Landau and I. Ya. Pomeranchuk, Dokl. Akad. Nauk SSSR **102**, 489 (1955).
[4]M. Gell-Mann and F. Low, Phys. Rev. **95**, 1300, 1954.
[5]J. D. Bjorken, Phys. Rev. **148**, 1467, 1966.
[6]A. M. Polyakov, Zh. Eksp. Teor. Fiz. **60**, 1572 (1971) [Sov. Phys.-JETP 33, 850 (1971)].
[7]J. D. Bjorken, Phys. Rev. **179**, 1547, 1969.
[8]V. N. Gribov and A. A. Migdal, Zh. Eksp. Teor. Fiz. **55**, 1498 (1968) [Sov. Phys.-JETP 28, 784 (1969)].
[9]K. G. Wilson, Phys. Rev. **179**, 1499, 1969.
[10]A. M. Polyakov, Zh. Eksp. Teor. Fiz. **59**, 542 (1970) [Sov. Phys.-JETP 32, 296 (1971)].
[11]L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR **95**, 497 (1954).
[12]V. V. Sudakov, Zh. Eksp. Teor. Fiz. **30**, 87 (1956) [Sov. Phys.-JETP 3, 65 (1956)].
[13]S. Adler and Wu-ki-Tung, Phys. Rev. **D1**, 1470, 1970.
[14]N. N. Bogolyubov and D. V. Shirokov, Vvedenie v teoriyu kvantovannykh poleĭ (Introduction to the Theory of Quantized Fields), Gostekhizdat, 1957, p. 370.
[15]M. Gell-Mann and F. Zachariasen, Phys. Rev. **123**, 1065, 1970.

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