DENSITY OF STATES IN INHOMOGENEOUS SUPERCONDUCTORS

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Inhomogeneities of the effective interaction between electrons in superconductors result in smearing of the state density maximum. The smearing alters the current-voltage characteristic of the tunnel junction and also the frequency dependence of electromagnetic field absorption in the superconductor.

1. INTRODUCTION

THE density of states in superconductors becomes infinite at an energy equal to the gap energy. This singularity becomes smeared out under the influence of magnetic impurities[1] or anisotropy of the energy gap[2]. In superconductors with large concentrations of nonmagnetic impurities, the influence of the anisotropy decreases. As a result, it turns out[3] that the singularity in the density of states becomes smeared out only in a narrow energy region of the order of $\Delta(\tau\Delta)^2\chi$, where τ is the free-path time and χ is the anisotropy parameter.

We consider below the smearing of the state-density singularity as a result of inhomogeneities of the effective interaction between the electrons. Such inhomogeneities are large in heterogeneous alloys, which are mixtures of particles with different compositions. The effective interaction also changes near dislocations and crystallite boundaries.

If the inhomogeneity dimension is small compared with the pair dimension, then only one parameter, which determines the width of the singularity smearing region, depends on the magnitude and dimension of the inhomogeneities. The form of the dependence of the density of states on the energy is universal. It has a maximum at an energy close to the average gap in the spectrum, and an exponential "tail" in the region of low energies.

At low temperatures, the density of states in a superconductor is determined directly from the current-voltage characteristic of a tunnel junction between the superconductor and the normal metal. The tunnel current between two superconductors should experience a jump at a voltage equal to the sum of the gaps in the spectrum. This jump is connected with the singularity in the density of states and becomes smeared out if this singularity is smeared out. A decrease of the peak in the density of states also leads to a noticeable change of the frequency dependence of the absorption of the electromagnetic radiation.

2. INHOMOGENEITIES OF LARGE SIZE

In homogeneous superconductors without a magnetic field and without magnetic impurities, the density of states does not depend on the concentration of the nonmagnetic impurities and is equal to

$$\rho(\omega) = \rho_0\omega\,\mathrm{sign}(\omega-\Delta)\,/\,(\omega^2-\Delta^2)^{1/2}, \qquad (1)$$

where ρ_0 is the energy-independent density of states in the normal metal. The density of states in the superconductor is equal to zero when $\omega < \Delta$ and becomes infinite if ω tends to Δ from above. Inhomogeneities in the effective interaction between the electrons lead to inhomogeneities in the ordering parameter Δ and smear out the singularity in formula (1).

If the dimension r_c of the inhomogeneities exceeds the pair dimension ξ, then the density of states at each point is determined by the value of Δ at the same point; this value is expressed in the usual manner in terms of the local value of the effective interaction. In this case the average density of states is obtained by averaging (1) over the possible values of Δ:

$$\rho = \int_0^\infty \rho(\omega,\Delta)W(\Delta)\,d\Delta, \qquad (2)$$

where $W(\Delta)$ is the distribution function of the random quantity Δ. Let us calculate the average density of states when the distribution function has a Gaussian form:

$$W(\Delta) = (2\pi\langle\Delta_1^2\rangle)^{-1/2}\exp(-\Delta_1^2/2\langle\Delta_1^2\rangle), \qquad (3)$$

where $\Delta_1 = \Delta - \langle\Delta\rangle$ and $\langle\Delta\rangle$ is the mean value of Δ. Substituting (3) in (2), we obtain for the case when $\Delta_1 \ll \langle\Delta\rangle$ and $|\omega-\Delta| \ll \langle\Delta\rangle$

$$\rho = \frac{\rho_0}{2}\left(\frac{\langle\Delta\rangle^2}{\langle\Delta_1^2\rangle}\right)^{1/4} D_{-1/2}\left(\frac{\langle\Delta\rangle-\omega}{\sqrt{\langle\Delta_1^2\rangle}}\right)\exp\left[-\frac{(\omega-\langle\Delta\rangle)^2}{4\langle\Delta_1^2\rangle}\right], \qquad (4)$$

where $D_{-1/2}$ is a parabolic-cylinder function.

Thus, the singularity in the density of states becomes smeared out in an energy region having a width on the order of $\langle\Delta_1^2\rangle^{1/2}$, and the maximum in the density of states shifts by an amount on the order of $\langle\Delta_1^2\rangle^{1/2}$ towards energies larger than $\langle\Delta\rangle$.

3. INHOMOGENEITIES OF SMALL SIZE. EQUATION FOR Δ

A more interesting case is that in which the dimension of the inhomogeneities is small compared with the pair dimension. Then the correction to the Green's functions turns out to be small everywhere except at the singular points. We consider first the case of a strongly "contaminated" superconductor, when the electron mean free path l is small compared with $\xi_0 = v/T_c$. In this case we can write a relatively simple system of equations for the Green's function[4]:

$$\Delta\alpha-\omega\beta+\frac{vl_{tr}}{6}\left\{\alpha\left(\frac{\partial}{\partial\mathbf{r}}-2ie\mathbf{A}\right)^2\beta-\beta\frac{\partial^2\alpha}{\partial\mathbf{r}^2}\right\}=0,$$

$$\alpha^2+|\beta|^2=1,\quad \Delta=g\pi T\sum_\omega\beta(\omega);$$

$$j=-\frac{ep_0^2l_{tr}}{6\pi}T\sum_\omega\left\{4e\mathbf{A}|\beta|^2+i\left(\beta^*\frac{\partial\beta}{\partial\mathbf{r}}-\beta\frac{\partial\beta^*}{\partial\mathbf{r}}\right)\right\},$$

$$\alpha=\frac{i}{\pi\rho_0}G(\mathbf{r},\mathbf{r}),\quad \beta=\frac{1}{\pi\rho_0}F(\mathbf{r},\mathbf{r}).$$

Formulas (5) and (6) were obtained in[4] in the Born approximation. With the aid of the results of[5] we can verify that they also hold true in the case of strong interaction between the electrons and the impurities.

In the homogeneous case and in the absence of a magnetic field ($\mathbf{A}=0$), Eqs. (5) and (6) have a solution that does not depend on the coordinates. To describe the inhomogeneities it is convenient to use the phenomenological device employed in[6,7]. We assume that the effective interaction constant g is a random function of the coordinates, and write it in the form

$$1/g=\langle 1/g\rangle+g_1. \tag{7}$$

The amount and dimensions of the inhomogeneities can be characterized by the correlation function

$$\varphi(\mathbf{r}-\mathbf{r}')=\langle g_1(\mathbf{r})g_1(\mathbf{r}')\rangle,\quad \varphi_\mathbf{k}=\int d^3\mathbf{r}\varphi(\mathbf{r})\exp(-i\mathbf{kr}). \tag{8}$$

The characteristic dimension r_c of the inhomogeneities is determined by the distances over which the function $\varphi(\mathbf{r})$ decreases. We shall assume below that the dimension of the inhomogeneities is smaller than or of the order of the pair dimension, $r_c \lesssim \xi \sim (vl/T_c)^{1/2}$. If $g_1 \ll 1$, then the corrections that must be introduced in Δ as a result of g_1 are small everywhere with the exception of a nrrrow temperature region near the transition temperature. In the absence of a magnetic field ($\mathbf{A}=0$) we seek the solution of the system (5) in the form

$$\Delta=\langle\Delta\rangle+\Delta_1,\quad \beta=\langle\beta\rangle+\beta_1,\quad \alpha=\langle\alpha\rangle+\alpha_1. \tag{9}$$

In the approximation linear in g_1 we get from (5), (6), and (8)

$$\Delta_1(\mathbf{k})=-\langle\Delta\rangle g_1(\mathbf{k})\Big/\left\{\pi T\sum_\omega\left[(\omega^2+\langle\Delta\rangle^2)^{-1/2}-\frac{\omega^2}{(\omega^2+\langle\Delta\rangle^2)(vl_{tr}\mathbf{k}^2/6+(\omega^2+\langle\Delta\rangle^2)^{1/2})}\right]\right\}. \tag{10}$$

Thus, $\Delta_1(\mathbf{r})$ is a random quantity whose correlation function is expressed in terms of the correlation function of the interaction constant by the relation

$$\langle\Delta_1(\mathbf{r})\Delta_1(\mathbf{r}')\rangle=\langle\Delta\rangle^2\int\frac{d^3\mathbf{k}}{(2\pi)^3}M_\mathbf{k}\exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}')], \tag{11}$$

where

$$M_\mathbf{k}=\varphi_\mathbf{k}\left\{\pi T\sum_\omega\left[(\omega^2+\langle\Delta\rangle^2)^{-1/2}-\frac{\omega^2}{(\omega^2+\langle\Delta\rangle^2)(vl_{tr}\mathbf{k}^2/6+(\omega^2+\langle\Delta\rangle^2)^{1/2})}\right]\right\}^{-2}. \tag{12}$$

When $\omega_D/T_c \gg (k\xi)^2 \gg 1$ we have with logarithmic accuracy

$$M_\mathbf{k}=\varphi_\mathbf{k}/\ln^2(k\xi)^2. \tag{13}$$

When $k\xi \ll 1$ we have

$$M_0=\varphi_0\left\{\pi T\sum_\omega\frac{\langle\Delta\rangle^2}{(\omega^2+\langle\Delta\rangle^2)^{3/2}}\right\}^{-2}. \tag{14}$$

It follows from (11)–(14) that the characteristic dimension over which Δ_1 changes coincides in order of magnitude with the dimension r_c of the inhomogeneities.

4. SELF-CONSISTENT FIELD REGION

Inhomogeneities in small amounts or of small dimensions lead to a small change of the parameter Δ, but change the form of the singularity in the density of states strongly. The density of states is expressed in terms of the imaginary part of the averaged Green's function by the relation

$$\rho=\rho_0\langle\operatorname{Im} i\alpha(-i\omega)\rangle. \tag{15}$$

In the case of inhomogeneities of small dimension $r_c \lesssim \xi$, the density of states can be obtained by the self-consistent-field method everywhere except in a narrow region near the end point of the spectrum. The deviations of the Green's functions α and β from the mean values are small in this case and can be obtained from formulas (5) and (9) by perturbation theory:

$$\alpha_1(\mathbf{k})=-\frac{\Delta_1(\mathbf{k})\langle\alpha\rangle\langle\beta\rangle}{vl_{tr}\mathbf{k}^2/6+\omega\langle\alpha\rangle+\langle\Delta\rangle\langle\beta\rangle}. \tag{16}$$

An equation for $\langle\alpha\rangle$ is obtained by averaging the general equation (5):

$$\langle\Delta\rangle\langle\alpha\rangle-\omega\langle\beta\rangle+\langle\Delta_1\alpha_1\rangle=0,\quad \langle\alpha\rangle^2+\langle\beta\rangle^2=1. \tag{17}$$

Substituting α_1 from (16) and Δ_1 from (9) in this expression, we obtain

$$\langle\alpha\rangle-\omega\langle\beta\rangle/\langle\Delta\rangle=\eta\langle\alpha\rangle\langle\beta\rangle,\quad \langle\alpha\rangle^2+\langle\beta\rangle^2=1. \tag{18}$$

The dimensionless parameter η, which characterizes the region of smearing of the singularity, is equal to

$$\eta=\frac{3\langle\Delta\rangle}{\pi^2vl_{tr}}\int_0^\infty dkM_k, \tag{19}$$

where M_k is determined by (12). In the derivation of (19) we made use of the fact that near the singularity the second and third terms in the denominator of (16) almost cancel each other. Using expression (13) for M_k, we obtain at $r_c \ll \xi$ a relation connecting the parameter η with the relative magnitude of the inhomogeneities $\varphi(0)$ and their dimension r_c:

$$\eta\sim\varphi(0)\left(\frac{r_c}{\xi\ln(\xi/r_c)^2}\right)^2. \tag{20}$$

Formula (18) can be obtained without assuming smallness of the mean free path l. Expression (19) for the parameter η becomes more complicated, and for isotropic scattering by the impurities it takes the form

$$\eta=\frac{\langle\Delta\rangle}{\pi^2v}\int_0^\infty dk\,k\frac{\operatorname{arctg} lk}{1-(lk)^{-1}\operatorname{arctg} lk}$$
$$\times\varphi_k\left\{\pi T\sum_\omega\left[\frac{2\omega^2}{(\omega^2+\langle\Delta\rangle^2)vk}\frac{\operatorname{arctg}(vk/(2\sqrt{\omega^2+\langle\Delta\rangle^2+\tau^{-1}}))}{1-(lk)^{-1}\operatorname{arctg}(vk/(2\sqrt{\omega^2+\langle\Delta\rangle^2+\tau^{-1}}))}-(\omega^2+\langle\Delta\rangle^2)^{-1/2}\right]\right\}^{-2}. \tag{21}$$

At small free path lengths $l \ll r_c, \xi$, formulas (19) and

(20) follow from (21). In the other limiting case $r_c \ll l, \xi$, we obtain

$$\eta = \frac{\langle\Delta\rangle}{2\pi v}\int_0^\infty \frac{dk\, k\varphi_k}{\ln^2(k\xi)^2} \sim \frac{\Delta r_c}{v\ln^2(\xi/r_c)^2}. \quad (22)$$

Equation (18) has the same form as the equation for the Green's function of a superconductor with paramagnetic impurities[1] or a dirty superconductor of small dimension in a magnetic field[8-10]. To obtain the density of states from formula (15), we replace ω in (18) by $-i\omega$ and make use of the fact that near the threshold, when $|\omega - \Delta| \ll \Delta$ and $\eta \ll 1$, the quantities α and β are large compared with unity. The system (18) reduces to a single cubic equation

$$i\eta\alpha^3 + \alpha^2(\omega/\langle\Delta\rangle - 1) - 1/2 = 0. \quad (23)$$

The frequency region where the first term of (23) is significant and the smearing of the singularity takes place is of the order of $\Delta\eta^{2/3}$. The maximum density of states is attained at an energy equal to the average gap $\omega = \langle\Delta\rangle$, and equals

$$\rho_{max} = \rho_0\sqrt{3}(2\eta)^{-1/3}/2. \quad (24)$$

At frequencies exceeding Δ by more than $\Delta\eta^{2/3}$, the density of states tends to its value in the homogeneous superconductor. From (15) and (23) we obtain

$$\rho = \rho_0 \frac{\omega\,\mathrm{sign}(\omega - \langle\Delta\rangle)}{(\omega^2 - \langle\Delta^2\rangle)^{1/2}}\left[1 - \frac{5}{16}\eta^2\left(\frac{\langle\Delta\rangle}{\omega - \langle\Delta\rangle}\right)^3\right]. \quad (25)$$

With further increase of the frequency, the correction determined by formula (25), which is proportional to the fourth power of the inhomogeneity g_1, becomes smaller than the quadratic correction, which is obtained by solving the system (5) and (6) by perturbation theory. In this region we have

$$\rho = \rho_0 \frac{\omega}{(\omega^2 - \langle\Delta^2\rangle)^{1/2}}\left[1 + \frac{3M_0}{8\pi}\left(\frac{3\langle\Delta\rangle}{vl_{tr}}\right)^{3/2}\left(\frac{\langle\Delta\rangle^2}{\omega^2 - \langle\Delta\rangle^2}\right)^{5/4}\right], \quad (26)$$

where M_0 is given by (14).

Of greater interest is the region of frequencies smaller than $\langle\Delta\rangle$. As seen from (23), at frequencies smaller than a certain value ϵ, all three roots of (23) become imaginary and the density of states is equal to zero. The gap in the excitation spectrum ϵ is obtained from the condition that two roots of (23) be equal, and is given by

$$\varepsilon = \langle\Delta\rangle(1 - 3/2\,\eta^{2/3}). \quad (27)$$

Near the threshold $\omega - \epsilon \ll \Delta\eta^{2/3}$ we obtain from (15) and (23)

$$\rho = \rho_0\left(\frac{2}{3}\right)^{1/2}\eta^{-2/3}\left(\frac{\omega - \varepsilon}{\langle\Delta\rangle}\right)^{1/2}. \quad (28)$$

In a narrow region near the threshold, the self-consistent-field method is not applicable and formula (28) is incorrect.

5. DENSITY OF STATES NEAR THE END OF THE SPECTRUM

In the region close to the threshold ϵ, the long-wave fluctuations of the function α become large. As will be shown below, the dimension of these fluctuations is $\sim \xi\,|(\epsilon - \omega)/\langle\Delta\rangle|^{-1/2}$. We denote by L a certain dimension satisfying the condition $r_c \ll L \ll \xi\,|(\epsilon - \omega)/\langle\Delta\rangle|^{-1/2}$. Fluctuations with a dimension smaller than L are themselves small and can be accounted for by perturbation theory. Averaging the system (5) over such fluctuations, we obtain

$$\langle\Delta\rangle\tilde{\alpha} - \omega\tilde{\beta} + \frac{vl_{tr}}{6}\left(\tilde{\alpha}\frac{\partial^2\tilde{\beta}}{\partial r^2} - \tilde{\beta}\frac{\partial^2\tilde{\alpha}}{\partial r^2}\right) + \tilde{\Delta}_1\tilde{\alpha} = \eta\langle\Delta\rangle\tilde{\alpha}\tilde{\beta},$$
$$\tilde{\alpha}^2 + \tilde{\beta}^2 = 1. \quad (29)$$

The tilde denotes here averaging over the short-wave fluctuations. The random quantity $\tilde{\Delta}_1$ has a correlation function (11), where M_k is determined by formula (14) when $kL \ll 1$, and decreases rapidly when $kL \gg 1$. The region $kL \lesssim 1$ makes a small contribution to the integral in (19), and the parameter η in (29) can be therefore assumed to be determined by formulas (19)—(22). Near the end of the spectrum we have $\beta = i\alpha(1 - (2\alpha^2)^{-1})$, $\tilde{\alpha} = -i\eta^{-1/3} - i\psi$, with $|\psi| \ll \eta^{-1/3}$. Expanding the system (23) in powers of ψ, we obtain the equation

$$\frac{vl_{tr}}{6}\frac{\partial^2\psi}{\partial r^2} + \frac{3}{2}\langle\Delta\rangle\eta^{2/3}\psi^2 = \eta^{-2/3}(\varepsilon - \omega + \tilde{\Delta}_1). \quad (30)$$

In this equation, only long-wave fluctuations are significant, and it can therefore be assumed that the correlation function of the random quantity $\tilde{\Delta}_1$ is proportional to the δ function

$$\langle\tilde{\Delta}_1(\mathbf{r})\tilde{\Delta}_1(\mathbf{r}')\rangle = \langle\Delta\rangle^2 M_0\delta(\mathbf{r} - \mathbf{r}'), \quad (31)$$

where M_0 is given by (14).

The density of states is expressed in terms of ψ by the relation

$$\rho = \rho_0\,\mathrm{Im}\,\langle\psi(\omega)\rangle. \quad (32)$$

When ω exceeds ϵ by a sufficiently large amount, we can neglect $\tilde{\Delta}_1$ in (30) compared with $\omega - \epsilon$ and with fluctuations of ψ. As a result we obtain expression (28) for the density of states (32).

Solving (30) by iteration with respect to $\tilde{\Delta}_1$, we obtain, taking into account the first correlation correction, the following expression for the density of states

$$\rho = \rho_0\left(\frac{2}{3}\right)^{1/2}\eta^{-2/3}\left(\frac{\omega - \varepsilon}{\langle\Delta\rangle}\right)^{1/2}\left[1 + \frac{3}{32\sqrt{2}\pi}\left(\frac{6s}{\omega - \varepsilon}\right)^{5/4}\right],$$
$$s = \langle\Delta\rangle\left(\frac{\langle\Delta\rangle M_0^{2/3}}{vl_{tr}}\right)^{6/5}. \quad (33)$$

For a region in which the results obtained by the self-consistent-field method are valid to exist it is necessary that the second term in the square brackets in (33) be small when $\omega - \epsilon \sim \Delta\eta^{2/3}$. Expressing M_0 and η from formulas (14) and (20) in terms of the amount of the inhomogeneities and their dimension r_c, we write this condition in the form

$$\varphi_0^{1/6}\,(r_c/\xi)^{4/3} \ll 1. \quad (34)$$

When the condition (34) is satisfied, almost the entire region where the singularity is smeared is described by formulas (15) and (23). The maximum of the density of states is given by formula (24). The edges of the peak are described by formulas (25) and (28). When the inequality inverse to (34) is satisfied, formulas (2) and (4) are valid.

Even if (34) is satisfied, the region near the threshold is not described by the self-consistent-field method, since the correction term in (33) increases and ceases to be small when

$$[(\omega-\varepsilon)/\Delta]^{5/4}\sim\varphi(0)\left(\frac{r_c}{\xi}\right)^3. \qquad (35)$$

It is of interest to find the density of states at energies much smaller than the threshold ϵ. At such energies, the density of states is different from zero only because of the regions in which the parameter Δ is noticeably smaller than its mean value. As seen from (30), when $\omega < \epsilon$ the parameter ψ is real almost everywhere. The imaginary part arises only as a result of regions where $-\tilde{\Delta}_1 > \epsilon - \omega$. The dimension r_0 of these regions should be sufficiently large to make the first term in (30) smaller than or of the order of the remaining terms. It follows therefore that $r_0 \sim \xi((\epsilon-\omega)/\langle\Delta\rangle)^{-1/4}$. The density of states is proportional to the number of such regions:

$$\rho\sim\rho_0\exp\left\{-\frac{1}{M_0\langle\Delta\rangle^2}\int\tilde{\Delta}_1{}^2d^3\mathbf{r}\right\}$$
$$\sim\rho_0\exp\left\{-\frac{\tilde{a}}{\varphi(0)}\left(\frac{\xi}{r_c}\right)^3\left(\frac{\varepsilon-\omega}{\langle\Delta\rangle}\right)^{5/4}\right\}. \qquad (36)$$

The calculation given in the Appendix for the numerical coefficient $\tilde{a}$ and the preexponential factor leads to the result

$$\rho=2{,}48\rho_0\eta^{-3/8}\left(\frac{\varepsilon-\omega}{\langle\Delta\rangle}\right)^{9/8}\left(\frac{vl_{tr}}{3\langle\Delta\rangle M_0^{2/3}}\right)^{3/4}$$
$$\times\exp\left\{-\frac{48\pi}{5M_0}\left(\sqrt{\frac{2}{3}}\frac{vl_{tr}}{3\langle\Delta\rangle}\right)^{3/2}\left(\frac{\varepsilon-\omega}{\langle\Delta\rangle}\right)^{5/4}\right\}. \qquad (37)$$

Formulas (33) and (37) give the limiting values of a certain universal function defined by Eqs. (30) and (31).

6. THE TUNNEL CURRENT-VOLTAGE CHARACTERISTIC

In the case of a weakly inhomogeneous superconductor, the dependence of the tunnel current on the voltage is expressed in the usual manner in terms of the density of states[11-13]:

$$eRJ={}^1/_2\int_{-\infty}^{\infty}\left(\operatorname{th}\frac{x}{2T}-\operatorname{th}\frac{x-eV}{2T}\right)\operatorname{Re}\langle\alpha^{(1)}(-ix+\delta)\rangle$$
$$\times\operatorname{Re}\langle\alpha^{(2)}(-i(x-eV)+\delta)\rangle dx, \qquad (38)$$

where the superscripts 1 and 2 pertain to the superconductors on both sides of the barrier, R is the contact resistance when both metals are in the normal state, and e is the electron charge. The expression for the tunnel current becomes much simpler if one metal is normal. In this case at T = 0 the derivative $\partial J/\partial V$ is proportional to the density of states in the superconductor. The temperature leads to a smearing of the singularity in the current-voltage characteristic. To observe the exponential "tail" in the density of states it is necessary that the temperature satisfy the condition

$$T\ll s. \qquad (39)$$

In the temperature region

$$s\ll T\ll\Delta\eta^{2/3} \qquad (40)$$

The exponential "tail" in the current-voltage characteristic is determined by the temperature, and the region of the maximum on the $\partial J/\partial V$ curve is described by formulas (24), (25), and (28). At temperatures

$$T\gg\Delta\eta^{2/3} \qquad (41)$$

the inhomogeneities lead only to slight corrections on the current-voltage characteristic. In the case of a tunnel junction of two superconductors, the singularities in the density of states lead to singularities in the current-voltage characteristic at all temperatures. For example, for homogeneous superconductors at $eV = \Delta_1 + \Delta_2$, there appears in the J(V) dependence a jump equal to[13]

$$J_+-J_-=\frac{\pi}{4eR}(\Delta_1\Delta_2)^{1/2}\left(\operatorname{th}\frac{\Delta_1}{2T}+\operatorname{th}\frac{\Delta_2}{2T}\right) \qquad (42)$$

and inhomogeneities lead to a smearing of the jump. At not too low temperatures, the exponential "tails" in the density of states make no appreciable contribution to the current, and we can use the self-consistent-field approximation (15) and (18) for the density of states. In this case, a gap ϵ exists in the excitation spectrum. The integral of (38) breaks up into several regions. In the integrals over the regions $x < \epsilon_1$ and $x > eV + \epsilon_2$ it is necessary to make the change of variables $x \to -x$ and $x - eV \to x$, as a result of which we obtain

$$2eRJ=\int_{\varepsilon_1}^{\infty}\left(\operatorname{th}\frac{x+eV}{2T}-\operatorname{th}\frac{x}{2T}\right)\frac{\rho_1(x)}{\rho_{10}}\frac{\rho_2(x+eV)}{\rho_{20}}dx$$
$$+\int_{\varepsilon_2}^{\infty}\left(\operatorname{th}\frac{x+eV}{2T}-\operatorname{th}\frac{x}{2T}\right)\frac{\rho_1(x+eV)}{\rho_{10}}\frac{\rho_2(x)}{\rho_{20}}dx$$
$$+\theta(eV-\varepsilon_1-\varepsilon_2)\int_{\varepsilon_1}^{eV-\varepsilon_2}\left(\operatorname{th}\frac{x}{2T}+\operatorname{th}\frac{eV-x}{2T}\right)\frac{\rho_1(x)}{\rho_{10}}\frac{\rho_2(eV-x)}{\rho_{20}}dx. \qquad (43)$$

The first two terms are smooth functions of the voltage and do not make a large contribution to the derivative $\partial J/\partial V$ at $eV \approx \epsilon_1 + \epsilon_2$. The last term differs from zero only when $eV > \epsilon_1 + \epsilon_2$. In the homogeneous case it leads to the jump (42). Inhomogeneities lead to a weakening of the singularity. If inhomogeneities exist only in one superconductor ($\eta_1 = 0$), then there is a jump in the derivative

$$R\frac{\partial J}{\partial V}=\frac{\pi}{4}\left(\operatorname{th}\frac{\varepsilon_1}{2T}+\operatorname{th}\frac{\varepsilon_2}{2T}\right)\eta_2^{-2/3}\left(\frac{\langle\Delta\rangle_1}{3\langle\Delta\rangle_2}\right)^{1/2}. \qquad (44)$$

In the case of inhomogeneities in both superconductors, the singularities are even weaker:

$$R\frac{\partial J}{\partial V}=\frac{\pi}{12}\left(\operatorname{th}\frac{\varepsilon_1}{2T}+\operatorname{th}\frac{\varepsilon_2}{2T}\right)(\eta_1\eta_2)^{-2/3}\frac{eV-\varepsilon_1-\varepsilon_2}{(\langle\Delta\rangle_1\langle\Delta\rangle_2)^{1/2}}. \qquad (45)$$

Formulas (44) and (45) are valid when $eV - \epsilon_1 - \epsilon_2 \ll \Delta\eta^{2/3}$, since they were derived on the basis of formula (28) for the density of states. In the opposite limiting case, the inhomogeneities lead only to a small deviation from (42). Another limitation is connected with the application of the self-consistent-field method. The singularities defined by formulas (44) and (45) become smeared in the region $eV - \epsilon_1 - \epsilon_2 \gtrsim s$. When $eV < \epsilon_1 + \epsilon_2$, the singular part of the current-voltage characteristic is determined by the "tail" of the spectrum.

7. ABSORPTION OF HIGH-FREQUENCY FIELD

Weak inhomogeneities at temperatures not very close to T_c lead to small corrections to the depth of penetration of the static magnetic field. Absorption of the high-frequency field is more sensitive to singularities in the density of states, and consequently to inhomogeneities.

We confine ourselves below to those frequencies and temperatures at which the self-consistent-field approximation is valid. In this case we can disregard the contribution made to the current by the change of the parameter Δ and we can replace the mean value of the product of the Green's functions, which enters in the equation for the current, by the product of the mean values. In the case of a strongly contaminated superconductor, the connection between the current and the vector potential **A** is local and is of the form

$$\mathbf{j} = -\frac{Ne^2\tau_{tr}}{m} Q(\omega)\mathbf{A}_1,$$

$$Q(\omega) = \int_{\Gamma_1} dx \left[1 + \beta\left(x - \frac{i\omega}{2}\right)\beta\left(x + \frac{i\omega}{2}\right) - \alpha\left(x + \frac{i\omega}{2}\right)\alpha\left(x - \frac{i\omega}{2}\right)\right] -$$

$$- \frac{1}{2i}\int_{\Gamma_2}\left(i - \operatorname{tg}\frac{x}{2T}\right)(\beta(x)\beta(x - i\omega) - \alpha(x)\alpha(x - i\omega))\,dx$$

$$+ \frac{1}{2i}\int_{\Gamma_3}\left(i + \operatorname{tg}\frac{x}{2T}\right)(\beta(x)\beta(x - i\omega) - \alpha(x)\alpha(x - i\omega))\,dx, \tag{46}$$

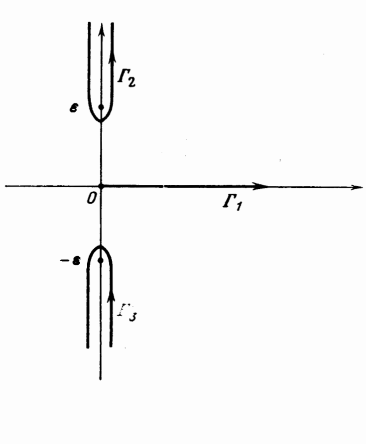
where the contours Γ_1, Γ_2, and Γ_3 are shown in the figure. At low frequencies $\omega \ll \Delta$, the absorption is determined by the density of the normal electrons, which is small at low temperatures and depends on the singularity in the density of states. Contributing to the imaginary part of Q, which determines the absorption, are only the integrals over the contours Γ_2 and Γ_3, a calculation of which at $T \ll \Delta\eta^{2/3}$ yields

$$\operatorname{Im} Q = -\frac{8\omega T}{3\Delta\eta^{4/3}}\operatorname{sh}\left(\frac{\omega}{2T}\right)K_1\left(\frac{\omega}{2T}\right)\exp\left(-\frac{\varepsilon}{T}\right), \quad \omega \ll \Delta\eta^{2/3},$$

$$\operatorname{Im} Q = -2\left(\frac{2\pi}{3\Delta}\right)^{1/2} T^{3/2}\eta^{-2/3}\frac{\rho(\omega+\varepsilon)}{\rho_0}\exp\left(-\frac{\varepsilon}{T}\right), \quad T \ll \omega \ll \Delta, \tag{47}$$

where K_1 is a cylindrical function, ϵ is the spectrum gap defined by (27), and the density of states $\rho(\omega)$ is determined by formulas (15), (23)–(25), and (28). The real part of Q, which equals $\pi\Delta$ at low temperatures and frequencies, is not sensitive to the inhomogeneities. The region of applicability of formula (47) is bounded by the condition $T \gg s$. At lower temperatures, the "tails" in the density of states become important and the self-consistent-field method cannot be used.

At all temperatures, the singularity in the density of states leads to a singularity in the absorption at the



threshold frequency. In the homogeneous case, Im Q has a kink at $\omega = 2\Delta$. The inhomogeneities weaken the singularity. The singular part of Im Q can readily be obtained for arbitrary temperatures and turns out to equal

$$\operatorname{Im} Q_{\text{sing}}(\omega) = -\frac{\pi}{12\Delta\eta^{2/3}}(\omega - 2\varepsilon)^2\theta(\omega - 2\varepsilon)\operatorname{th}\frac{\varepsilon}{2T},$$

$$\omega - 2\varepsilon \ll \Delta\eta^{2/3}. \tag{48}$$

That part of Im Q which depends smoothly on the frequency is not sensitive to inhomogeneities at $T \gg \Delta\eta^{2/3}$ and is exponentially small at $T \ll \Delta$. The singularity (48) becomes smeared out in the frequency region $\omega - 2\epsilon \lesssim s$. With increasing frequency, the influence of the inhomogeneities weakens. If the condition inverse to (48) is satisfied, then the inhomogeneities lead to a small correction that increases when the threshold is approached:

$$\operatorname{Im} Q^{(1)} = -\frac{3\pi\Delta}{4\sqrt{2}}\left(\frac{\Delta}{\omega - 2\Delta}\right)^{1/3}\eta\operatorname{th}\frac{\Delta}{2T} \qquad \Delta\eta^{2/3} \ll \omega - 2\Delta \ll \Delta. \tag{49}$$

We note that when $\omega < 2\Delta$ the real part of Q is subject to a correction equal to that given in (49).

CONCLUSION

Inhomogeneities in a superconductor lead to a smearing of the singularity in the density of states. If the dimension of the inhomogeneities is larger than the pair dimension, then the local density of states is determined by the local value of the ordering parameter Δ. In this case the shape of the curve describing the frequency dependence of the average density of states depends on the probability of different values of the effective-interaction constant. In the opposite case of small inhomogeneities, the shape of the peak of the density of states is universal and depends only on the parameter η, which is expressed in terms of the amount and dimension of the inhomogeneities. At frequencies smaller than the average threshold, there is an exponential "tail" in the density of states.

The smearing of the singularity in the density of states can be observed experimentally with the aid of the tunnel effect and with the aid of absorption of high-frequency radiation in superconductors. At low temperatures, the voltage dependence of the tunnel current between a superconductor and a normal metal makes it possible to reconstruct the dependence of the density of states on the energy. As a function of the voltage, the tunnel current between two homogeneous superconductors has a jump at a voltage equal to the sum of the gaps in the excitation spectrum of the superconductors. This jump exists at any temperature. The smearing of the singularity in the density of states leads to a smearing of the jump of the function J(V).

The form of the singularity in the density of states can be obtained by measuring the dependence of the absorption of the high-frequency radiation on the frequency. At low temperatures and frequencies satisfying the inequality $T \ll \omega \ll \Delta$, the intensity of the absorption is proportional to the density of states. At all temperatures, the form of the singularity in the density of states determines the dependence of the absorption on the frequency at frequencies close to double the gap.

It should be borne in mind that a systematic change in the effective interaction and in the ordering parameter Δ is possible near the surface of the superconductor or near the contact. This change was assumed to be small compared with the random variation, and was disregarded.

APPENDIX

We rewrite Eqs. (30) and (31) for the complex quantity ψ in the form of a system of equations for the real functions x and y:

$$\partial^2 x / \partial \mathbf{R}^2 + x^2 - y^2 = f(\mathbf{R}) + \gamma,$$
$$\partial^2 y / \partial \mathbf{R}^2 + 2xy = 0,$$
$$\langle f(\mathbf{R}) f(\mathbf{R}') \rangle = (2\pi)^{3/2} \delta(\mathbf{R} - \mathbf{R}'). \quad \text{(A.1)}$$

We have changed over here to the dimensionless variables

$$\mathbf{r} = (^2/_3)^{2/5} \frac{M_0^{1/5}}{\sqrt{2\pi}} \left(\frac{\pi v l_{tr}}{3 \langle \Delta \rangle M_0^{2/5}} \right)^{4/5} \mathbf{R},$$
$$\psi = (x + iy) (^2/_3)^{1/5} \eta^{-2/3} \left(\frac{3 \langle \Delta \rangle M_0^{2/3}}{\pi v l_{tr}} \right)^{3/5},$$
$$\Delta_1(r) = \left(\frac{3}{2} \right)^{3/5} \left(\frac{3 \langle \Delta \rangle M_0^{2/3}}{\pi v l_{tr}} \right)^{6/5} f(R),$$
$$\varepsilon - \omega = \left(\frac{3}{\pi} \sqrt{\frac{3}{2}} \right)^{6/5} s\gamma. \quad \text{(A.2)}$$

The density of states is proportional to the mean value $\langle y \rangle$, which is determined by a functional integral,

$$\langle y \rangle = \int \delta f(\mathbf{R}) y \{ f(\mathbf{R}) \} \exp \left\{ - \frac{1}{2 (2\pi)^{3/2}} \int d^3 \mathbf{R} f^2(\mathbf{R}) \right\}. \quad \text{(A.3)}$$

The distribution of the random quantity f in the energy region under consideration can be regarded as Gaussian, since the significant fluctuations are made up of a large number of small inhomogeneities. From the system (A.1) it follows that $\langle y \rangle$ is a certain universal function of the parameter γ. Let us find the asymptotic behavior of this function at large positive γ. The function $f(\mathbf{R})$ can be expressed in terms of y(R) with the aid of the system (A.1). When $\gamma \gg 1$, the main contribution to the integral (A.3) is made by the vicinity of the function $f\{y_0(\mathbf{R})\}$, in which the argument of the exponential in formula (A.3) has a minimum. At the minimum we have $y \to 0$ and the argument of the exponential takes the form

$$I(y) = \frac{1}{2(2\pi)^{3/2}} \int d^3 \mathbf{R} \left\{ \gamma + \frac{\partial^2}{\partial \mathbf{R}^2} \left(\frac{1}{2y} \frac{\partial^2 y}{\partial \mathbf{R}^2} \right) - \left(\frac{1}{2y} \frac{\partial^2 y}{\partial \mathbf{R}^2} \right)^2 \right\}^2 \quad \text{(A.4)}$$

Variation of the functional (A.4) with respect to y results in an equation whose solution is

$$y_0 \sim f_0(\mathbf{R}) = -6\gamma \,\mathrm{sh} \left(\frac{\gamma^{1/4} R}{\sqrt{2}} \right) \Big/ \frac{\gamma^{1/4} R}{\sqrt{2}} \mathrm{ch}^3 \left(\frac{\gamma^{1/4} R}{\sqrt{2}} \right). \quad \text{(A.5)}$$

Substituting this expression for y_0 in (A.4), we obtain

$$I(y_0) = \frac{48}{5\sqrt{\pi}} \gamma^{5/4}. \quad \text{(A.6)}$$

When $\gamma \gg 1$, the density of states is exponentially small and the argument of the exponential is determined by the formula (A.6). To find the preexponential factor we use a method proposed in[14], where the asymptotic behavior of the density of states was obtained for a particle in a random potential.

The origin in formula (A.5) is fixed, and we choose it for each $f(\mathbf{R})$ in such a way as to approximate $f(\mathbf{R})$ by means of the function $f_0(\mathbf{R} - \mathbf{R}_0)$ i.e., we determine $\mathbf{R}_0$ from the condition minimizing the functional

$$D(\mathbf{R}'|f) = \int d^3 \mathbf{R} [f(\mathbf{R}) - f_0(\mathbf{R} - \mathbf{R}')]^2, \quad \frac{\partial D}{\partial R} \bigg|_{\mathbf{R}' = \mathbf{R}_0} = 0. \quad \text{(A.7)}$$

The mean value $\langle y \rangle$ can be written in the form

$$\langle y \rangle = \left\langle \int_\sigma y \{ f(\mathbf{R}) \} \delta(\mathbf{R}' - \mathbf{R}_0(f(\mathbf{R}))) d^3 \mathbf{R}' \right\rangle$$
$$= \left\langle \int_\sigma y \{ f(\mathbf{R}) \} \delta(\nabla D(\mathbf{R}'|f)) \, |\det \nabla_j \nabla D(\mathbf{R}'|f)| \, d^3 \mathbf{R}' \right\rangle. \quad \text{(A.8)}$$

The dimension of the integration region σ in (A.8) is chosen such as to include a single well containing the bound state y_0. When $\gamma \gg 1$ this can always be done, since the distance between neighboring wells increases exponentially with increasing γ, and the dimension of the well, as follows from (A.5), decreases.

We interchange the order of the averaging and integration with respect to $\mathbf{R}'$ in (A.8). With $\mathbf{R}'$ fixed, we expand the function $f(\mathbf{R})$ in a series in the complete orthonormal set of functions $\varphi_n(\mathbf{R})$:

$$f(\mathbf{R}) = \sum_{n=0} \xi_n \varphi_n(\mathbf{R}). \quad \text{(A.9)}$$

We choose the functions φ_0 and φ_1 in the form

$$\varphi_0(\mathbf{R}) = -a f_0(\mathbf{R} - \mathbf{R}'), \quad \varphi_1 = b \, \partial f_0(\mathbf{R} - \mathbf{R}') / \partial \mathbf{R},$$
$$a^{-2} = \int d^3 \mathbf{R} f_0^2(\mathbf{R}) = 2(2\pi)^{3/2} I_0, \quad b^{-2} = \frac{1}{3} \int d^3 \mathbf{R} (\nabla f_0(\mathbf{R}))^2$$
$$= \frac{128\sqrt{2}}{7} \pi \gamma^{7/4}. \quad \text{(A.10)}$$

It follows from (A.7), (A.9), and (A.10) that

$$\nabla D = 2\xi_1 / b, \quad |\det \nabla \nabla D| = 8 b^{-6}. \quad \text{(A.11)}$$

The first relation in (A.11) is exact, and the second is valid in the principal approximation in the parameter γ^{-1}. Substituting the expressions (A.9)–(A.11) in (A.8), we obtain

$$\langle y \rangle = b^{-3} \int d^3 \mathbf{R}' \int_{-\infty}^{\infty} \prod_{n=0}^{\infty} \frac{d\xi_n}{(2\pi)^{3/4}} \delta(\xi_1) y \{ f(\mathbf{R}) \}$$
$$\times \exp \left[-\frac{1}{2} \sum_{n=0}^{\infty} \frac{\xi_n^2}{(2\pi)^{3/2}} \right]. \quad \text{(A.12)}$$

In the principal approximation, the coordinate dependence of $y\{f(\mathbf{R})\}$ is given by (A.5). To determine the coefficient in this formula, we multiply both sides of the first equation in (A.1) by $\varphi_0(\mathbf{R})$ and integrate with respect to $\mathbf{R}$. As a result we obtain

$$y(\mathbf{R}) = \varphi_0(\mathbf{R}) [-(a^{-1} + \xi_0)]^{1/2} \left(\int d^3 \mathbf{R} \varphi_0^3(\mathbf{R}) \right)^{-1/2}. \quad \text{(A.13)}$$

Integration with respect to ξ_n with $n \gg 2$ in (A.12) yields unity in the principal approximation, integration with respect to ξ_1 is carried out with the aid of a δ function, the integration with respect to ξ_0 is between the limits $-\infty$ and $-a^{-1}$, and in the principal approximation with respect to γ^{-1} we obtain

$$\langle y \rangle = \frac{16}{21} \left(\frac{10}{7} \right)^{1/2} \pi^{1/4} \gamma^{9/8} \left\{ {}^4/_9 \zeta(3) + \frac{31}{5\pi^2} \zeta(5) \right.$$
$$\left. - \frac{511}{\pi^6} \zeta(9) \right\}^{-1/2} \exp \left[- \frac{48 \gamma^{5/4}}{5\sqrt{\pi}} \right]. \quad \text{(A.14)}$$

Substituting this expression for ⟨ y⟩ into (32) and (A.2), we obtain expression (37) for the density of states.

[1] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksp. Teor. Fiz. 39, 1781 (1960) [Sov. Phys.-JETP 12, 1243 (1961)].
[2] Y. Masuda, Phys. Rev. 126, 1271, 1962.
[3] P. Hochenberg, Zh. Eksp. Teor. Fiz. 45, 1208 (1963) [Sov. Phys.-JETP 18, 834 (1964)].
[4] K. D. Usadel, Phys. Rev. Lett. 25, 507, 1970.
[5] A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 55, 2262 (1968) [Sov. Phys.-JETP 28, 1200 (1969)].
[6] A. I. Larkin, ibid. 58, 1466 (1970) [31, 784 (1970)].
[7] A. I. Larkin and Yu. N. Ovchinnikov, ibid. (in press).
[8] K. Maki, Progr. Theor. Phys. 29, 603, 1963; 31, 731, 1964.
[9] A. I. Larkin, Zh. Eksp. Teor. Fiz. 48, 232 (1965) [Sov. Phys.-JETP 21, 153 (1965)].
[10] Yu. N. Ovchinnikov, ibid. 57, 894 (1969) [30, 489 (1970)].
[11] I. C. Fischer and I. Giaever, J. App. Phys. 32, 172, 1961.
[12] I. Giaever and K. Megerle, Phys. Rev. 122, 1101, 1961.
[13] A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 51, 1535 (1966) [Sov. Phys.-JETP 24, 1035 (1967)].
[14] J. Zittartz and J. S. Langer, Phys. Rev. 148, 741, 1966.

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225