EXACT SOLUTION OF THE PROBLEM OF THE BROADENING OF THE HYDROGEN SPECTRAL LINES IN THE ONE-ELECTRON THEORY

V. S. LISITSA and G. V. SHOLIN

Submitted February 24, 1971

Zh. Eksp. Teor. Fiz. 61, 912–921 (September, 1971)

A general theory of the Stark broadening of the hydrogen spectral lines is developed within the framework of the one-electron (binary) approximation, without involving any of the restrictions of the impact or quasistatic treatments. The theory is based on the exact solution of the dynamical problem of the behavior of the radiating hydrogen atom in the field of a charged particle which is passing by. The four-dimensional symmetry properties of the hydrogen atom are used in order to solve this problem. A general analytic expression is obtained for the complete line contour; in the limiting cases of small and large frequency shifts this expression contains the results of the impact and quasistatic theories, respectively. An important feature of the present solution is that it provides an analytic description of the intermediate frequency region. The present treatment also retains the clarity of visualization characteristic of simple qualitative models of line broadening.

1. The theory of the Stark broadening of the spectral lines of hydrogen in a plasma[1-3] developed in recent years has obtained, on the whole, rather good experimental verification (see[4]). One can regard the basic assumptions of the theory as being reliably established: the quasistatic nature of the ionic broadening and the impact nature of the electronic broadening. At the same time, conditions can be realized experimentally in which the indicated assumptions of the theory are violated.[5-7] Therefore, it is of interest to develop a theory of broadening which would not be limited by either the impact or the quasistatic approximations. Such a generalization of the theory can apparently be reached most simply in the so-called one-electron scheme,[2] in which it is assumed that the effects of broadening coming from separate collisions are additive. However, within the framework of this scheme the profile of the line can be determined only if the exact solution of the dynamical problem, concerning the behavior of the radiating atom in the field of the charged particle passing by, is known. Hitherto such a program has not been carried out, and all of the calculations of the line contours in the region of the transition from the impact to the quasistatic limit have been based on more or less successful simplified models.[8-10] Such a situation is explained by the presence of a number of theoretical difficulties which are primarily related to taking account of the effects due to the rotation of the vector of the electric microfield created by the perturbing particle. It is clear, therefore, that even the very latest of the indicated models[10] contains, as will be evident below, a number of little-justified oversimplifications associated with taking the effects of rotation into consideration.

The goal of the present article is the development of a one-electron version of a general theory of the Stark broadening of the hydrogen lines, in which the effects mentioned above are correctly taken into consideration. In order to do this, the complete solution of the dynamical problem of the behavior of the hydrogen atom during collision with a charged particle is utilized. Following[11], a system of coordinates rotating with the perturbing field is introduced. In this coordinate system the problem turns out to formally reduce to the problem of the hydrogen atom in crossed (variable) electric and magnetic fields. One is able to obtain the complete solution of such a problem by using the four-dimensional symmetry properties of the hydrogen atom and a generalization of the method of Demkov et al.[12] to the case of time-dependent electric and magnetic fields. The resulting expression for the profile of the line is expressed in the form of a single integral of confluent hypergeometric functions. The obtained expression for the profile of the line contains, in the limit of small frequency shifts, the results of the impact theory (see[1,2]), whereas for the distant wings of the line a transition to the results of the quasistatic theory is achieved. It is important that, in contrast to[9,10], such a transition can be traced with complete account of the effects of rotation. An important feature of the result obtained here is an analytic description of the intermediate region of broadening, where up till now only interpolation formulas[13,14] have been available.

One should discuss the work of Vidal, Cooper, and Smith in more detail, since their formulation of the problem is very similar to the present treatment. In fact, article[10] also utilizes a transformation to a certain rotating coordinate system in which the Hamiltonian describing the interaction of the radiating atom with the perturbing electric field is diagonalized. However, this change to a rotating frame is not associated with the angle of rotation of the electric field vector but with an angle defined by a certain integral of this field (see formula (VIII.13) in[10]), which generally does not appear in the interaction Hamiltonian. As a consequence of this the wave functions used in[10] are not eigenfunctions of the perturbed Hamiltonian, so that in actual fact the desired diagonalization is not achieved in[10]. It is interesting that, in spite of the introduction of similar methods of solution, the authors of article[10] still had to use interpolation in order to calculate the resulting contour. We shall verify below that the utilization of complete solutions of the dynamical problem simultaneously gives a more compact description of the profile of the line, that is, more compact than the description

obtained by using approximate solutions.

2. We start from the expression for the intensity $I(\omega)$ of the dipole radiation of an atom associated with the transition from an upper level (a) to the lowest level (b):[1,2]

$$I(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tau e^{i\omega\tau}\Phi(\tau), \qquad (1)$$

where ω denotes the observed frequency, and $\Phi(\tau)$ is the correlation function of the dipole moments d of the atom

$$\Phi(\tau)=\left\{\sum_{a',b'}\sum_{k}\rho_{a'}(-1)^{k}\langle a'(t)|d_{k}'|b'(t)\rangle\langle b'(0)|d_{-k}'|a'(0)\rangle\right\}_{av}. \qquad (2)$$

Here the indices a′ and b′ denote evaluating the matrix elements with respect to the wave functions of the atom, defined in a certain fixed coordinate system, $d'_{\pm k}$ are the spherical components[1)] of the vector in this system; $\rho_{a'}$ is the density matrix for the initial states of the atom; the symbol $\{\ldots\}_{av}$ denotes averaging over the ensemble of perturbing particles in the plasma, where the motion of the perturbing particles is assumed to be given.

As follows from Eqs. (1) and (2), the emission spectrum of the atom is determined by the evolution of the vector **d**. Changes of **d** occur in a plasma, owing to the influence of the electric microfield associated with the ions and electrons which surround the atom. If we confine our attention to the one-electron (binary) approximation, i.e., if we assume, following[2,13], that the intensity distribution in the line can be derived by summing the intensities from individual collisions, then the character of the evolution of **d** will only depend on the parameters characterizing the flight of an individual perturbing charged particle. Assuming that the particle is moving along a straight line trajectory with a velocity **v** and with its initial position denoted by $\mathbf{r}_0$, we write the electric field created by this particle in the form

$$\mathbf{F}(t)=e\frac{\mathbf{r}_0+\mathbf{v}t}{|\mathbf{r}_0+\mathbf{v}t|^3}.$$

During the collision process the field changes in magnitude and, in addition, is rotated by 180° in the plane formed by the vectors $\mathbf{r}_0$ and **v** (the collision plane). If the rotation of the vector **F**(t) occurs sufficiently slowly, then the vector **d**(t) follows after it, at all times preserving its component along **F**. However, in the case of rapid flights, as we shall see, the component of the atom's angular momentum **L** along the normal to the collision plane is conserved. The described nature of the "quantization" of the atom leads in a natural way to the introduction of a rotating coordinate system:[11] At each moment of time the x axis of this system is directed along the vector **F**(t), and the z axis is directed along the normal to the collision plane. The transformation from the fixed to the rotating frame is determined by three Euler angles φ_0, θ_0, ψ_0 associated with the initial orientation of the field (**F**(0)), and also by the angle $\psi_1(t)$ of rotation of the field **F**(t) during the time t ($\psi_1(0)=0$). Then by introducing the rotation operator in the rotating system[11]

$$R(t)\equiv R[\varphi_0,\theta_0,\psi_0+\psi_1(t)]$$
$$=\exp\{iL_z\varphi_0\}\exp\{iL_y\theta_0\}\exp\{iL_z(\psi_0+\psi_1)\},$$

one can easily write down the relation between the wave functions $\chi(t)$ in the rotating system and the wave functions $\chi'(t)$ in the fixed system:

$$\chi'(t)=R(t)\chi(t). \qquad (3)$$

The corresponding relation between the components d_k and d'_k has the form

$$R^{+}d_{k}'R=\sum_{m}D_{km}^{(1)}d_{m}, \qquad (4)$$

where $D^{(1)}\equiv D^{(1)}[\varphi_0,\theta_0,\psi(t)]$ are the matrices of the representation of the three-dimensional rotation group.[15]

Let us change in Eq. (2) to the rotating coordinate system, by utilizing Eqs. (3) and (4). In this connection the symbol $\{\ldots\}_{av}$ denotes averaging over the three Euler angles φ_0, θ_0, ψ_0 and also over the parameters characterizing the flight of the particle in the collision plane. After substituting (3) and (4) into (2) we can verify that the averaging over the Euler angles only extends over the product of the D-functions. Carrying out this averaging in analogy to[11], we obtain

$$\Phi(\tau)=\sum_{a,b}\left\{\rho_{a}\sum_{k=-1}^{+1}e^{-ik\psi_1(t)}\langle a(t)|d_{k}|b(t)\rangle\langle b(0)|d_{-k}|a(0)\rangle\right\}_{av}. \qquad (5)$$

3. The central feature of the present investigation is the determination of the wave functions $\chi(t)$ in the rotating system. The Schrödinger equation for $\chi(t)$ is obtained from the equation for $\chi'(t)$ with Eq. (3) taken into account:[11]

$$i\hbar\frac{\partial\chi}{\partial t}=[H_0+d_xF(t)+\hbar L_z\dot{\psi}_1(t)]\chi\equiv[H_0+V(t)]\chi, \qquad (6)$$

where H_0 is the Hamiltonian of the free atom, and F(t) $\equiv |\mathbf{F}(t)|$.

From Eq. (6) it follows that both electrostatic (d_xF) and "magnetic" ($\hbar L_z\dot{\psi}_1$) interactions exist in the rotating system. The latter designation is justified by the fact that there is a complete analogy between the interaction $\hbar L_z\dot{\psi}_1$ and the interaction of an atom with a magnetic field in the absence of spin: $\mu_0 L_z H$ ($\mu_0=e\hbar/2mc$), so that the third term in the Hamiltonian (6) can be regarded as the interaction with a certain effective magnetic field $\mathbf{H}_{eff}=\hbar\dot{\psi}_1/\mu_0$, which appears in the rotating system.[2)] Thus, the problem has been reduced to finding the energy levels and wave functions of the hydrogen atom in mutually perpendicular (variable) electric and magnetic fields.

The possibility of an exact solution of this problem is based on the utilization of the degeneracy, specific for hydrogen, with respect to the orbital quantum number *l*, said degeneracy being closely related to the presence in a Coulomb field of an additional integral of the motion—namely, the Runge-Lenz vector*

$$\mathbf{A}=\frac{1}{2m}([\mathbf{pL}]-[\mathbf{Lp}])-\frac{e^2\mathbf{r}}{r}$$

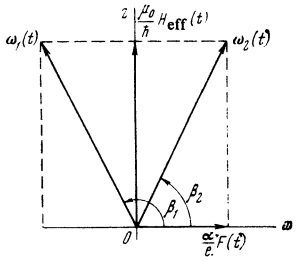
(see[17,18,12]). States pertaining to a fixed principal quantum number n are responsible for the effects of broadening. But, as is well known, it is precisely for such states that it is possible to use the symmetry

---

1) We use the notation of Wigner, [15] according to which $\sqrt{2}d_{-1}=d_x+id_y=-\sqrt{2}(d_{+1})^*$.

2) An analogy exists here with the treatment used in magnetic resonance problems. [16]

*$[\mathbf{pL}]\equiv\mathbf{p}\times\mathbf{L}$.



The mutual disposition of the vectors $(\alpha/e)\mathbf{F}(t)$, $(\mu_0/\hbar)\mathbf{H}_{eff}(t)$, $\omega_1(t)$, and $\omega_2(t)$ in the rotating coordinate system.

properties (corresponding to the rotation group O_4[17,18]) of the hydrogen atom. Therefore, following the usual procedure[17,18] we introduce new "angular momentum operators" $\mathbf{J}_1$ and $\mathbf{J}_2$:

$$\mathbf{J}_1 = {}^1/_2(\mathbf{L}+\mathbf{A}), \quad \mathbf{J}_2 = {}^1/_2(\mathbf{L}-\mathbf{A}). \tag{7}$$

Then V(t) in Eq. (6) can be represented in the form

$$V(t) = \mathbf{dF}(t) + \hbar\mathbf{L}\dot{\psi}_1 = \hbar\mathbf{J}_1\omega_1(t) + \hbar\mathbf{J}_2\omega_2(t), \tag{8}$$

where

$$\omega_{1,2}(t) = \frac{\mu_0}{\hbar}\mathbf{H}_{eff}(t) \mp \frac{\alpha}{e}\mathbf{F}(t), \tag{9}$$

and $\alpha \equiv (3/2)ne^2a_0/\hbar$ (a_0 denotes the Bohr radius).

In formulas (8) and (9) it is assumed that the vectors $\mathbf{F}$ and $\dot{\psi}_1$ ($\mathbf{H}_{eff}$) are directed, respectively, along the x and z axes of the rotating coordinate system (see the accompanying figure).

The subsequent solution consists in the construction of wave functions $u_{nn'n''}$ which diagonalize the Hamiltonian (8). These wave functions correspond to a definite projection of $\mathbf{J}_1$ on $\boldsymbol{\omega}_1$ (characterized by the quantum number n′) and a definite projection of $\mathbf{J}_2$ on $\boldsymbol{\omega}_2$ (characterized by the quantum number n″). In the case of constant $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ the functions $u_{nn'n''}$ can be obtained from the usual parabolic wave functions $u_{ni_1i_2}$ (where i_1 and i_2 are the quantum numbers corresponding to the projections of $\mathbf{J}_1$ and $\mathbf{J}_2$ on the x axis) by means of rotations through the angles β_1 and β_2, determining the axis of quantization of the atom (see the figure).

In our case there is an essential complication due to the dependence of the vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ on the time. One can show, however, that the direction of $\boldsymbol{\omega}_{1,2}(t)$ does not change during the collision process. In fact, from a direct investigation of the geometry of the trajectory in the collision plane it follows that

$$\operatorname{tg}\beta_2 = \mu_0 H_{eff}(t)/\hbar\frac{\alpha}{e}F(t) = \dot{\psi}_1(t)/\frac{\alpha}{e}F(t) = \rho v/\alpha \equiv 1/\delta, \tag{10}$$

where ρ is the impact parameter characterizing the flight path of the particle.

Thus, in the process of collision on the atom there are selected "directions of quantization," determined by the angle β_2, which depends on a single characteristic dimensionless parameter δ. Relation (10) shows that in the case of close and slow collisions ($\delta \gg 1$) the direction of the axis of quantization coincides with the direction of the axis of quantization coincides with the direction of the electric field, but in the case of fast and distant collisions ($\delta \ll 1$)—it coincides with the direction of the "magnetic field." The boundary value of the impact parameter corresponds to the Weisskopf radius α/v.[1] Conservation of the direction of quantization during the collision process means the absence of transitions between states having different values of n′ and n″. What has been said immediately permits us to generalize to our case the results which were obtained for constant $\mathbf{F}$ and $\mathbf{H}$, by treating the dependence on the time as dependence on a parameter, since the wave functions $u_{nn'n''}$ diagonalize our Hamiltonian. Thus, for the energy $E_{nn'n''}$ we obtain

$$E_{nn'n''}(t) = \langle nn'n''|H_0 + d_xF(t) + \hbar L_z\dot{\psi}_1(t)|nn'n''\rangle$$
$$= \hbar\omega_0 + \hbar(n'+n'')|\omega_{1,2}(t)| = \hbar\omega_0 + \hbar(n'+n'')\frac{\sigma}{\delta}\frac{\alpha}{e}F(t)$$
$$(\sigma \equiv \sqrt{1+\delta^2}), \tag{11}$$

which gives the following result for the wave function $\chi(t)$

$$\chi(t) = u_{nn'n''}\exp\left[-i\omega_0 t - i(n'+n'')\frac{\sigma}{\delta}\frac{\alpha}{e}\int^t F(t')\,dt'\right], \tag{12}$$

where the $u_{nn'n''}$ are obtained, as indicated, from $u_{ni_1i_2}$ by means of simple rotations:[12]

$$u_{nn'n''} = \sum_{i_1i_2} D^{((n-1)/2)}_{n'i_1}(0, \beta_1, 0)\, D^{((n-1)/2)}_{n''i_2}(0, \beta_2, 0)\, u_{ni_1i_2} \tag{13}$$
$$(\beta_1+\beta_2 = \pi, \quad \operatorname{tg}\beta_2 = 1/\delta).$$

Substitution of expression (12) into Eq. (5) gives the general solution for the problem which has been posed:[3)]

$$\Phi(\tau) = \sum_{n_a'n_a''n_b'n_b''}\sum_k \{|\langle n_an_a'n_a''|d_k|n_bn_b'n_b''\rangle|^2 \exp[-i\omega_0\tau - ik\psi_1(\tau)$$
$$- i(n_0'+n_a'')\sigma_a\eta_a(\tau) + i(n_b'+n_b'')\sigma_b\eta_b(\tau)]\}_{av}, \tag{14}$$

where

$$\eta_c(\tau) \equiv \frac{\alpha_c}{\delta_c e}\int_0^\tau F(t)\,dt;$$

the subscripts a(b) indicate whether the states belong to the upper (or lower) level.

4. From Eq. (14) it follows that the evolution of the correlation function, just like in the adiabatic model, is related to the amplitude F(t) of the electric field. Thus, the problem turns out to be analogous to the adiabatic theory with peculiarly determined components. The effects of non-adiabaticity reduce to the appearance of a dependence of the amplitudes of these components [associated with the D-functions in (13)] on the parameters characterizing the flight path, and it also leads to a certain complication of the phase factor.

We emphasize that the results obtained here are not connected with the impact or quasistatic approximations which are usually used in the theory of line broadening. As to the possibility of using Eq. (14) for specific calculations, then they in any event can be rather simply carried out for the principal terms of the Lyman, Balmer, and Paschen series, for which the dimension of the D-matrices in Eq. (13) is not too large. Below we shall carry out such calculations for the L_α line. In this connection certain results will be of a general nature.

In the case of the L_α line ($n_a = 2$, $n_b = 1$) expression (14) takes the form (the lowest state is denoted by the symbol 0)

$$\Phi(\tau) = \sum_{n',n''=1/2}^{-1/2}\sum_{k=-1}^{+1}\{|\langle 2n'n''|d_k|0\rangle|^2\exp[-i\omega_0\tau - ik\psi_1(\tau) + i(n'+n'')\sigma\eta(\tau)]\}_{av}. \tag{15}$$

[3)] The density matrix is obviously diagonal in the states $u_{nn'n''}$, corresponding to equalibrium of the atom with the medium. In what follows we shall omit it.

In evaluating the matrix elements in (15) it is convenient in (13) to change from parabolic wave functions $|2i_1i_2\rangle$ to spherical wave functions $|2\,l\mathrm{m}\rangle$:[17,18]

$$|2i_1i_2\rangle = \sum_{l,m} C[1/2, 1/2, l; i_1, i_2, m]\,|2lm\rangle, \qquad (16)$$

where C[1/2, 1/2, l; i_1, i_2, m] is the usual Clebsch-Gordan coefficient. The functions $|2\,l\mathrm{m}\rangle$, just like the functions $|2i_1i_2\rangle$ in Eq. (16), correspond to the x axis of quantization, whereas the spherical components d_k were referred to the z axis. Therefore, introducing an additional rotation through an angle $-\pi/2$ around the y axis, we find

$$\langle 2n'n,'|d_k|0\rangle = \sum_{\substack{i_1,\ i_2\\ mm'}} D^{(1/2)}_{n'i_1}(0,\beta_1,0)\,D^{(1/2)}_{n'i_2}(0,\beta_2,0)\,D^{(1)}_{mm'}\left(0,-\frac{\pi}{2},0\right) \times C[1/2, 1/2, 1; i_1, i_2, m]\langle 21m'|d_k|0\rangle. \qquad (17)$$

Utilization of the specific form of the matrix elements in (16) reduces the calculation to elementary trigonometry. Thus, for example, we have

$$\langle 2\,1/2\,1/2|d_{-1}|0\rangle = \langle 2\,-1/2\,-1/2|d_{+1}|0\rangle = \frac{a_0'}{2\sqrt{3}}(1+\sin\beta_2),$$
$$\langle 2\,-1/2\,1/2|d_{-1}|0\rangle = \langle 2\,1/2\,-1/2|d_{+1}|0\rangle = \frac{a_0'}{2\sqrt{3}}\cos\beta_2, \qquad (18)$$

where $a_0' = ea_0 2^{15/2}/3^{9/2}$. Then substituting (17) into (15), and with (10) taken into consideration, we obtain the final expression for $\Phi(\tau)$:

$$\Phi(\tau) = \left|\frac{a_0'}{2\sqrt{3}}\right|^2 e^{-i\omega_0\tau}\left\{\left(\frac{\sigma+1}{\sigma}\right)^2 e^{-i(\sigma-1)\eta(\tau)} + \left(\frac{\sigma-1}{\sigma}\right)^2 e^{-i(\sigma+1)\eta(\tau)} + 2\left(\frac{\delta}{\sigma}\right)^2 e^{-i\eta(\tau)} + 2 + (\text{c.c.})\right\}_{\text{av}} \qquad (19)$$

where the symbol (c.c.) denotes the complex conjugate of the expression written out.

The result (19) permits us to describe very intuitively a picture of Stark splitting. In fact, each of the side components of the L_α line is split in two (corresponding to the shifts $\sigma \pm 1$), and the central line is split into three components, two of which are symmetric with respect to the third unperturbed component. The amplitudes of the outer side-band components (proportional to $(\sigma-1)^2/\sigma^2$) decrease with increasing distance from the center of the line. A similar picture of the splitting of the Stark components is observed upon investigating the behavior of the hydrogen atom in a rotating (constant in magnitude) electric field.[19,20] The corresponding results follow from Eq. (19) in the case F = const and upon replacing the parameter δ by the ratio of the Stark frequency of splitting, $(\alpha/e)F$, to the angular velocity Ω of rotation of the field. It is interesting to note that the described picture leads to the appearance of seven components of the line L_α, whereas the number of states is given by $n^2 = 4$. The solution of this apparent contradiction consists in the fact that the effect of the atom's rotation following after the field F(t) (compare with[11]) leads to the appearance of additional "Raman" shifts of the frequency.

Let us consider the limiting expressions for $\Phi(\tau)$ which follow from (19) in the case of large and small values of δ. For $\delta \gg 1$ the quantity $\sigma \approx \delta$ and from Eq. (19) we obtain

$$\Phi(\tau) \approx \left|\frac{a_0'}{\sqrt{6}}\right|^2 e^{-i\omega_0\tau}\{e^{-i\delta\eta(\tau)} + e^{-i\eta(\tau)} + 1 + (\text{c.c.})\}_{\text{av}}. \qquad (20)$$

The first term in Eq. (20) corresponds to the usual Stark side component of the line upon taking into account only one phase modulation (see, for example,[1,11]); the second term corresponds to a small splitting of the central components of the line owing to amplitude modulation.[11]

The correlation function corresponding to an unperturbed atom ($\Phi(\tau) = \text{const}\cdot e^{-i\omega_0\tau}$) is obtained from Eq. (18) in the limit $\delta \to 0$, just as should happen.

In order to obtain the spectrum $I(\omega)$ it is necessary to carry out averaging over the parameters of the flight in Eq. (19) and find, according to Eq. (1), its Fourier transform. In this connection the symbol $\{\ldots\}_{\text{av}}$ in (19) means, as has already been indicated, the average with respect to the parameters of flight in the collision plane, that is

$$\{\ldots\}_{\text{av}} = 2\pi\rho d\rho N v_0 dt_0, \qquad (21)$$

where t_0 denotes the time of nearest approach.[4)] In order to determine the dependence of $\eta(\tau)$ on the parameters ρ and t_0, let us write F(t) in terms of these variables:

$$F(t) = \frac{e}{\rho^2 + v_0^2(t-t_0)^2}.$$

This gives

$$\eta(\tau) = \frac{a}{e\delta}\int_0^\tau F(t)\,dt = \operatorname{arctg}\frac{v_0(\tau-t_0)}{\rho} + \operatorname{arctg}\frac{v_0 t_0}{\rho}. \qquad (22)$$

Let us consider a typical integral ($\equiv \Lambda$) which appears upon averaging with respect to t_0 and taking the Fourier transform of (19):

$$\Lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{i\Delta\omega\tau}\int_{-\infty}^{\infty} dt_0\, e^{-i\epsilon\eta(\tau)}. \qquad (23)$$

Here $\Delta\omega \equiv \omega - \omega_0$, ϵ denotes one or the other coefficient associated with the alternating phase [$\epsilon = \sigma \pm 1$, 1; see Eq. (19)]. Substituting (22) into (23) and introducing the new variable $\tau' = \tau - t_0$, we verify that the integrals over t_0 and τ' turn out to be complex conjugates of each other, so that Eq. (23) reduces to the following form:

$$\Lambda = \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} dt\, e^{i\Delta\omega t - i\varepsilon\varphi(t)}\right|^2, \qquad (24)$$

where $\varphi(t) = \tan^{-1}(v_0 t/\rho)$. Integrating (24) by parts and using the relation[21]

$$\int_0^{\pi/2}\cos(a\operatorname{tg}\varphi - \gamma\varphi)\,d\varphi = \frac{\pi}{2}\frac{W_{\gamma/2,\,1/2}(2a)}{\Gamma(1+\gamma/2)}, \qquad (25)$$

where $W_{\lambda,\mu}(z)$ denotes the Whittaker function,[21] and Γ is the gamma function, we obtain

$$\Lambda = \frac{\pi}{8}\frac{\varepsilon^2}{\Delta\omega^2}\left[\frac{W_{\varepsilon/2,\,1/2}(2\rho v_0^{-1}\Delta\omega)}{\Gamma(1+\varepsilon/2)}\right]^2. \qquad (26)$$

In what follows we shall use a more compact way of writing Eqs. (25) and (26) in terms of the so-called Bateman function:[21]

$$k_\gamma(a) = W_{\gamma/2,\,1/2}(2a)\,/\,\Gamma(1+\gamma/2).$$

The result (26) shows that the intensity distribution

[4)] Here and below we assume, as is usually done in the theory of line broadening, that the velocity v_0 of the particles is given and is equal, for example, to its most probable Maxwellian value.

can be expressed in terms of a universal function of the frequency. In fact, using (26) for the Fourier inversion of (19), introducing the dimensionless frequency shift $\beta = \Delta\omega\alpha/v_0^2$, and determining the intensity distribution $I(\beta)$ from the relation $|a_0'/2\sqrt{3}|^2 I(\beta)d\beta = I(\omega)d\omega$, we obtain

$$I(\beta) = \frac{1}{\pi} h \frac{\gamma(\beta)}{\beta^2}, \tag{27}$$

where $h \equiv N(\alpha/v_0)^3$ is the characteristic dimensionless parameter of the problem[5] (see[1,14]); $\gamma(\beta)$ is a universal function which plays the role of a "variable linewidth"

$$\gamma(\beta) = \frac{\pi^3}{2}\int_0^\infty \frac{dx}{x(1+x^2)}\left[k^2_{\sqrt{1+x^{-2}}-1}(x\beta) + k^2_{\sqrt{1+x^{-2}}+1}(x\beta) + 2x^2k_1^2(x\beta)\right]. \tag{28}$$

Thus, the solution given by Eqs. (27) and (28) is expressed in the form of a single integral of tabulated functions (see, for example,[22]).

Formula (28) contains the results of the impact and quasistatic theories as limiting cases. Thus, for $\beta \ll 1$ the third term gives the major contribution in (28). A simple calculation with the relation

$$W_{1/2,\,1/2}(z)/\Gamma(3/2) = 2\pi^{-1}z[K_1(z) + K_0(z)], \tag{29}$$

taken into account, where K_1 and K_0 are Macdonald functions, gives the following result for $\gamma(\delta)$ (to within logarithmic accuracy):

$$\gamma(\beta) = -4\pi \ln \beta. \tag{30}$$

The presence of the logarithmic cutoff factor in Eq. (30) corresponds to taking the incompleteness of the perturbing flight paths into account.[14,23]

For $\beta \gg 1$ the analysis of the functions appearing in the integrand of Eq. (28) shows that the range of effective values x_{eff}, giving the major contribution to the integral, turns out to be $\sim 1/\sqrt{\beta} \ll 1$, so that one can use the asymptotic expression for $W_{\lambda\mu}(z)$ for large values of the first subscript (λ) and for large values of the argument (see[21]). A simple calculation leads to $\gamma(\beta) = 2\pi^2\beta^{-1/2}$ and

$$I(\beta) = 2\pi h/\beta^{5/2}, \tag{31}$$

that is, it leads to a quasistatic distribution of the intensity in the wing of the line.

5. The one-electron approximation does not include those cases when, at the instant when the electron is passing near the radiating electron, a slowly varying electric field due to the ions also exists. It is impossible to take account of the presence of the ionic field by formally introducing a finite splitting of the Stark levels, as is done in[10], since this field changes the geometrical picture of the collision in a major fashion: The vector of the electric field will no longer simply be rotated through 180° in the collision plane, but it will describe a more complicated loop in space. This case requires special consideration.

The domain of applicability of the result (27) and (28), obtained in the one-electron approximation, is strictly speaking determined by the inequality[2]

$$\beta \gg h. \tag{32}$$

[5] In the case under consideration of the dominance of binary collisions, $h \ll 1$.

On the other hand, the range of applicability of the impact approximation corresponds to $\beta \ll 1$. Since $h \ll 1$, there is a broad region of overlap of the two approximations and there is justification to use the result (27), (28) not only in the one-electron scheme, but in the general formulas of the impact theory, thereby also encompassing the center of the line.[6]

Expression (28) for the profile of the line L_α can also be obtained by the method of the secular equation. Such a calculation was first carried out long ago by Spitzer.[13] However, this approach has not subsequently received any appreciable development, which obviously is related to the sharp increase in the complexity of solving the secular equations for increasing values of n. As was shown above, utilization of the four-dimensional symmetry properties of the hydrogen atom enables us to diagonalize the Hamiltonian without resorting to a solution of the secular equation. Thanks to this, the generalization of the result (28) to the case of other lines reduces, in accordance with (14), to only making the appropriate change of the parameters appearing in the indices of the Bateman function, and to an increase in the number of terms consistent with an increase in the number of Stark components.

In conclusion we note that the method developed here can also be extended to the case of broadening of the spectral lines of the ions. In the Coulomb field of the ions, the trajectories of the perturbing particles will no longer be straight lines; however, the concept of the plane of the collisions retains its meaning.

The authors thank V. I. Kogan for helpful discussions and valuable advice.

[6] In this connection it is necessary to replace the "dynamical" cutoff of the width (30) by a cutoff based on considerations of the Debye screening. [1,2]

[1] I. I. Sobel'man, Vvedenie v teoriyu atomnykh spektrov (Introduction to the Theory of Atomic Spectra), Fizmatgiz, 1963.

[2] M. Baranger, in Atomic and Molecular Processes, edited by D. R. Bates, Academic Press, 1962, Chap. 13 (Russ. Transl., "Mir", 1964).

[3] P. Kepple and Hans P. Griem, Phys. Rev. 173, 317 (1968).

[4] W. L. Wiese, in Plasma Diagnostic Techniques, edited by R. H. Huddlestone and S. L. Leonard (Academic Press, Inc., New York, 1965), Chap. 6 (Russ. Transl., Mir, 1967).

[5] G. Boldt and W. S. Cooper, Z. Naturforsch. 19a, 968 (1964).

[6] R. C. Elton and H. R. Griem, Phys. Rev. 135, A1550 (1964).

[7] H. Schlüter and C. Avila, Astrophys. J. 144, 785 (1966).

[8] Hans R. Griem, Astrophys. J. 136, 422 (1962).

[9] Hans R. Griem, Phys. Rev. 140, A1140 (1965).

[10] C. R. Vidal, J. Cooper, and E. W. Smith, JQSRT 10, 1011 (1970).

[11] G. V. Sholin, V. S. Lisitsa, and V. I. Kogan, Zh. Eksp. Teor. Fiz. 59, 1390 (1970) [Sov. Phys.-JETP 32, 758 (1971)].

[12] Yu. N. Demkov, B. S. Monozon, and V. N. Ostrovskiĭ, Zh. Eksp. Teor. Fiz. 57, 1431 (1969) [Sov. Phys.-

JETP 30, 775 (1970)].
[13] Lyman Spitzer, Phys. Rev. 55, 699 (1939); 56, 39 (1939); 58, 348 (1940).
[14] V. I. Kogan, v sb. Fizika plazmy i problema upravlyaemykh termoyadernykh reaktsiĭ (in the collection: Plasma Physics and the Problem of Controlled Thermonuclear Reactions), Vol. 4, 1958, p. 258.
[15] Eugene P. Wigner, Group Theory and its Application to the Quantum Mechanics of Atomic Spectra, Academic Press, 1959 (Russ. Transl., IIL, M. 1961).
[16] Charles P. Slichter, Principles of Magnetic Resonance, Harper and Row, 1963 (Russ. Transl., Mir, 1967).
[17] J. W. B. Hughes, Proc. Phys. Soc. (London) 91, 810 (1967).
[18] A. I. Baz', Ya. B. Zel'dovich, and A. M. Perelomov, Rasseyanie, reaktsii i raspady v kvantovoĭ mekhanike (Scattering, Reactions, and Decays in Quantum Mechanics), Fizmatgiz, 1966.
[19] Tsutomu Ishimura, J. Phys. Soc. Japan 23, 422 (1967).
[20] V. S. Lisitsa, Optika i spektroskopiya, in press.
[21] Higher Transcendental Functions, edited by Arthur Erdélyi, McGraw-Hill Book Company, 1953, Vol. 1 (Russ. Transl., Nauka, 1966).
[22] Tablitsy funktsiĭ Uitteker (Tables of Whittaker Functions), V. Ch. Akad. Nauk SSSR, 1964.
[23] M. Lewis, Phys. Rev. 121, 501 (1961).

Translated by H. H. Nickle
92