INFLUENCE OF KINETIC EFFECTS ON WAVE PROPAGATION IN AN INHOMOGENEOUS PLASMA

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Nonlocal, linear, and nonlinear effects (of the echo type in a homogeneous plasma) which lead to anomalous reflection and penetration of the waves into a plasma are investigated. The effects considered exist only in an inhomogeneous plasma.

INTRODUCTION

WHILE the fields of waves propagating in a plasma are determined by the currents of the plasma particles, they nevertheless do not duplicate the particle motion, and vice versa. This difference makes possible unique effects of reflection and penetration of electromagnetic waves into the plasma.

As they absorb the wave energy, the plasma particles receive, by the same token, information concerning the perturbation; this information is contained in the microscopic oscillations of the distribution function. In other words, the system has a unique "memory." By virtue of the foregoing difference, the oscillations of the distribution function can be transported by the particles into a plasma region which for some reason (for example, Landau damping or opacity) is not accessible to the initial wave, and in this region, under certain conditions, the information stored in the "memory" of the system is again reproduced.

In a homogeneous plasma, this effect is well known as the electromagnetic-wave echo, which was investigated in[1-5]. The region of localization of the initial waves was determined in[1-3] by the Landau damping, while in[4,5] (where echo due to external sources with frequencies belonging to the region of opacity of the plasma was considered) it was determined by the depth of the skin layer.

It must be emphasized that in a homogeneous plasma, when the phase velocities and the unperturbed particle velocities are constant, the initial perturbation attenuates irreversibly in time, owing to the velocity scatter. We have in mind a process analogous to the decay of a macroscopic density resulting from an initial perturbation of the distribution function; this decay is due to the random diffusion of the particles (see[6]). The echo effect in a homogeneous plasma is therefore nonlinear.

In the case of an inhomogeneous plasma, when the wave in each section of the plasma interacts resonantly only with some definite group of particles, effects of the echo type can be linear. Thus, for example, in[7], they investigated the effect of nonlocal linear wave reflection. For a weakly inhomogeneous plasma, when the usual reflection coefficient is exponentially small, the coefficient of nonlocal reflection can be comparable with unity.

In the present paper we investigate a number of new nonlocal effects that exist only in an inhomogeneous plasma and are connected with the presence of "memory" in the system.

These include the following:

1) Linear echo, appearing when an extraordinary wave propagates along an inhomogeneous external magnetic field. It is important that the incident wave is again regenerated behind the opacity barrier and propagates in the previous direction.

2) A proper longitudinal echo from two transverse sources situated in the opacity region of an inhomogeneous plasma.

3) Echo at the summary frequency from two longitudinal sources in an inhomogeneous isotropic plasma in second order in the field amplitude. We note that in a homogeneous isotropic plasma, the echo is produced only at the difference frequency.

In all cases, in order for wave regeneration to be effective, it is necessary in essence that the phase of the distribution function vary in a regular fashion and have at a certain point of space an extremum with respect to the particle energies; this corresponds to the appearance of macroscopic currents.

The effects considered below are essentially kinetic and lead to anomalous penetration and reflection of the waves in an inhomogeneous plasma. As shown in[8], the echo is sensitive to random changes of the phase of the distribution function, which result from Coulomb collisions or microturbulences. Therefore the analysis presented below is valid in a sufficiently rarefied plasma, when the characteristic lengths of the investigated processes are small compared with the effective mean free path of the particles forming the echo (for details see[8]).

1. REGENERATION OF EXTRAORDINARY WAVES PROPAGATING ALONG AN INHOMOGENEOUS MAGNETIC FIELD

We consider the propagation of an extraordinary wave of frequency ω along an inhomogeneous magnetic field $\mathbf{H}_0 = H(z)\mathbf{e}_z$ having the form of a hump, with

$$H_{min} < mc\omega / e < H_{max}. \tag{1.1}$$

We neglect the inhomogeneity of the plasma density and also the variation of the particle velocity components as a result of the inhomogeneity of the magnetic field. As will be seen below, this limitation is not principal.

Under these conditions, we obtain from the linearized kinetic equation for the perturbation of the distribution function and from Maxwell's equations for the electric field $E_\omega(z)$ of the extraordinary wave, using the method of characteristics,

$$\frac{c^2}{\omega^2}\frac{d^2E_\omega(z)}{dz^2}+E_\omega(z)=\frac{\omega_{pe}^2}{i\omega}\int_0^\infty\frac{dv}{v}F(v)\left\{\int_{-\infty}^{z}dz'\,E_\omega(z')\cdot\right.$$
$$\left.\times\exp\left[\frac{i}{v}\int_{z'}^{z}\Omega(z'')\,dz''\right]+\int_z^\infty dz'E_\omega(z')\exp\left[-\frac{i}{v}\int_{z'}^{z}\Omega(z'')\,dz''\right]\right\}, \tag{1.2}$$

where $\Omega(z)=\omega-\omega_{He}(z)$ and $F(v)$ is the unperturbed distribution function of the particles with respect to the longitudinal velocities.

In the case of weak inhomogeneity of $H(z)$, we seek the solution of (1.2), in analogy with[7], in the WKB approximation in the form

$$E_\omega(z)=A_\omega(z)\exp\left(i\int_a^z k(z')\,dz'\right).$$

Then, integrating in (1.2) by parts with respect to z', accurate to terms of second order of smallness in the WKB parameter, we have

$$A_\omega(z)\Lambda(\omega,k)-i\left(\frac{\partial\Lambda(\omega,k)}{\partial k}\right)^{1/2}\frac{d}{dz}\left[A_\omega(z)\left(\frac{\partial\Lambda(\omega,k)}{\partial k}\right)^{1/2}\right]=0. \tag{1.3}$$

Here

$$\Lambda(\omega,k)=\varepsilon_\perp(\omega,k)-(ck/\omega)^2, \tag{1.4}$$

where

$$\varepsilon_\perp(\omega,k)=1+\frac{\omega_{pe}^2}{\omega}\int_{-\infty}^{+\infty}\frac{dvF(v)}{v\alpha_\omega(z,v)}\qquad\left(\alpha_\omega(z,v)=k-\frac{\Omega(z)}{v}\right). \tag{1.5}$$

From (1.3) we find a dispersion equation for the extraordinary wave

$$\Lambda(\omega,k)=0, \tag{1.6}$$

which determines the wave vector $k(z)\equiv k_\omega(H(z))$ and the pre-exponential factor

$$A_\omega(z)=\left(\frac{\partial\Lambda(\omega,k_\omega(z))}{\partial k_\omega(z)}\right)^{-1/2}.$$

In the integration by parts in (1.2) it was assumed that $\alpha_\omega(z,v)\neq 0$, but the equality $\alpha_\omega(z,v)=0$ (which is the condition for cyclotron resonance) determines the contribution of the particles that are resonant with the wave at the point z and have a velocity v. This contribution can readily be taken into account by adding to (1.3) the current of the resonant particles, which enters in the first term of formula (1.3) in the form of the anti-Hermitian part of $\Lambda(\omega,k)$. Then Eq. (1.6) determines the complex wave vector $k_\omega(z)=q_\omega(z)+i\kappa_\omega(z)$, where $|\kappa_\omega|\ll|q_\omega|$.

We must make here the following remark. The curly brackets in the right-hand side of (1.2) contain the perturbation of the distribution function. It is seen from this equation that although the wave does attenuate as a result of cyclotron absorption (or as a result of reflection from the opacity region), the oscillations of the distribution function propagate with a phase factor in the exponential

$$\theta=\int^z\alpha_\omega(z'',v)\,dz''$$

further into the plasma region where there is no wave. In a homogeneous plasma, when integrating with respect to the velocities v, the phase factor θ leads to thermal diffusion of the perturbation in accordance with the law

$$E_\omega(z)\sim\exp\left(-\frac{3}{4}\left|\frac{\Omega z}{v_T}\right|^{2/3}\right).$$

In an inhomogeneous plasma, since α_ω depends on z, the phase θ can have an extremum with respect to v. Therefore integrating with respect to the particle velocities gives rise to a macroscopic current that regenerates the wave.

Bearing such a case in mind, we represent the field $E_\omega(z)$ as the sum of incident and regenerated waves:

$$E_\omega(z)=E_i(z,\omega)+E_R(z,\omega)=A_\omega(z)\exp\left(i\int_a^z k_\omega(z')\,dz'\right)$$
$$+R_\omega(z)\exp\left(i\int_{z_c}^z k_\omega(z')\,dz'\right), \tag{1.7}$$

where

$$A_\omega(z)=\frac{\Pi_\omega(z)}{\Pi_\omega(a)},\qquad\Pi_\omega(z)=\left[\frac{\partial\Lambda(\omega,k_\omega(z))}{\partial k_\omega(z)}\right]^{-1/2}$$

and z_c is the wave-regeneration point. Then, assuming that the following inequalities are satisfied for the wave amplitudes

$$|A_\omega(z)|\gg|R_\omega(z)|\quad(z<z_c),\qquad|A_\omega(z)|\ll|R_\omega(z)|\quad(z>z_c)$$

and substituting (1.7) in (1.2), we obtain, when account is taken of formula (1.3) for the regenerated wave,

$$E_R(z,\omega)=\frac{\omega_{pe}^2\Pi_\omega(z)}{\omega\Pi_\omega(a)}\int_0^\infty\frac{dv}{v}F(v)\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\,\Pi_\omega(x)\Pi_\omega(y)$$
$$\times\exp\left[i\int_a^x k_\omega(z')\,dz'+i\int_y^z k_\omega(z')\,dz'-\frac{i}{v}\int_y^x\Omega(z')\,dz'\right]. \tag{1.8}$$

The integrals with respect to x and y in (1.8) are determined by the contribution of the saddle points $z_{1,2}(v)$ $(z_1<z_2)$, at which there is satisfied the cyclotron-resonance condition

$$\operatorname{Re}\alpha_\omega(z_n(v),v)=0\qquad(n=1,2),$$

corresponding to absorption of a wave by the particles with velocity v at the point $z_1(v)$, and then emission at the point $z_2(v)$. Thus, the field $E_R(z,\omega)$ is generated by the nonlocal part of the resonant current excited by the field of the initial wave at the point $z_1(v)$.

The result of integration with respect to x and y in (1.8) is conveniently represented in the form

$$E_R(z,\omega)=\frac{2\Pi_\omega(z)}{\Pi_\omega(a)}\int_0^\infty dv\left[\varkappa_1(v)\varkappa_2(v)\frac{dz_1(v)}{dv}\frac{dz_2(v)}{dv}\right]^{1/2}\exp(i\psi_\omega(z,v)). \tag{1.9}$$

We have used here the notation

$$\psi_\omega(z,v)=\int_a^z k_\omega(z')\,dz'-\int_{z_1(v)}^{z_2(v)}\alpha_\omega(z',v)\,dz'+\gamma(v),$$
$$\gamma(v)=\frac{\varkappa_1^2(v)}{2\beta_1(v)}-\frac{\varkappa_2^2(v)}{2\beta_2(v)};\qquad\beta_n(v)=\left(\frac{\partial}{\partial z}\left[q_\omega(z)-\frac{\Omega(z)}{v}\right]\right)\Big|_{z=z_n(v)}.$$
$$\varkappa_n(v)\equiv\varkappa_\omega(z_n(v)),\qquad\varkappa_\omega(z)=-\frac{\pi\omega_{pe}^2\Pi_\omega^2(z)}{\omega q_\omega(z)}F\left(\frac{\Omega(z)}{q_\omega(z)}\right).$$

The integrand in (1.9) contains a rapidly-oscillating function. Therefore the value of the integral in (1.9) is determined by the contribution of the saddle point $v=v_0$; this point is determined from the equation

$$\frac{d\theta(v)}{dv}\equiv\left(\frac{d}{dv}\operatorname{Re}\int_{z_1(v)}^{z_2(v)}\alpha_\omega(z,v)\,dz\right)\Big|_{v=v_0}=0. \tag{1.10}$$

We then obtain ultimately

$$E_R(z,\omega)=\left[\frac{8\pi}{\rho}\varkappa_1(v_0)\varkappa_2(v_0)\frac{dz_1(v_0)}{dv_0}\frac{dz_2(v_0)}{dv_0}\right]^{1/2}\frac{\Pi_\omega(z)}{\Pi_\omega(a)}\times\exp\left[-i\frac{\pi}{4}+i\psi_\omega(z,v_0)\right], \tag{1.11}$$

where

$$\psi_\omega(z,v_0)=i\int_a^{z_1(v_0)}\varkappa_\omega(z')\,dz'+i\int_{z_2(v_0)}\varkappa_\omega(z')\,dz'-\theta(v_0)+\gamma(v_0),$$

$$\rho=d^2\theta(v_0)/dv_0^2.$$

Using (1.11), we find the ratio of the energy fluxes in the regenerated and initial waves, which determines the efficiency of the regeneration:

$$d_\omega[z,a]\equiv\frac{S_R(z,\omega)}{S_i(z,\omega)}=\left|\frac{8\pi}{\rho}\varkappa_1(v_0)\varkappa_2(v_0)\frac{dz_1}{dv_0}\frac{dz_2}{dv_0}\right|\exp\left[-2\int_a^{z_1(v_0)}\varkappa_\omega(z')\,dz'-2\int_{z_2(v_0)}^{z}\varkappa_\omega(z')\,dz'\right], \tag{1.12}$$

or, in order of magnitude

$$d_\omega(z,a)\sim[\varkappa(v_0)(L/q_\omega)^{1/2}]^2\exp\left(-2\int_a^{z_1(v_0)}\varkappa_\omega(z')\,dz'-2\int_{z_2(v_0)}^{z}\varkappa_\omega(z')\,dz'\right), \tag{1.13}$$

where L is the characteristic dimension of the inhomogeneity of $\alpha_\omega(z, v_0)$. As seen from (1.13), the effect increases with increasing inhomogeneity length L, and when the dimension of the cyclotron-resonance region $(L/q_\omega)^{1/2}$ for particles with velocity v_0 becomes larger than the attenuation length $1/\kappa(v_0)$, the effect reaches a maximum, which can be of the order of unity. However, it is impossible to use formula (1.11) here, since it is obtained under the assumption that the dependence of γ on v is inessential.

Let us examine in greater detail the conditions under which the given effect exists. First, formula (1.10) (the phase-coherence condition) reduces to the form

$$\int_{z_1(v_0)}^{z_2(v_0)}\Omega(z')\,dz'=0.$$

It follows therefore that the function $\Omega(z) = \omega - \omega_{He}$ should pass through zero. Then the wave interacts resonantly in the region $\omega_{He} < \omega$ with the particles traveling in the direction of the wave propagation, and in the region $\omega_{He} > \omega$ it interacts with the particles traveling in the opposite direction. Therefore in the region $\omega_{He} > \omega$ the particles (unless they are reflected) generate a reflected wave. Thus, at the points $z_{1,2}(v_0)$ we should have $\omega_{He}(z_{1,2}(v_0)) < \omega$. Excluding narrow regions in which $|\Omega(z)| \ll |k_\omega v_{Te}|$, we use for the refractive index the hydrodynamic expression

$$(ck_\omega/\omega)^2=1-\omega_{pe}^2/\omega\Omega.$$

We can then show that the effect investigated above exists if the magnetic field has the form of a hump, with

$$eH_{min}/mc\omega<1-{}^3/_2(\omega_{pe}/\omega)^2,\quad H_{max}>mc\omega/e$$

and, in addition, $3\sqrt{3}(\omega_{pe}/\omega)^2 < 2v_0/c$.

Let us consider a concrete example of the profile of the magnetic field in the region of the hump:

$$H(z)=H_0(1-z^2/L^2)\quad(1<eH_0/mc\omega<{}^3/_2).$$

If we take $(\omega_{pe}/\omega)^2 \ll 2(eH_0/mc\omega - 1)$, then we have

$$z_{1,2}(v_0)=\mp L\sqrt{3\left(1-\frac{mc\omega}{eH_0}\right)},\quad v_0=2c\left(\frac{eH_0}{mc\omega}-1\right),$$

$$q_0=\frac{\omega}{c},\quad \varkappa_0=\frac{\pi\omega_{pe}^2}{2\omega}F(v_0).$$

Formula (1.12) then assumes the form

$$d[z_c,z_1(v_0)]=4\pi(\varkappa_0\sqrt{L_*/q_0})^2,\ L_*=L(v_0/c)^{1/2}.$$

As seen from the expression for $z_{1,2}(v_0)$, the field is transported through a distance on the order of the length of the magnetic-field inhomogeneity.

In addition to the case investigated above, there can be observed also an effect of regeneration of a wave by particles reflected from a magnetic mirror. In this case the particles absorb the extraordinary wave in the region $\omega_{He} < \omega$, and are reflected in the magnetic mirror, where $\omega_{He} > \omega$. If the conditions of phase coherence and cyclotron resonance are satisfied for the particles in the region $\omega_{He} > \omega$ after reflection, then the particles will emit a wave traveling in the same direction as before, i.e., into the region of the strong magnetic field. In this case the phase-coherence condition takes the form

$$\frac{\partial\theta}{\partial\mathscr{E}}\equiv\left(\frac{\partial}{\partial\mathscr{E}}\oint_{\xi_1}^{\xi_2}\frac{dz\,\Omega(z)}{v_\parallel(z,\mathscr{E},v_\perp)}\right)\Bigg|_{\xi_{1,2}=z_{1,2}(\mathscr{E},v_\perp)}=0. \tag{1.14}$$

Here $\mathscr{E}$ and $v_\perp$ are the energy and transverse velocity of the particle, $z_{1,2}$ are the points of cyclotron resonance $\Omega(z_{1,2}) = \pm q_\omega(z_{1,2})|v_\parallel(z_{1,2})|$, and the integration contour circles around the particle reflection point. As seen from formula (1.14), the phase of the distribution function θ depends on the transverse velocity. Therefore in this case an additional mixing is produced by the thermal scatter with respect to the transverse velocities, and it is necessary to stipulate that the phase have an extremum with respect to the transverse velocities. Otherwise the wave regeneration will have the character of an above-the-barrier effect. A case is also possible when the dependence of θ on the transverse velocity is insignificant and arithmetic addition of resonant particle currents with different transverse velocities takes place. It is easily seen that the calculations performed in this section pertain precisely to the case of a weak dependence of the phase θ on the transverse velocity.

2. PROPER ECHO OF TRANSVERSE WAVES IN AN INHOMOGENEOUS ISOTROPIC PLASMA

It was shown in Sec. 1 that an anomalous penetration of an extraordinary wave into a plasma situated in an inhomogeneous magnetic field is possible already in the linear approximation, owing to the presence of "memory" in the system. It is of interest in this connection to investigate the nonlinear effects, particularly the possibility of nonlinear "clearing" of an inhomogeneous plasma. We consider the case of an isotropic plasma. It is well known that in this case the electromagnetic signal does not penetrate into the plasma region where $\omega_{pe}(z) > \omega$. We now show that penetration actually is possible if account is taken of the effect of the nonlinear echo. We take two transverse sources

$$j_{ext}(z,t)=e_y\sum_{s=1}^{2}j_s\delta(z-a_s)\cos\omega_s t$$

in a plasma that is inhomogeneous along the z axis.

For simplicity we assume that the bulk of the plasma is almost cold and has an inhomogeneous density $N(z)$, to which a hot component with Maxwellian velocity distribution and homogeneous density $n_0(n_0 \ll N)$ is added. The frequencies of the external sources ω_1 and ω_2 ($\omega_1 < \omega_2$) belong to the opacity region, and the density $N(z)$ decreases monotonically with increasing z. As will be shown below, under these conditions, in second order in the amplitude of the external sources, an echo current of the hot component is produced at the difference frequency $\omega_3 = \omega_2 - \omega_1$, and excites natural oscillations of the cold plasma component in the transparency region, where $\omega_3^2 > 4\pi e^2 N(z)/m$.

To calculate the echo we take the perturbations in the form

$$E(z,t) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} d\omega E(z,\omega) e^{-i\omega t}.$$

In the linear approximation, the fields satisfy the equations

$$\frac{d^2}{dz^2} E_y^{(1)}(z,\omega) + \frac{\omega^2}{c^2} \varepsilon(z,\omega) E_y^{(1)}(z,\omega) = \frac{4\pi\omega}{ic^2} j_{\text{ext}}(z,\omega),$$

$$H_x^{(1)}(z,\omega) = \frac{ic}{\omega} \frac{d}{dz} E_y^{(1)}(z,\omega), \tag{2.1}$$

where $\epsilon(z,\omega) = 1 - 4\pi e^2 N(z)/m$. The solution of Eqs. (1.1) in the case of weak inhomogeneity is

$$E_y^{(1)}(z,\omega) = {}^1/_2 i \sum_{s=1}^{2} E_s[\delta(\omega-\omega_s) - \delta(\omega+\omega_s)] \exp(-\varkappa_s|z-a_s|),$$

$$H_x^{\cdot}(z\,\omega) = {}^1/_2 \sum_{s=1} H_s[\delta(\omega-\omega_s) + \delta(\omega+\omega_s)] \theta(z-a_s) \exp(-\varkappa_s|z-a_s|),$$

$$E_s = \frac{(2\pi)^{3/2} j_s \omega_s}{\varkappa_s c^2}, \quad H_s = \frac{(2\pi)^{3/2} j_s}{c}, \quad \varkappa_s = \left| \frac{\omega_s^2}{c^2} \varepsilon(a_s,\omega_s) \right|^{1/2};$$

$$\theta(\xi) = \begin{cases} 1 & (\xi > 0) \\ -1 & (\xi < 0) \end{cases}. \tag{2.2}$$

We write out the linearized kinetic equation for the distribution function of the hot components:

$$-i\omega f_1(z,\mathrm{v},\omega) + v_z \frac{\partial}{\partial z} f_1(z,\mathrm{v},\omega) = \frac{e}{m} E_y^{(1)}(z,\omega) \frac{\partial F}{\partial v_y}, \tag{2.3}$$

where

$$F = \frac{1}{2\pi v_T^2} \exp\left(-\frac{v_y^2 + v_z^2}{2v_T^2}\right).$$

In the case $a_1 < a_2$, the echo point z_c lies to the right of the sources ($a_1 < a_2 < z_c$) and the echo current is produced by the hot-component particles with $v_z > 0$. For these particles we obtain from (2.3), taking (2.2) into account,

$$f_1(z,\mathrm{v},\omega) = \frac{ie}{2m} \frac{\partial F}{\partial v_y} \sum_{s=1}^{2} E_s[\delta(\omega-\omega_s) - \delta(\omega+\omega_s)] \tag{2.4}$$

$$\times \int_{-\infty}^{z} dz' \exp\left[i\frac{\omega}{v_z}(z-z') - \varkappa_s|z'-a_s|\right].$$

Then, neglecting the field $E_2 \sim n_0/N$ in the kinetic equation for the hot-component distribution function $f_2(z, v, \omega)$, we readily obtain the echo current

$$j_2(z,\omega) = \frac{ieE_1H_2}{8\pi^2 mc} \varkappa_1 d \frac{\omega_p^2}{\omega_1\omega_2} \delta(\omega-\omega_3) \int_0^\infty du \exp\left(-\frac{u^2}{2} + i\frac{Z}{u}\right), \tag{2.5}$$

where $d = a_2 - a_1$, $Z = (\omega_3/v_T)(z - z_c)$, $z_c = a_2 + d\omega_1/\omega_3$ is the echo point, and $\omega_p^2 = 4\pi e^2 n_0/m$.

To obtain the echo proper we take into account the hydrodynamically-weak thermal motion of the cold plasma component. Then the longitudinal echo field $E_z^{(2)}(z,\omega)$ satisfies the equation

$$\frac{d^2}{dz^2} E_z^{(2)} + \frac{\omega^2}{V_T^2} \varepsilon(z,\omega) E_z^{(2)} = \frac{4\pi\omega}{iV_T^2}\left[j_2 + \frac{V_T^2}{\omega^2} \frac{d^2}{dz^2} j_2\right]. \tag{2.6}$$

Here V_T is the velocity of the cold component of the plasma, and $j_2(z,\omega)$ is specified by means of formula (2.5). From (2.5) and (2.6) we readily see, first, that the plasma wave is emitted forward from the vicinity of the echo point. We consider two cases in the calculation of the echo.

a) The echo point z_c lies in the transparency region far from the plasma-wave reflection point, at which $\epsilon(z,\omega_3) = 0$. Then $E_z^{(2)}(z,\omega)$ is given by $(z > z_c)$

$$E_z^{(2)}(z,\omega) = \frac{H_1H_2}{iH_*} \left(\frac{\omega_p\Omega_p}{\omega_2\omega_3}\right)^2 \frac{v_\varphi}{v_T} \exp\left(-\frac{v_\varphi^2}{2v_T^2}\right) \delta(\omega-\omega_3)$$

$$\times \left(\frac{\varepsilon_0}{\varepsilon_3}\right)^{1/4} \exp\left(i\frac{\omega_3}{V_T}\int_{z_c}^{z} \sqrt{\varepsilon_3(z')}\,dz'\right), \tag{2.7}$$

where

$$H_* = \frac{2mc^2\varepsilon_0}{ed_*}, \quad d_* = d\frac{\omega_2}{\omega_3}, \quad \varepsilon_0 = \varepsilon(z_c,\omega_3),$$

$$\Omega_p^2 = \frac{4\pi e^2}{m} N(z_c), \quad \varepsilon_3(z) = \varepsilon(z,\omega_3),$$

$$v_\varphi = V_T \varepsilon_0^{-1/2}.$$

b) The echo point z_c coincides with the reflection point. Putting $\epsilon(z,\omega_3) = (z - z_c)/L$, we obtain in place of (2.7)

$$E_z^{(2)}(z,\omega) = \frac{H_1H_2}{H_*}\left(\frac{\omega_p}{\omega_2}\right)^2 R\left(\frac{v_\varphi^3}{v_T^3}\right) \delta(\omega-\omega_3) \left(\frac{\varepsilon_0}{\varepsilon_3}\right)^{1/4}$$

$$\times \exp\left(i\frac{\omega_3}{V_T}\int_{z_c}^{z}\sqrt{\varepsilon_3(z')}\,dz' - \frac{3}{4}i\pi\right), \tag{2.8}$$

where the role of ϵ_0 is played by the quantity $(v_T/\omega_3 L)^{2/3}$. In (2.8) we have

$$R(\gamma) = \pi^{-1/2}\int_0^\infty d\xi \exp\left(-\frac{\xi^2}{2} + \frac{i\gamma}{3\xi^3}\right). \tag{2.9}$$

When $\gamma^{2/5} \ll 1$, the integral in (2.9) is equal to $2^{-1/2}$.

From a comparison of (2.7) and (2.8) we see that for small V_T, the echo field increases resonantly if the echo point approaches the point of reflection of the plasma wave.

Let us make a few remarks concerning the effect considered here. Just as in the case of a homogeneous plasma, the echo occurs at the difference frequency and with the same distance from the sources. Qualitatively, the difference lies in the fact that the natural oscillations are excited precisely because of the inhomogeneity of the density, which decreases towards the plasma boundary.

Since the distance from the echo to the sources can be made larger than the characteristic width of the "hump" of the density, it is possible to transmit information concerning the oscillations from sources lying in the transparency region to the opposite boundary of the plasma. It should be noted here that a longitudinal plasma wave propagating almost normally to the plasma boundary is transformed, with a transformation coefficient on the order of unity, into a transverse wave that is radiated into vacuum[9].

3. ECHO AT SUMMARY FREQUENCY IN AN INHOMOGENEOUS ISOTROPIC PLASMA

In an inhomogeneous plasma, the possible types of frequency spectra of the echo increase considerably. Whereas in the homogeneous plasma the only possible echo is at the difference frequency (at least up to second order in the field amplitude inclusive), in an inhomogeneous plasma, as shown in Sec. 1, a linear echo at the same frequency, i.e., regeneration of the wave with conservation of the frequency, is possible.

Let us show now that a nonlinear echo at the summary frequency of the external signals is possible in an inhomogeneous isotropic plasma. This effect is analogous basically to the linear echo considered in Fig. 1, since in both cases the inhomogeneity of the plasma is essential in order to satisfy the phase-coherence condition.

Let us consider by way of an example two longitudinal external sources

$$j_{\text{ext}}(x,t)=\sum_{s=1}^{2} j_s\delta(x-a_s)\cos\omega_s t, \tag{3.1}$$

where the particles traveling in the positive x direction interact at the point $a_{1,2}$ with the external current (3.1) and acquire (or give up) a certain energy. We shall assume that $a_1 < a_2 < x_{\mathscr{E}}$ $(v(x_{\mathscr{E}}) = 0)$. Then after these particles are refletted from the potential $\Phi(x)$ that contains the inhomogeneous plasma, the phase of the second-approximation contribution function is given by

$$\psi(\mathscr{E},x)=\omega_1\left(\int_{a_1}^{x_{\mathscr{E}}}\frac{dx'}{v(x',\mathscr{E})}+\int_{x}^{x_{\mathscr{E}}}\frac{dx'}{v(x',\mathscr{E})}\right)+\omega_2\left(\int_{a_2}^{x_{\mathscr{E}}}\frac{dx'}{v(x',\mathscr{E})}+\int_{x}^{x_{\mathscr{E}}}\frac{dx'}{v(x',\mathscr{E})}\right),$$

$$v(x,\mathscr{E})=\sqrt{2[\mathscr{E}-\Phi(x)]}. \tag{3.2}$$

Under the condition of phase coherence $\partial\psi/\partial\mathscr{E} = 0$, a macroscopic current of plasma particles is produced and excites natural oscillations of the plasma at the frequency $\omega_3 = \omega_1 + \omega_2$, if the resonance point $x_3(\mathscr{E})$ $(\omega_3 = k(\omega_3, x_S)v(x_S, \mathscr{E}))$ is located near the phase-coherence point of the particles with energy $\mathscr{E}$. The most interesting case in this example corresponds to $\omega_{1,2} < \omega_{pe}(x) < \omega_3$, when the fields are subject to the skin effect near the points $a_{1,2}$ in first order in the amplitude of the sources (3.1).

The system of equations for the perturbations of the distribution function $f_1(x, v, t)$ and of the electric field $E_1(x, t)$ has the following form in the case of longitudinal oscillations:

$$\frac{\partial f_1}{\partial t}+v\frac{\partial f_1}{\partial x}-\frac{d\Phi}{dx}\frac{\partial f_1}{\partial v}=\frac{e}{m}vE_1\frac{dF}{d\mathscr{E}},$$

$$\frac{\partial E_1}{\partial t}+4\pi(j_1+j_{\text{ext}})=0, \tag{3.3}$$

where $\mathscr{E} = \frac{1}{2}v^2 + \Phi(x)$ is the normalized energy of the electrons, and $F(\mathscr{E})$ is the equilibrium distribution function, satisfying the condition

$$\int_{-\infty}^{+\infty} dv F(v^2/2)=1.$$

The current $j_1(x, t)$ is given by the expression

$$j_1(x\ t)=-en_0\int_{\Phi(x)}^{\infty} d\mathscr{E}\,(f_+-f_-),$$

where f_+ and f_- stand for $f_1(x, v, t)$ at $v > 0$ and $v < 0$, respectively.

Let us calculate the effect of the echo at double the frequency from one external source

$$j_{\text{ext}}=j\delta(x-a)\cos\omega t,$$

with $\omega < \omega_{pe}(x) < \omega$. In the linear approximation we can put

$$E_1(x)=\frac{2\pi j}{i\omega}\delta(x-a). \tag{3.4}$$

From (3.3) we readily obtain with the aid of (3.4) the linear increment to the distribution function. Repeating the procedure, we obtain the following expression for the distribution of the particles with $v < 0$ in second order, in the region $a < x < x_{\mathscr{E}}$,

$$f_2(x,\mathscr{E})=-\frac{1}{2}(2\pi)^{3/2}\left(\frac{ej}{m\omega}\right)^2\exp\left(2i\omega\int_a^{x_{\mathscr{E}}}\frac{dx'}{v(x',\mathscr{E})}+2i\omega\int_x^{x_{\mathscr{E}}}\frac{dx'}{v(x',\mathscr{E})}\right)\frac{d^2F}{d\mathscr{E}^2}. \tag{3.5}$$

With the aid of (3.5), the integral equation for the echo field $E_2(x)$ takes the form

$$\begin{aligned}E_2(x)=\frac{i\omega_{pe}^2}{2\omega}\int_{\Phi(x)}^{\infty}d\mathscr{E}\frac{dF}{d\mathscr{E}}\Bigg\{&\int_{-\infty}^{x}dx'E_2(x')\exp\left(2i\omega\int_{x'}^{x}\frac{dx''}{v}\right)\\&+\int_{x}^{x_{\mathscr{E}}}dx'E_2(x')\exp\left(2i\omega\int_{x}^{x'}\frac{dx''}{v}\right)\\-\exp\left(2i\omega\int_{x}^{x_{\mathscr{E}}}\frac{dx'}{v}\right)&\int_{-\infty}^{x_{\mathscr{E}}}dx'E_2(x')\exp\left(2i\omega\int_{x'}^{x}\frac{dx''}{v}\right)\Bigg\}+\frac{2\pi}{i\omega}j_2(x),\end{aligned} \tag{3.6}$$

where $\omega_{pe}^2 = 4\pi e^2 n_0/m$; and

$$j_2(x)=-en_0\int_{\Phi(x)}^{\infty}d\mathscr{E}f_2(x,\mathscr{E})$$

is the nonlinear echo current. We seek the field of the echo proper at the frequency 2ω in the form

$$E_2(x)=C\Pi_2(x)\exp\left(-i\int_{x_c}^{x}k_2(x')dx'\right). \tag{3.7}$$

Here $k_2(x) \equiv k(2\omega, \Phi(x))$ is the wave vector of the longitudinal oscillations of the frequency 2ω, $\Pi_2(x) = [\partial\epsilon(2\omega, k_2(x))/\partial k_2(x)]^{-1/2}$, and $\epsilon(2\omega, k_2)$ is the dielectric constant at the frequency 2ω.

The procedure for calculating $E_2(x)$ is perfectly analogous to that used in Sec. 1. As a result we obtain for the constant C the expression

$$C=-\frac{2\pi}{\omega}\int_{-\infty}^{+\infty}dx\Pi_2(x)j_2(x)\exp\left(i\int_{x_c}^{x}k_2(x')dx'\right). \tag{3.8}$$

Substituting expression (3.5)–(3.7) in (3.8) we obtain

$$C=-\frac{2e}{m\omega}\left(\frac{\pi}{2}\right)^{3/2}\left(\frac{j\omega_{pe}}{\omega}\right)^2\int_0^{\infty}d\mathscr{E}\frac{d^2F}{d\mathscr{E}^2}R(\mathscr{E}) \tag{3.9}$$

$$\times\exp\left[i\int_a^{x_{\mathscr{E}}}\left(k_2+\frac{2\omega}{v}\right)dx'-i\int_{x_s}^{x_{\mathscr{E}}}\left(k_2-\frac{2\omega}{v}\right)dx+i\int_{x_c}^{a}k_2dx\right],$$

where

$$R(\mathscr{E})=\int_{-\infty}^{x_{\mathscr{E}}}dx\Pi_2(x)\exp\left[i\int_{x_s}^{x}\left(k_2-\frac{2\omega}{v}\right)dx'\right]\quad(2\omega=k_2(x_s)v(x_s,\mathscr{E})).$$

The integral in the expression for $R(\mathscr{E})$ is obtained by the saddle-point method.

From formula (3.9) we readily see that in order for the echo to be produced it is necessary to satisfy the phase-coherence condition

$$\left.\frac{\partial \tau(x, \mathscr{E})}{\partial \mathscr{E}}\right|_{x=x_s} \equiv \left.\left(\frac{\partial}{\partial \mathscr{E}}\left[\int_a^{x_\mathscr{E}} \frac{dx'}{v} + \int_x^{x_\mathscr{E}} \frac{dx'}{v}\right]\right)\right|_{x=x_s} = 0, \quad (3.10)$$

which separates the particles with energy $\mathscr{E} = \mathscr{E}_0$. Finally we get

$$E_2(x) = \frac{eE_0^2 \varkappa_2(\mathscr{E}_0)}{m v_T^2} \left(\frac{{}^1/{}_2\pi}{1+\alpha}\right)^{1/2} v_T^2 \frac{d \ln F'(\mathscr{E}_0)}{d\mathscr{E}_0} \frac{\Pi_2(x)}{\Pi_2(x_c)} \quad (3.11)$$

$$\times \exp\left[i\psi_0 - i\int_{x_c}^{x} k_2(x')\,dx'\right],$$

where

$$E_0 = \frac{2\pi j}{v_T}, \quad \varkappa_2(\mathscr{E}_0) \equiv \operatorname{Im} k_2(x_c), \quad \alpha = 1 + \left.\frac{8\omega^3}{k_2^3(x_c)} \frac{d\mathscr{E}}{dx_s} \frac{d^2\tau}{d\mathscr{E}^2}\right|_{\mathscr{E}=\mathscr{E}_0},$$

$$x_c = x_s(\mathscr{E}_0), \quad \psi_0 = 2\omega\tau(x_c, \mathscr{E}_0) - \left.\frac{4\omega^2\varkappa_2^2(\mathscr{E}_0)}{k_2^3(x_c)}(1+\alpha)\frac{dx_s}{d\mathscr{E}}\right|_{\mathscr{E}=\mathscr{E}_0}.$$

Formula (3.11) was obtained under the condition that the echo point x_c lies to the right of the source. On the other hand, if $x_c < a$, we obtain in place of (3.5) $(0 = 2\omega \int_a^{x_\mathscr{E}} dx/v)$.

$$f_2(x, \mathscr{E}) = -\frac{1}{2}(2\pi)^{3/2}\left(\frac{ej}{m\omega}\right)^2 \left[(1 - e^{i\theta} + e^{2i\theta})\frac{d^2F}{d\mathscr{E}^2} - \frac{dF}{d\mathscr{E}}\frac{d}{d\mathscr{E}}e^{i\theta}\right]$$

$$\times \exp\left(2i\omega\int_x^a \frac{dx}{v}\right).$$

We see therefore that the investigated effect takes place also in the region $x < a$.

Let us consider a concrete example of a containing potential $\Phi(x)$ in the form

$$\Phi(x) = \begin{cases} \Phi_0 x/L & (x > 0) \\ 0 & (x < 0) \end{cases}.$$

We then obtain for the echo-wave amplitude at the frequency 2ω the expression $(x < 0)$

$$A = \left(\frac{2}{\pi}\right)^{1/2} \frac{eE_*^2}{m\omega v_T} \left(\frac{\omega_{pe}}{\omega}\right)^2 \left(\frac{v_\varphi^2}{2v_T^2}\right)^2 \exp\left(-\frac{v_\varphi^2}{2v_T^2}\right),$$

where $v_\varphi = 2\omega/k_2$, $E_* = \pi j/v_T\sqrt{3}$, $x_c = -a - 2Lv_\varphi^2/\Phi_0 < 0$. We note that the method of harmonic generation in the inhomogeneous plasma indicated in the present section may turn out to be quite effective if the energy $\mathscr{E}_0$ of the particles that excite the echo at the double frequency is of the order of the average thermal energy. In addition, since the point x_c can easily be shifted by changing the plasma and wave parameters, this method is also convenient for diagnostic purposes.

In conclusion, we note the following. The echo effects considered in the present paper are connected with the possibility of producing phase coherence, i.e., of focusing of the particles by the plasma inhomogeneity. However, at sufficiently short distances, as already noted in[7], information concerning the wave motion can be transmitted also in the absence of focusing, owing to the incomplete cancellation of the currents of the individual particles, meaning incomplete phase mixing. In fact, the change of the phase velocity of the wave at the distance d is equal to $\Delta v_d \sim v_\varphi d/L$ where L is the inhomogeneity length of the phase velocity. It can then be seen readily that the condition for the absence of phase mixing when the resonant particles pass through an opacity barrier of width b, located at a distance d from the source, has for $b \sim d$ the form

$$d < (\lambda L)^{1/2} \ll L.$$

This estimate is meaningful when $\kappa(\lambda L)^{1/2} < 1$, when the inhomogeneity-induced resonance detuning Δv_d between the wave and the particles exceeds the absorption line width with respect to velocity, $\Delta v_\kappa \sim v_\varphi \lambda\kappa \ll v_\varphi$. In the opposite case $\kappa(\lambda L)^{1/2} > 1$, the phase mixing of the resonant particles occurs over the wave attenuation length $\kappa^{-1} \ll L$. Thus, in the absence of particle focusing by the inhomogeneity, the echo effects occur at distances from the source that are much smaller than the inhomogeneity length L.

It is seen from the foregoing analysis that besides the known types of wave transformation there exist in an inhomogeneous plasma also a nonlocal wave transformation (both linear and nonlinear), pertaining to different dispersion branches. An investigation of these effects, however, is outside the scope of the present paper.

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