VISCOUS VORTEX FLOW IN SUPERCONDUCTORS WITH PARAMAGNETIC IMPURITIES

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An expression for the conductivity of a vortex lattice in fields $H \ll H_{c2}$ is obtained on the basis of the nonstationary equations for superconductors with paramagnetic impurities. The case of high temperatures (small paramagnetic impurity concentrations) leads effectively to a conductivity greater than in the low temperature region. This agrees qualitatively with the experiments with ordinary alloys. Macroscopic electrodynamics equations are written down for vortex-structure superconductors.

1. INTRODUCTION AND FORMULATION OF PROBLEM

As demonstrated by a number of experiments (see, e.g.,[1]), superconductors of the second kind (alloys) have a finite resistance in a magnetic field exceeding the lower critical value H_{c1}. The nature of these dissipative effects is due to motion of the Abrikosov vortex structure[2] under the influence of the Lorentz force resulting from the flow of current[3-5]. Attempts to study this very interesting question were based mainly on phenomenological models (see[3-8]). With the exception of the work of Bardeen and Stephen[5], an analogy is usually suggested with the model of an ideal charged liquid, which apparently has little bearing even on the case of a pure superconductor of the second kind, owing to the microscopic dissipative effects. The latter are due to the existence of gapless excitations in the center of the vortex filament[9]. In particular, an ideal model would give a large Hall angle, in direct contradiction to an experiment[10] in which the Hall effect turned out to be of the same order as in the normal metal. Bardeen and Stephen[5] attempted to take phenomenological account of these spectrum singularities and were thus able to obtain the correct value of the resistance.

The motion of the vortex structure was investigated on the basis of the BCS microscopic theory of superconductivity only in the vicinity of the upper critical field H_{c2} (see[11]), where the resistance due to this motion constitutes a small correction. The general case calls for the use of equations that describe the dynamic properties of the superconductors. A theory that includes nonstationary processes within the framework of the BCS microscopic theory of superconductivity was constructed in[12]. In the present article we confine ourselves to a study of superconducting alloys with paramagnetic impurities, where it turns out that the problem of vortex motion admits of an analytic solution. Ordinary alloys or a pure superconductor of the second kind can be handled in the same manner, but additional difficulties arise, connected with the complex picture of the excitation spectrum in the core of the vortex[9,13].

An attempt to solve the problem in alloys with paramagnetic impurities was made by Baba and Maki[14]. The results obtained below for the resistance in fields $H \ll H_{c2}$ do not agree with the results of Baba and Maki. The point is that in[14] all the expressions for the physical quantities, for example for the current density, are expanded simultaneously in terms of the electric field intensity and the magnitude of the superconducting parameter Δ. For $H \ll H_{c2}$ this, of course, is incorrect, since the dissipation of the current (the resistance) is still small and is due only to the motion of the structure if one deals with small currents. Finally, Baramidze[15] considered the spectrum of the "oscillations" of an individual filament in the same model (the spectrum has here a diffuse character). Some inaccuracies in Baramidze's result will be pointed out in Sec. 2.

We assume throughout that the parameter of the Ginzburg-Landau theory is $\kappa \gg 1$. Because of this, the distribution of all the quantities (fields and currents) can be investigated separately far from the vortex filaments and close to them. The matching of the resultant expressions gives the solution of our problem. Such an approach is valid so long as the distance d between individual filaments is large compared with the dimension of the core of the vortex. In other words, the induction $B = n\phi_0 \ll H_{c2}$ (n is the density of the filaments and ϕ_0 is the flux quantum).

The equations have the simplest form in the case of a superconductor with large paramagnetic-impurity concentration, $\tau_s T_c \ll 1$[12]. Putting $\Delta = |\Delta| e^{i\theta}$, $\mathbf{Q} = \mathbf{A} - (c/2e)\nabla\theta$ and $\mu = \dot{\theta} + 2e\varphi$, where ω and $\mathbf{A}$ are the scalar and vector potentials and θ is the phase of the ordering parameter, we write the equations in the form

$$|\dot{\Delta}| + \frac{\tau_s}{3}\left[-\pi^2(T_c^2 - T^2) + \frac{|\Delta|^2}{2}\right]|\Delta| - D\nabla^2|\Delta| + \frac{4e^2}{c^2}DQ^2|\Delta| = 0,$$

$$\frac{c}{2e}|\Delta|^2\mu + D\,\mathrm{div}[|\Delta|^2\mathbf{Q}] = 0, \qquad \mathbf{j} = \sigma\mathbf{E} - \frac{2\sigma\tau_s}{c}|\Delta|^2\mathbf{Q}.$$

Here $D = v_F l/3$ is the diffusion coefficient and τ_s the time between collisions of the electron with the impurity atoms relative to the spin flip. In addition, we have the relations

$$\mathrm{div}\,\mathbf{j} = 0, \quad \mathrm{rot}\,\mathbf{Q} = \mathbf{H}, \quad \mathrm{rot}\,\mathbf{H} = \frac{4\pi}{c}\mathbf{j}, \quad \mathbf{E} = -\frac{1}{c}\dot{\mathbf{Q}} - \frac{1}{2e}\nabla\mu.$$

The convenience of this notation lies in the fact that the combinations of $\mathbf{Q}$ and μ are gauge-invariant.

We introduce the following dimensionless variables:

$$|\Delta| = \Delta_\infty \Delta', \quad \omega = 2\tau_s \Delta_\infty^2 \omega', \quad \mathbf{r} = \delta \mathbf{r}',$$

$$\mathbf{Q} = \frac{\phi_0}{2\pi\xi} \mathbf{Q}', \quad \mathbf{j} = \frac{c\phi_0}{8\pi^2\delta^2\xi} \mathbf{j}', \quad \mu = 2\tau_s \Delta_\infty^2 \mu'.$$

Here $\phi_0 = \pi\hbar c/e$ is the flux quantum, $\Delta_\infty^2 = 2\pi^2(T_c^2 - T^2)$, $\delta = (8\pi\sigma\tau_S\Delta_\infty^2)^{-1/2}c$—the depth of penetration, and $\xi = (6D/\tau_S\Delta_\infty^2)^{1/2}$—the coherence radius, with $\kappa = \delta/\xi = c(48\pi D\sigma)^{-1/2} \gg 1$ as the parameter.

In terms of this notation, the equations take the form (the primes designating the reduced quantities will henceforth be omitted):

$$12\dot{\Delta} + (\Delta^2 - 1 + Q^2)\Delta - \frac{1}{\varkappa^2}\nabla^2\Delta = 0, \tag{1}$$

$$12\Delta^2\mu + \frac{1}{\varkappa}\operatorname{div}[\Delta^2\mathbf{Q}] = 0, \tag{2}$$

$$\mathbf{j} = \operatorname{rot}\operatorname{rot}\mathbf{Q} = -\dot{\mathbf{Q}} - \frac{1}{\varkappa}\nabla\mu - \Delta^2\mathbf{Q}. \tag{3}$$

The condition $\operatorname{div}\mathbf{j} = 0$ jointly with (2) yields

$$12\Delta^2\mu - \frac{1}{\varkappa^2}\nabla^2\mu = \frac{1}{\varkappa}\operatorname{div}\dot{\mathbf{Q}}. \tag{4}$$

We write out also the connection between the fields and the potentials:

$$\mathbf{H} = \operatorname{rot}\mathbf{Q}, \quad \mathbf{E} = -\dot{\mathbf{Q}} - \frac{1}{\varkappa}\nabla\mu.$$

In the case of an immobile lattice of vortex filaments, $\dot{\mathbf{Q}} = \dot{\Delta} = \mu = 0$ and Eqs. (1)—(3) turn into the ordinary Ginzburg-Landau equations. The solutions of the static equations for an isolated vortex were obtained in[2]. In cylindrical coordinates with z axis directed parallel to the magnetic field $\mathbf{H}_0$ we have, for a vortex containing one flux quantum, $\theta = \varphi$ (φ is the polar angle) and $\Delta = f(r)$, where f satisfies the relation

$$\frac{1}{\varkappa^2}\left(\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right) + (1-f^2)f - Q_0^2 f = 0, \tag{5}$$

$\mathbf{Q}$ has only the component $Q_\varphi \equiv Q_0$, satisfying the equation

$$\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(rQ_0) - f^2 Q_0 = 0. \tag{6}$$

Since the scale of variation of f is of the order of $1/\kappa$, which is much smaller than the scale of variation of Q_0 (the latter is of the order of unity), we get from (5) and (6) $Q_0 = -\kappa^{-1}K_1(r)$; f is not expressed in analytic form, but when $r \gg 1/\kappa$ we have $f = 1 - 1/2\kappa^2r^2$, and when $r \ll 1/\kappa$ we have $f \propto \kappa r$.

Equations (2) and (4) are satisfied identically.

2. FIELD IN THE VICINITY OF AN INDIVIDUAL MOVING FILAMENT

In the presence of a current, the vortex lattice begins to move and becomes deformed. So long as the current is small, the rates of motion and of deformation are also small. We choose on one of the filaments a point with which we connect the cylindrical coordinate system (ρ, φ, z), directing the z axis along the undeformed filament. Then the coordinates of the remaining points along the filament are characterized by a two-dimensional deformation vector $\mathbf{u}(z, t)$. The coordinates along the remaining filaments $\boldsymbol{\rho}_{0i} + \mathbf{u}_i(z, t)$, where $\boldsymbol{\rho}_{0i}$ is the equilibrium position of the i-th filament in the absence of deformations.

Let us consider first the picture of motion in the vicinity of one vortex. For an individual filament, the zeroth approximation $(\tilde{f}_0, \tilde{\mathbf{Q}}_0)$ corresponds to a displacement and rotation of the unperturbed solutions

$$\tilde{f}_0(r) = f(\rho) - (\mathbf{u}(z,t)\,\nabla)f(\rho),$$
$$\tilde{\mathbf{Q}}_0 = \mathbf{Q}_0(\rho) - (\mathbf{u}(z,t)\,\nabla)\mathbf{Q}_0 + [\delta\boldsymbol{\varphi}\mathbf{Q}_0], \tag{7}$$

where

$$\delta\boldsymbol{\varphi} = \left[\mathbf{n}\frac{\partial\mathbf{u}}{\partial z}\right]. \tag{8}^*$$

Here f and $\mathbf{Q}_0 = (0, Q_0, 0)$ are the solutions of Eqs. (5) and (6), and the vector n is directed along the z axis. The solutions (7) satisfy identically the system of equations (1)—(4) if we neglect the dependence of u on t and $\partial\mathbf{u}/\partial z$ on z. We shall seek corrections to solutions (7) and (8) in the form

$$\Delta = \tilde{f}_0 + \chi, \quad \mathbf{Q} = \tilde{\mathbf{Q}}_0 + \mathbf{q}, \tag{9}$$

where the quantities χ and q are proportional, as we shall show below, to $\partial\mathbf{u}/\partial t$ and to $\kappa^{-2}\partial^2\mathbf{u}/\partial z^2$. We note immediately that from (4) it follows that μ is small in this approximation, since the zeroth approximation $\tilde{\mathbf{Q}}_0$ satisfies the condition $\operatorname{div}\tilde{\mathbf{Q}}_0 = 0$. We can similarly verify the smallness of the component q_z. The term $\dot{\Delta}$ in (1) is transformed into $(\dot{\mathbf{u}}\cdot\nabla)f$ and give the right-hand side, from the form of which we conclude that the dependence of the solution $(\chi, \mathbf{q})$ on the angles φ is conveniently sought in a form proportional to $e^{i\varphi}$. Omitting the intermediate steps, we present the equations obtained for the components $(\chi, \mathbf{q})$ from (1)—(3) after substituting in them (7)—(9) (we put henceforth $u = u_x - iu_y$):

$$\frac{1}{\varkappa^2}\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\chi}{d\rho}\right) - \frac{\chi}{\rho^2}\right] - (3f^2-1)\chi - Q_0^2\chi - 2fQ_0q_\varphi$$
$$= \left(12\dot{u} - \frac{\partial^2 u}{\partial z^2}\right)\psi, \tag{10a}$$

$$2\psi q_\rho + f\left[\frac{q_\rho}{\rho} + \frac{dq_\rho}{d\rho}\right] + \frac{if}{\rho}q_\varphi + \frac{2iQ_0\chi}{\rho} = i\frac{\partial^2 u}{\partial z^2}Q_0, \tag{10b}$$

$$\left(f^2 + \frac{1}{\rho^2}\right)q_\rho + \frac{i}{\rho}\left(\frac{dq_\varphi}{d\rho} + \frac{q_\varphi}{\rho}\right) = i\frac{\partial^2 u}{\partial z^2}\frac{1}{\rho}\frac{d}{d\rho}(\rho Q_0). \tag{10c}$$

We have left out of the last of these equations the term resulting from $\dot{\mathbf{Q}}$ in (3), which is small when $\kappa \gg 1$. By putting $\chi = \psi w$ we transform Eq. (10a):

$$2Q_0Q_0'fw + \frac{1}{\varkappa^2}\left[\left(2\psi' + \frac{\psi}{\rho}\right)w' + \psi w''\right] - 2fQ_0q_\varphi = \left(12\dot{u} - \frac{1}{\varkappa^2}\frac{\partial^2 u}{\partial z^2}\right)\psi. \tag{11}$$

We have used here the following relation for the function $\psi = df/d\rho$:

$$\frac{1}{\varkappa^2}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\psi}{d\rho}\right) - \frac{1}{\varkappa^2\rho^2}\psi - (3f^2-1)\psi - Q_0^2\psi = 2Q_0Q_0'f, \tag{12}$$

which is derived from (5) by differentiation. Multiplying now (11) by $\rho\psi$, (10b) by $i\rho^2 fQ_0'$ and then adding, we obtain the integral of the system for a solution that is finite at zero:

$$-\left[12\dot{u} - \frac{1}{\varkappa^2}\frac{\partial^2 u}{\partial z^2}\right]\int_0^\rho \rho\psi^2 d\rho - \frac{\partial^2 u}{\partial z^2}\int_0^\rho \rho f^2(\rho Q_0^2)' d\rho$$
$$= \rho\psi^2 w' + i\rho^2 f^2 Q_0' q_\rho - \rho Q_0 f^2 q_\varphi.$$

Recognizing that the function ψ decreases rapidly when

*$[\mathbf{n}\dot{\mathbf{u}}] \equiv \mathbf{n} \times \dot{\mathbf{u}}$.

$\rho \gg \kappa^{-1}$, and $f \to 1$, we can write in the region $\kappa^{-1} \ll \rho \ll 1$[1])

$$-12\dot{u}\int_0^\infty \rho\psi^2 d\rho + \frac{\partial^2 u}{\partial z^2}\left[\frac{1}{\varkappa^2}\int_0^\infty \rho\psi^2 d\rho - \int_0^\rho \rho f^2(\rho Q_0^2)' d\rho\right] = \frac{1}{\varkappa}(q_\varphi + iq_\rho), \quad (13')$$

where $\int_0^\infty \rho\psi^2 d\rho = \gamma \approx 0.247$. The term in the square brackets can be rewritten in the form

$$\int_0^\infty\left[\frac{1}{\varkappa^2}\left(\frac{df}{d\rho}\right)^2 - f^2\frac{d}{d\rho}(\rho Q_0^2)\right]\rho d\rho + \int_\rho^\infty \rho(\rho Q_0^2)' d\rho = \frac{\varepsilon}{2\pi} + \int_\rho^\infty \rho(\rho Q_0^2)' d\rho,$$

where, according to[2], $\epsilon \approx 2\pi\kappa^{-2}(\ln \kappa + 0.08)$ is the energy per unit length of the vortex. The obtained relation can be written in vector form. Since $\mathbf{j}_1 = -\mathbf{q}$ when $\rho \gg \kappa^{-1}$, we find that

$$\mathbf{j}_1 = -\mathbf{q} = 6\varkappa\gamma[\mathbf{n}\dot{\mathbf{u}}] - \frac{\varkappa\varepsilon}{4\pi}\left[\mathbf{n}\frac{\partial^2\mathbf{u}}{\partial z^2}\right] - \frac{\varkappa}{2}\int_\rho^\infty (\rho Q_0^2)'\rho d\rho\left[\mathbf{n}\frac{\partial^2\mathbf{u}}{\partial z^2}\right]. \quad (14)^*$$

When $\rho \ll 1$ we have, accurate to terms $\sim\rho^2$,

$$\int_\rho^\infty \rho(\rho Q_0^2)' d\rho \approx -\frac{1}{\varkappa^2}K_0(\rho) - \frac{1}{2\varkappa^2}.$$

Formula (14) gives an expression for the correction to the current near the vortex; this correction is necessitated by the motion and deformation of the vortex.

3. FIELD AT LARGE DISTANCES FROM THE CORE OF THE VORTEX FILAMENT

To find the current produced by all vortices in the vicinity $\kappa^{-1} \ll r \ll d$ of a given vortex, we proceed as follows. The microscopic magnetic field $\mathbf{h}$ outside the core ($r \gg \kappa^{-1}$) is described by the Londons' generalized equation (cf., e.g.,[7]). In the usual units

$$\delta^2 \,\mathrm{rot}\,\mathrm{rot}\,\mathbf{h} + \mathbf{h} = \Phi_0 \sum_j \int d\mathbf{S}_j \delta(\mathbf{R} - \mathbf{R}_j).$$

Here $d\mathbf{S}_j$ is an element of length along the j-th filament, $\delta(\mathbf{R} - \mathbf{R}_j)$ a three-dimensional δ function, and $\mathbf{R}_j$ a three-dimensional vector drawn to a point on the j-th filament.

From this we easily obtain an expression for the magnetic field

$$\mathbf{h}(\mathbf{R}) = \frac{\Phi_0}{4\pi\delta^2}\sum_j \int d\mathbf{S}_j \frac{1}{|\mathbf{R} - \mathbf{R}_j|}\exp\left[-\frac{|\mathbf{R} - \mathbf{R}_j|}{\delta}\right]$$

and for the current

$$\mathbf{j}(\mathbf{R}) = -\frac{c\Phi_0}{16\pi^2\delta^2}\sum_j \int\left[d\mathbf{S}_j \,\mathrm{grad}\left(\frac{1}{|\mathbf{R} - \mathbf{R}_j|}\exp\left\{-\frac{|\mathbf{R} - \mathbf{R}_j|}{\delta}\right\}\right)\right]$$

[1])Baramidze [15] investigated the oscillations of an isolated filament in the same model, using the condition of translational invariance of the initial equations. He found that the dispersion law is given by $\omega \backsim -i(k^2/\kappa^2)\ln(k/\kappa)$ for all k. It is seen from (13) that by setting q_ρ, q_φ, and w' approach zero for an isolated filament as $\rho \to \infty$, we obtain $\omega \backsim -i(k^2/\kappa^2)\ln\kappa$. Baramidze's result is valid only if $\kappa \gg k \gg 1$. The condition of Galilean invariance was used earlier by Pitaevskii in a paper on oscillations of a vortex in a weakly-nonideal Bose gas [16].

*$[\mathbf{n}\dot{\mathbf{u}}] \equiv \mathbf{n} \times \dot{\mathbf{u}}$.

where $\mathbf{R}_j = (\rho_{0j} + \mathbf{u}_j;\, z_j)$. We put $\Phi[(\mathbf{R} - \mathbf{R}_j)^2] = |\mathbf{R} - \mathbf{R}_j|^{-1}\exp(-|\mathbf{R} - \mathbf{R}_j|/\delta]$. We expand (15) in powers of a small displacement $\mathbf{u}_j$, after first writing $\mathbf{j}(\mathbf{R})$ in the form

$$\mathbf{j}(\mathbf{R}) = -\frac{c\Phi_0}{8\pi^2\delta^2}\sum_j \int [d\mathbf{S}_j(\mathbf{R} - \mathbf{R}_j)]\Phi'[(\mathbf{R} - \mathbf{R}_j)^2].$$

Since $d\mathbf{S}_j = (\mathbf{n} + \partial\mathbf{u}_j/\partial z)dz_j$, where $\mathbf{n} \parallel \mathbf{H}_0$, we get

$$\mathbf{j}(\mathbf{R}) = -\frac{c\Phi_0}{8\pi^2\delta^2}\sum_j\left\{\int[\mathbf{n}\rho_{0j}]\Phi'[(\mathbf{R} - \mathbf{R}_{0j})^2]dz_j - \int[\mathbf{n}\mathbf{u}_j]\Phi'[(\mathbf{R} - \mathbf{R}_{0j})^2]dz_j - 2\int[\mathbf{n}\rho_{0j}](\mathbf{u}_j\rho_{0j})\Phi''[(\mathbf{R} - \mathbf{R}_{0j})^2]dz_j + \int\left[\frac{\partial\mathbf{u}_j}{\partial z}\mathbf{n}\right]z_j\Phi'[(\mathbf{R} - \mathbf{R}_{0j})^2]dz_j + \int\left[\frac{\partial\mathbf{u}_j}{\partial z}\rho_{0j}\right]\Phi'[(\mathbf{R} - \mathbf{R}_{0j})^2]dz_j.\right. \quad (16)$$

Assuming that $\mathbf{u}$ varies slowly over distances on the order of δ (for a loose lattice, $d \gg \delta$, it is necessary to stipulate that $\mathbf{u}$ vary over distances on the order of d in a plane perpendicular to the constant magnetic field), we represent $\mathbf{u}_j$ in the form

$$\mathbf{u}_j = \mathbf{u}^0 + (\rho_{0j\alpha}\nabla_\alpha)\mathbf{u} + z_j\frac{d\mathbf{u}}{dz} + \frac{1}{2}(\rho_{0j\alpha}\rho_{0j\beta}\nabla_\alpha\nabla_\beta)\mathbf{u} + \left(\rho_{0j\alpha}\nabla_\alpha z_j\frac{\partial}{\partial z}\right)\mathbf{u} + \frac{1}{2}z_j^2\frac{\partial^2\mathbf{u}}{\partial z^2}$$

(α and β are two-dimensional vector indices). For the selected vortex we have

$$\mathbf{u} = z\frac{\partial\mathbf{u}}{\partial z} + \frac{1}{2}z^2\frac{\partial^2\mathbf{u}}{\partial z^2}.$$

Substituting in (16) the expansion for $\mathbf{u}$ and integrating with respect to z from $-\infty$ to $+\infty$ and summing over a lattice with hexagonal symmetry, we obtain near the selected vortex (we can put $\rho = 0$ in the regular terms)

$$\mathbf{j}_1 = \frac{c\Phi_0}{16\pi^2\delta^2}\left\{[\mathbf{n}\nabla\,\mathrm{div}\,\mathbf{u}]\sum_j{}'\frac{1}{2}\rho_{0j}^4\int\Phi''(\mathbf{R}_{0j}^2)dz + [\mathbf{n}\nabla^2\mathbf{u}]\sum_j{}'\left[\frac{1}{4}\rho_{0j}^4\int\Phi''(\mathbf{R}_{0j}^2)dz + \frac{1}{2}\rho_{0j}^2\int\Phi'(\mathbf{R}_{0j}^2)dz\right] + \left[\mathbf{n}\frac{\partial^2\mathbf{u}}{\partial z^2}\right]\sum_j{}'\left[\frac{1}{2}\int\Phi(\mathbf{R}_{0j}^2)dz - \frac{1}{2}\rho_{0j}^2\int\Phi'(\mathbf{R}_{0j}^2)dz\right] - \frac{1}{2}\mathrm{rot}\frac{\partial\mathbf{u}}{\partial z}\sum_j{}'\left[\rho_{0j}^2\int\Phi'(\mathbf{R}_{0j}^2)dz\right] + \left[\mathbf{n}\frac{\partial^2\mathbf{u}}{\partial z^2}\right]\left[\frac{1}{2}\int\Phi(\mathbf{R}^2)dz - \frac{1}{2}\rho^2\int\Phi'(\mathbf{R}^2)dz\right]\right\}. \quad (17)$$

Here $\mathbf{j}_1$, as in (14), in a correction to the distribution of the current of the undeformed lattice. Since the coordinate system was chosen with its origin coinciding at any given instant with the undeformed vortex, there is no current corresponding to the shift in (17). The selected vortex drops out of the sum. The last term in (17) corresponds precisely to the increments to the velocity field of the selected filament. The sums in (17) are expressed in terms of the elastic constants of a triangular lattice of vortex filaments, obtained by Labusch[17]. We ultimately obtain

$$\mathbf{j}_1 = \frac{c}{n\Phi_0}\left\{(c_{11} - c_{66})[\mathbf{n}\nabla\,\mathrm{div}\,\mathbf{u}] + c_{66}[\mathbf{n}\nabla^2\mathbf{u}] + c_{44}\left[\mathbf{n}\frac{\partial^2\mathbf{u}}{\partial z^2}\right] + (c_{12} - c_{66})\frac{\partial}{\partial z}\,\mathrm{rot}\,\mathbf{u}\right\} - \frac{c\Phi_0}{16\pi^2\xi^2}\frac{\varepsilon}{2\pi}\left[\mathbf{n}\frac{\partial^2\mathbf{u}}{\partial z^2}\right] + \frac{c\Phi_0}{16\pi^2\delta^2}\left[\mathbf{n}\frac{\partial^2\mathbf{u}}{\partial z^2}\right]\left\{K_0\left(\frac{\rho}{\delta}\right) + \frac{\rho}{2\delta}K_1\left(\frac{\rho}{\delta}\right)\right\}, \quad (18)$$

where the constants c are given by

$$c_{11}=\frac{B^2}{4\pi}\frac{\partial H}{\partial B}+\frac{1}{8\pi}\int_0^B x^2H''(x)dx,$$

$$c_{44}=\frac{BH(B)}{4\pi},\quad c_{66}=\frac{1}{8\pi}\int_0^B x^2H''(x)dx.$$

In dimensionless units ($c = \frac{1}{2}\kappa(\phi_0/2\pi\delta^2)^2c'$) we have

$$\mathbf{j}_1'=\frac{1}{B'}\Big\{(c_{11}'-c_{66}')[\mathbf{n}\nabla\operatorname{div}\mathbf{u}]+c_{66}'[\mathbf{n}\nabla^2\mathbf{u}]+c_{44}'\Big[\mathbf{n}\frac{\partial^2\mathbf{u}}{\partial z^2}\Big]+(c_{12}'-c_{66}')\frac{\partial}{\partial z}\operatorname{rot}\mathbf{u}\Big\}+\frac{1}{2\varkappa}\Big[\mathbf{n}\frac{\partial^2\mathbf{u}}{\partial z^2}\Big]\Big\{K_0(\rho)+\frac{\rho}{2}K_1(\rho)-\frac{\varkappa^2\varepsilon}{2\pi}\Big\}. \qquad (18')$$

Expression (18′) must be matched to (14) in the region $\kappa^{-1}\ll\rho\ll 1$. As a result we obtain an equation describing the motion of the vortex structure:

$$\frac{1}{B'}\Big\{(c_{11}'-c_{66}')[\mathbf{n}\nabla\operatorname{div}\mathbf{u}]+c_{66}'[\mathbf{n}\nabla^2\mathbf{u}]+c_{44}'\Big[\mathbf{n}\frac{\partial^2\mathbf{u}}{\partial z^2}\Big]\Big\}=6\varkappa\gamma[\mathbf{n}\dot{\mathbf{u}}]. \qquad (19)$$

In the usual units we have

$$6\gamma H_{c2}\frac{\sigma}{c^2}B[\mathbf{n}\dot{\mathbf{u}}]=(c_{11}-c_{66})[\mathbf{n}\nabla\operatorname{div}\mathbf{u}]+c_{66}[\mathbf{n}\nabla^2\mathbf{u}]+c_{44}\Big[\mathbf{n}\frac{\partial^2\mathbf{u}}{\partial z^2}\Big]. \qquad (20)$$

The component along the z axis in (18′) does not enter in the condition (14).

4. RESISTANCE OF A SAMPLE WITH VORTEX FILAMENTS

For macroscopic problems it is necessary to relate the fields E and H as well as the macroscopic current $\mathbf{j}_e$ flowing through the sample with the lattice deformation. To this end we write down the thermodynamic Gibbs potential (see[18]) accurate to terms of order $(\nabla_i u_k)^2$ and $(\partial u_i/\partial z)^2$:

$$F=F_0+\int\Big\{\Big[\frac{1}{2}(c_{11}-c_{66})\Big(\frac{\partial u_i}{\partial x_i}\Big)^2+c_{66}\Big(\frac{\partial u_k}{\partial x_i}\Big)^2+c_{44}\Big(\frac{\partial u_i}{\partial z}\Big)^2\Big]-\frac{\mathbf{H}-\mathbf{H}_0}{4\pi}\mathbf{B}\Big\}dV;$$

here F_0 corresponds to the undeformed lattice, and i, k = 1, 2. Deformation changes the induction B by an amount

$$\Delta\mathbf{B}=B_0\frac{\partial\mathbf{u}}{\partial z}-B_0\mathbf{n}\frac{\partial u_i}{\partial x_i},\quad B_0=n\phi_0 \qquad (21)$$

By varying the potential over the deformation vector u, we obtain

$$\delta F=\int\Big\{-\Big[(c_{11}-c_{66})\nabla\operatorname{div}\mathbf{u}+c_{66}\nabla^2\mathbf{u}+c_{44}\frac{\partial^2\mathbf{u}}{\partial z^2}\Big]+\frac{B_0}{4\pi}\Big[\frac{\partial\mathbf{H}}{\partial z}-\nabla(H_z-H_0)\Big]\Big\}\delta\mathbf{u}\,dV+\int\Big(c_{44}\frac{\partial\mathbf{u}}{\partial z}-\frac{B_0}{4\pi}\mathbf{H}\Big)\delta\mathbf{u}\,ds+\int\Big[(c_{11}-c_{66})\operatorname{div}\mathbf{u}\,\delta_{ik}+c_{66}\frac{\partial u_k}{\partial x_i}+\frac{B_0}{4\pi}(H_z-H_0)\delta_{ik}\Big]\delta u_k d\sigma_i,$$

where ds is a surface element perpendicular to the magnetic field H_0, and $d\sigma$ is parallel to H_0.

The condition that the potential be a minimum yields

$$(c_{11}-c_{66})\nabla\operatorname{div}\mathbf{u}+c_{66}\nabla^2\mathbf{u}+c_{44}\frac{\partial^2\mathbf{u}}{\partial z^2}=\frac{B_0}{4\pi}\Big[\frac{\partial\mathbf{H}}{\partial z}-\nabla(H_z-H_0)\Big] \qquad (22)$$

(all the vectors and the vector differentiations are two-dimensional). The boundary conditions supplementing (2) are

$$\nu_k(c_{11}-c_{66})\operatorname{div}\mathbf{u}+c_{66}\nu_i\frac{\partial u_k}{\partial x_i}+c_{44}\nu_z\frac{\partial u_k}{\partial z}=\nu_z\frac{B_0H_k}{4\pi}-\nu_k\frac{B_0(H_z-H_0)}{4\pi} \qquad (23)$$

($\nu = (\nu_i;\nu_z)$ is the vector normal to the surface).

Calculating curl H, we readily see that

$$\Big[\mathbf{n}\frac{\partial\mathbf{H}}{\partial z}\Big]-[\mathbf{n}\nabla(H_z-H_0)]=\frac{4\pi}{c}\mathbf{j}_e,$$

therefore

$$(c_{11}-c_{66})[\mathbf{n}\nabla\operatorname{div}\mathbf{n}]+c_{66}[\mathbf{n}\nabla^2\mathbf{u}]+c_{44}\Big[\mathbf{n}\frac{\partial^2\mathbf{u}}{\partial z^2}\Big]=\frac{B_0}{c}\mathbf{j}_e. \qquad (24)$$

Comparing (24) with (20) we see that

$$\mathbf{j}_e=6\gamma H_{c2}\sigma[\mathbf{n}\mathbf{v}]/c.$$

The vortex motion velocity $\mathbf{v}=\dot{\mathbf{u}}$ is perpendicular to the current, and consequently there is no Hall effect in this approximation.

Motion of the vortex lattice gives rise to an electric field E, which can be determined from Maxwell's equation curl $\mathbf{E}=-c^{-1}\partial\mathbf{B}/\partial t$, where B is expressed in terms of u by means of (21):

$$\mathbf{E}=B_0[\mathbf{n}\mathbf{v}]/c. \qquad (25)$$

Thus, the superconductor in a field $B\ll H_{c2}$ has a finite conductivity

$$\sigma_{\text{eff}}=6\gamma\sigma H_{c2}/B_0. \qquad (26)$$

Experiments[1] yielded $\sigma_{\text{eff}}/\sigma\approx H_{c2}/B_0$.

Equation (22) has the form of the equilibrium conditions of an elastically deformed lattice $\partial\sigma_{ik}/\partial x_k = F_i$, where the role of the external force is played by the Lorentz force $\mathbf{F}=c^{-1}\mathbf{j}_e\times\mathbf{B}_0$. Taking the force acting on one vortex, we find that $\mathbf{F}=\eta\mathbf{v}$, where the "viscosity" coefficient is

$$\eta=6\gamma\sigma\phi_0c^{-2}H_{c2}.$$

The derived equations make it possible to describe the reflection of an electromagnetic wave from the surface of a superconductor with a vortex structure. Let us consider a plane wave incident normally on the surface of the superconductor. The intensity of the alternating electric and magnetic fields will be denoted by $E_\sim$ and $H_\sim$. Two geometries are possible: $H_\sim \parallel H_0$ and $H_\sim \perp H_0$. These are shown in Figs. 1 and 2. The z axis is always parallel to H_0.

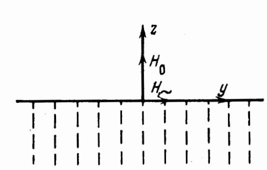

FIG. 1. Alternating magnetic field perpendicular to the vortex filaments (the dashed lines represent the vortex filaments).

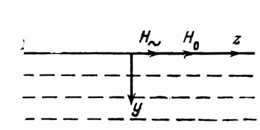

FIG. 2. Alternating magnetic field parallel to the vortex filaments.

Let first $H_\sim \perp H_0$, and then $H_{\sim z} = 0$. In terms of (22) and (23) this means that $H_z - H_0 = 0$. It follows from the boundary conditions (23) that on the surface $z = 0$ (the superconductor fills the half-space $z < 0$) we have

$$c_{44}\frac{\partial \mathbf{u}}{\partial z} = \frac{B_0 \mathbf{H}_\sim}{4\pi}.$$

By virtue of the homogeneity in x and y, all the quantities depend only on z. From (22) it follows that

$$c_{44}\frac{\partial^2 \mathbf{u}}{\partial z^2} = \frac{B_0}{4\pi}\frac{\partial \mathbf{H}_\sim}{\partial z}.$$

Since $H_\sim$ is continuous on the boundary, the magnetic field inside the superconductor is

$$H_\sim = \frac{4\pi c_{44}}{B_0}\frac{\partial \mathbf{u}}{\partial z}.$$

Expression (21) yields

$$\mathbf{B}_\sim = B_0 \partial \mathbf{u} / \partial z,$$

here $B_\sim$ is the induction produced by the alternating field. Comparing the last two formulas, we see that $B_\sim = \mu_\perp H_\sim$, where $\mu_\perp = B_0^2/4\pi c_{44} = B_0/H_0$.

If $H_\sim \parallel H_0$, then $H_z - H_0 = H_{\sim z}$ in (22) and (23). From the boundary conditions we find that when $y = 0$

$$\frac{\partial u_x}{\partial y} = \frac{\partial u_x}{\partial z} = \frac{\partial u_y}{\partial z} = 0;$$

$$c_{66}\frac{\partial u_y}{\partial y} + (c_{11} - c_{66})\operatorname{div}\mathbf{u} = -\frac{B_0}{4\pi}H_{\sim z}.$$

By virtue of the homogeneity in x and z, all the quantities depend only on y. From (22) it follows that $u_x = 0$ and

$$c_{11}\frac{\partial^2 u_y}{\partial y^2} = -\frac{B_0}{4\pi}\frac{\partial H_{\sim z}}{\partial y}.$$

This means that inside the superconductor

$$-\frac{B_0}{4\pi}H_{\sim z} = c_{11}\frac{\partial u_y}{\partial y}.$$

The induction is $B_{\sim z} = -B_0 \partial u_y/\partial y = H_{\sim z} B_0^2/4\pi c_{11}$. Thus, the "magnetic permeability" is $\mu_\parallel = B_0^2/4\pi c_{11}$.

We can now write our equations in the form of Maxwell's equations

$$\operatorname{rot}\mathbf{E}_\sim = -\frac{1}{c}\frac{\partial B_\sim}{\partial t}, \quad \operatorname{rot}\mathbf{H}_\sim = \frac{4\pi}{c}\mathbf{j}_\sim,$$

and $j_\sim = \sigma_{\text{eff}} E_\sim$ and $B_\sim = \mu H_\sim$, where μ is equal to $\mu_\perp$ or $\mu_\parallel$, and σ_{eff} is determined by (26). Proceeding in the usual manner, we can obtain the "skin" depth of penetration of the alternating field $\lambda_{\perp,\parallel}$, where

$$\lambda_\perp = (1+i)\frac{c}{\sqrt{8\pi\mu_\perp\sigma_{\text{eff}}\omega}} = (1+i)\frac{c}{\sqrt{48\pi\gamma\sigma\omega}}\left(\frac{H_0}{H_{c2}}\right)^{1/2},$$

$$\lambda_\parallel = (1+i)\frac{c}{\sqrt{8\pi\mu_\parallel\sigma_{\text{eff}}\omega}} = (1+i)\frac{c}{\sqrt{12\gamma\sigma\omega}}\left(\frac{c_{11}}{B_0 H_{c2}}\right)^{1/2}.$$

The impedance of the superconductor $\zeta = -\omega\lambda/c$ is, respectively

$$\zeta_\perp = (1-i)\sqrt{\frac{\omega}{48\pi\gamma\sigma}}\left(\frac{H_0}{H_{c2}}\right)^{1/2}, \quad \zeta_\parallel = (1-i)\sqrt{\frac{\omega}{12\gamma\sigma}}\left(\frac{c_{11}}{B_0 H_{c2}}\right)^{1/2}.$$

The last formulas are valid if $\lambda \gg \delta$. In addition, the depth of penetration should be large compared with the distances between filaments.

5. MOTION OF VORTICES IN A SUPERCONDUCTOR WITH SMALL PARAMAGNETIC-IMPURITY CONCENTRATION

So far we have considered the case of large paramagnetic-impurity concentrations, corresponding to a near-zero critical temperature. At finite temperatures, as is well known, the so-called anomalous terms in the nonstationary equations for the superconductors become significant. To ascertain their role, let us consider the limiting case of low paramagnetic-impurity concentrations $\tau_s T_c \gg 1$, when the anomalous terms are large. We assume for simplicity that the concentration of the nonmagnetic impurities is large ($\kappa \gg 1$). The corresponding equations were obtained by Éliashberg[12]. They can be written in the form

$$\frac{\pi}{8T_c}\{-|\dot{\Delta}| + D\nabla^2|\Delta| - DQ^2|\Delta|\} + \left[\frac{T_c - T}{T_c} - \frac{7\zeta(3)|\Delta|^2}{8(\pi T_c)^2}\right]|\Delta| + |\Delta| U_1 = 0$$

$$\dot{U}_1 - D\nabla^2 U_1 = -\frac{\pi\tau_s}{8T_c}|\Delta||\dot{\Delta}|.$$

$$\frac{\pi}{8T_c}\{-|\Delta|^2\mu - D\operatorname{div}[|\Delta|^2\mathbf{Q}]\} + |\Delta|^2\overline{U}_\gamma = 0$$

$$\overline{U}_\gamma = -\frac{i\tau_1}{4\tau_s}\int\frac{\gamma_\varepsilon d\varepsilon}{\varepsilon^2 + \tau_s^{-2}},$$

$$\gamma_\varepsilon - D\nabla^2\gamma_\varepsilon = \frac{1}{\tau_1}\frac{1}{2T_c}\operatorname{ch}^{-2}\left(\frac{\varepsilon}{2T_c}\right)\frac{2i|\Delta|^2\mu}{\tau_s(\varepsilon^2 + \tau_s^{-2})} - \frac{i}{\tau_1}\frac{1}{2T_c}\operatorname{ch}^{-2}\left(\frac{\varepsilon}{2T_c}\right)(D\operatorname{div}\dot{\mathbf{Q}} + D\nabla^2\mu) - \frac{2|\Delta|^2}{\tau_s(\varepsilon^2 + \tau_s^{-2})}\gamma_\varepsilon,$$

$$\mathbf{j} = \sigma\mathbf{E} - \frac{mp_0 e}{4\pi c}\frac{|\Delta|^2}{T_c}D\mathbf{Q}.$$

A simplifying circumstance in this rather complicated system is the fact that owing to the electroneutrality condition $\operatorname{div}\mathbf{j} = 0$ the equations for γ_e and μ contain in the right-hand side terms proportional to $\operatorname{div}\dot{\mathbf{Q}}$. It can therefore be readily shown, as was already noted in Sec. 2, that μ, γ_e, and $\overline{U}_\gamma$ are small. An important role is therefore played only by the anomalous part of $\dot{U}_1$. Changing over to the dimensionless variables of the Ginzburg-Landau theory, and also introducing the time scale $t = \omega_0^{-1}t'$, where $\omega_0^{-1} = 14\zeta(3)/\pi\Delta_\infty^2$, and putting $U_1 = (7\zeta(3)\Delta_\infty^2/8\pi^2T_c^2)U_1'$, we obtain a dimensionless system in which Eq. (1) is replaced by the following two equations (the primes have been omitted, just as in the system (1)–(4)):

$$\frac{1}{\varkappa^2}\nabla^2\Delta - (\Delta^2 - 1 + Q^2)\Delta = a\dot{\Delta} - U_1\Delta, \tag{27}$$

$$a\dot{U}_1 - \frac{1}{\varkappa^2}\nabla^2 U_1 = -a\nu(\dot{\Delta^2}). \tag{28}$$

Here $a = \pi^4/14\zeta(3) \approx 5.8$ and

$$\nu = \frac{\pi^3}{14\zeta(3)}\tau_s T_c \gg 1. \tag{29}$$

The reasoning now follows that of Sec. 2. Replacing $\dot{\Delta}$ by $(\dot{\mathbf{u}}\cdot\nabla)\Delta$, we solve Eqs. (27) and (28) jointly with Eq. (3) and with the condition $\operatorname{div}[\Delta^2\mathbf{Q}] = 0$ by perturbation theory. For the component $U_1 \propto e^{i\varphi}$ we have from (28)

$$\frac{1}{\varkappa^2}\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d}{d\rho}(\rho U_1)\right) = 2a\nu\dot{u}f\psi,$$

whence, stipulating that U_1 decrease as $\rho \to \infty$ and be finite at zero, we readily get

$$U_1 = \frac{\varkappa^2}{\rho} a v \dot{u} \int_0^\rho (f^2 - 1) \rho \, d\rho.$$

This expression is contained in the right-hand side of (27) together with the term $a\dot{u}\psi$. We put

$$\gamma_a = -\varkappa^2 \int_0^\infty f \frac{df}{d\rho} \left[\int_0^\rho (f^2 - 1) \rho' \, d\rho' \right] d\rho$$

(the constant γ_a is positive and is of the order of unity).

We see thus that all the formulas obtained in the preceding sections can be used also for a superconductor with low paramagnetic-impurity concentration, by replacing 12γ with $a(\gamma + \nu\gamma_a) \approx a\nu\gamma_a$. By the same token, the vortex velocity decreases in order of magnitude by a factor $\nu \gg 1$: the anomalous terms suppress strongly the motion of the vortex filaments and the associated energy dissipation. In particular, the effective conductivity of a superconductor with a vortex structure increases:

$$\sigma_{\text{eff}} = \frac{a\gamma_a}{2} \nu \frac{H_{c2}}{B_0} \sigma = \frac{a^2 \gamma_a}{2\pi} (\tau_s T_c) \frac{H_{c2}}{B_0} \sigma. \tag{30}$$

It is interesting to note in this connection that in[1], within the limits of the experimental accuracy, the ratio $\sigma_{\text{eff}}/\sigma = H_{c2}(0)/H$ had no temperature dependence. Equation (3) contains $H_{c2}(T)$ at a temperature close to T_c, but the smallness of $H_{c2}(T)$ is essentially compensated by the factor $\tau_s \tau_c \gg 1$.

In conclusion we note that all our results are valid only at sufficiently low vortex velocities. In order for perturbation theory to be valid in the case of a superconductor with a large paramagnetic-impurity concentration it is necessary to have $(\mathbf{v} \cdot \nabla) f \ll f$, i.e., $v \ll 1/\kappa$. For a superconductor with low paramagnetic-impurity concentration it is necessary to have $U_1 f \ll f$, i.e., $v \ll 1/\kappa\nu$. This imposes the condition $j_e \ll H_{c2} c / 4\pi\delta\kappa$ on the magnitude of the current density.

In the case when the anomalous terms are large, perturbation theory is initially invalid for the function Δ with increasing current. We performed no calculations in this region, but qualitatively the picture is such as if the large viscosity were to cause the core of the vortex to become strongly deformed in the direction of motion.

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