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ASYMPTOTIC SOLUTION OF THE PROBLEM OF THE ANOMALOUS RESISTANCE OF A COLLISIONLESS PLASMA

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We consider the temporal evolution of the distribution functions of charged particles in a uniform plasma in an external electric field **E**. We show that in the limit as $t \to \infty$ the distribution functions change in a self-similar way with all velocities increasing linearly with time. We establish that in the case when the current is parallel to an external magnetic field the directed electron velocity **v** changes essentially in the same way as in the case of free acceleration: $\mathbf{v} = \alpha(e\mathbf{E}/m)t$, where α is a numerical coefficient which is of the order of, but smaller than, unity. If the current is at right angles to the external magnetic field the "runaway" effect disappears while the ratio of the directed electron velocity to their thermal velocity becomes much smaller than unity.

1. INTRODUCTION

LET there be in a uniform plasma a uniform constant electric field **E** which is parallel to the magnetic field and is sufficiently strong so that we can neglect binary collisions. The electrons are then freely accelerated until the velocity of their directed motion becomes equal to the threshold for the excitation of sound-wave-like oscillations after which an instability occurs in the plasma which leads to a damping of the electrons and the occurrence of the so-called "anomalous resistance."[1)] After a period of the order of a few inverse increments the system turns into the threshold state and ever afterwards it remains in that state.

The basic problem which must be answered by a theory of the anomalous resistance is the following one: what is the stationary value of the conduction electron velocity and how does it change in time due to the heating of the plasma.

Important results in the theory of the anomalous resistance were obtained in [4], where the anomalous resistance was studied at not too long times starting from the moment that the external electric field is switched on. In that paper it was shown that in the initial stage of the heating the conduction electron velocity is of the same order of magnitude as the ion sound velocity and this prediction of the theory was verified in a number of experiments.[4-7] At the same time it was established in [4] that with time by far the majority of the electrons goes over into a regime of continuous acceleration and until the plasma temperature, roughly speaking, is doubled, practically all electrons are "runaway" electrons. Beyond that the theory developed in [4] becomes inapplicable.

It is clear that the more the plasma is heated the more it "forgets" its initial state and ultimately the evolution of the particle distribution functions (and also the vibrational spectrum) takes on a universal character which is independent of the initial conditions. Following [8], we shall call the corresponding solution an asymptotic one.

[1)]The phenomenon of the anomalous resistance was connected with an ion-sound instability in refs. [1-3].

The problem stated here can be solved on the basis of quasi-linear equations.[2)] We write these equations in a system of coordinates which is moving with the velocity of the freely accelerated ions $-e\mathbf{E}t/M$ (we assume that the electric field is equal to $-\mathbf{E}$, i.e., that the electrons are accelerated in the direction of the vector **E**):

$$\frac{\partial f_e}{\partial t} + \left(\frac{1}{m} + \frac{1}{M}\right) e\mathbf{E}\frac{\partial f_e}{\partial \mathbf{v}} = \frac{\partial}{\partial v_\alpha} D_{\alpha\beta} \frac{\partial f_e}{\partial v_\beta}, \tag{1}$$

$$\frac{\partial f_i}{\partial t} = \frac{m^2}{M^2} \frac{\partial}{\partial v_\alpha} D_{\alpha\beta} \frac{\partial f_i}{\partial v_\beta}, \tag{2}$$

where

$$D_{\alpha\beta} = \frac{8\pi^2 e^2}{m^2} \int \frac{k_\alpha k_\beta}{k^2} W \delta(\omega - \mathbf{k}\mathbf{v})\, d^3\mathbf{k}$$

$W \equiv W(\mathbf{k}, t)$ is the spectral density of the electrostatic energy of the vibrations, while $\omega \equiv \omega(\mathbf{k}, t)$ is the vibrational frequency which in the threshold regime is determined by the equation

$$\varepsilon(\omega, \mathbf{k}) \equiv 1 + \frac{4\pi e^2}{mk^2} \int \frac{d^3\mathbf{v}}{\omega - \mathbf{k}\mathbf{v}} \mathbf{k} \frac{\partial}{\partial \mathbf{v}} \left(f_e + \frac{m}{M} f_i\right) = 0. \tag{3}$$

We also give here the formula for the instability increment $\gamma(\mathbf{k}, t)$:

$$\gamma = \left[\frac{\partial \varepsilon}{\partial \omega}\right]^{-1} \frac{4\pi^2 e^2}{mk^2} \int \mathbf{k} \frac{\partial}{\partial \mathbf{v}} \left(f_e + \frac{m}{M} f_i\right) \delta(\omega - \mathbf{k}\mathbf{v})\, d^3\mathbf{v}.$$

We neglect the influence of the magnetic field (assuming that the electron plasma frequency is considerably larger than the electron cyclotron frequency).

As we have not taken into account in Eqs. (1) and (2) the loss of energy to the plasma particles, the theory given in what follows can be applied only to those experiments where the energetic plasma lifetime is large compared to the heating time[3)] but it is just this situation which must clearly be realized in any apparatus

[2)]The role of non-linear processes decreases asymptotically with time as the ratio of the vibrational energy to the particle kinetic energy is proportional to t^{-1}, as we shall see in what follows. We note that the influence of the non-linear processes on the initial stages of the problem so far is far from clear. [9,10]

[3)]In many existing experiments this condition is not satisfied and there is therefore no self-similarity in the sense of Sec. 2 of the present paper while the analysis of the anomalous resistance itself is made difficult by the necessity to take heat transfer into account.

which is of interest for thermonuclear studies (otherwise all expenditure of energy goes into "heating" the walls).

2. SELF-SIMILAR VARIABLES

To find the threshold solution we must find that function $W(\mathbf{k}, t) \geq 0$ such that the functions f_e and f_i determined from Eqs. (1) and (2) will give $\gamma = 0$ when $W > 0$ and $\gamma \leq 0$ when $W = 0$. The presence of an asymptotic regime corresponds to the possibility to change in this problem to self-similar variables. It then follows from simple dimensional considerations that the particle velocities must be measured in units eEt/m and the wave vectors of the vibrations in units $m\omega_{pe}/eEt$.

Using the normalization conditions for the functions f_e and f_i we can write

$$f_{e,i}(\mathbf{v},t) = n\left(\frac{m}{eEt}\right)^3 g_{e,i}(\mathbf{u}), \quad \mathbf{u} = \frac{m\mathbf{v}}{eEt}, \tag{4}$$

where the functions $g_{e,i}$ are normalized to unity:

$$\int g_{e,i}\, d^3\mathbf{u} = 1.$$

The spectral density of the electrostatic energy of the vibrations has in the asymptotic regime the form

$$W(\mathbf{k},t) = \frac{mn}{2\pi\omega_{pe}^4}\left(\frac{eE}{m}\right)^5 t^4 \mathscr{W}(\mathbf{q}), \quad \mathbf{q} = \frac{\mathbf{k}eEt}{m\omega_{pe}}, \tag{5}$$

where the factor t^4 appears in order that Eqs. (1) and (2) when expressed in terms of the self-similar variables $\mathbf{u}$ and $\mathbf{q}$ should not contain the time while the coefficient in front of the function $\mathscr{W}$ which contains various symbols is chosen on the grounds of convenience (so that $\mathscr{W}$ would be a dimensionless function while the diffusion tensor written in terms of the self-similar variables would not contain a factor involving symbols).

Substituting Eqs. (4) and (5) into (1) and (2) we obtain a set of quasilinear equations in terms of the self-similar variables:

$$-3g_e - \mathbf{u}\frac{\partial g_e}{\partial \mathbf{u}} - \mathbf{n}\frac{\partial g_e}{\partial \mathbf{u}} = \frac{\partial}{\partial u_\alpha}\mathscr{D}_{\alpha\beta}\frac{\partial g_e}{\partial u_\beta}, \tag{6}$$

$$-3g_i - \mathbf{u}\frac{\partial g_i}{\partial \mathbf{u}} = \frac{m^2}{M^2}\frac{\partial}{\partial u_\alpha}\mathscr{D}_{\alpha\beta}\frac{\partial g_i}{\partial u_\beta}, \tag{7}$$

where

$$\mathscr{D}_{\alpha\beta} = \int \frac{q_\alpha q_\beta}{q^2}\mathscr{W}\delta(\omega - \mathbf{q}\mathbf{u})\, d^3\mathbf{q};$$

$\mathbf{n}$ is a unit vector in the direction of $\mathbf{E}$. We also write out the dispersion relation (3) in terms of the self-similar variables:

$$\varepsilon(\omega, \mathbf{q}) \equiv 1 + \frac{1}{q^2}\int \frac{d^3\mathbf{u}}{\omega - \mathbf{q}\mathbf{u}}\mathbf{q}\frac{\partial}{\partial \mathbf{u}}\left(g_e + \frac{m}{M}g_i\right) = 0$$

and the condition $\gamma = 0$:

$$\int \mathbf{q}\frac{\partial}{\partial \mathbf{u}}\left(g_e + \frac{m}{M}g_i\right)\delta(\omega - \mathbf{q}\mathbf{u})\, d^3\mathbf{u} = 0$$

(the frequency is measured in units ω_{pe}). In some cases it turns out to be convenient to use a spherical system of coordinates u, θ, φ and q, θ', φ' in velocity and in wave-vector space. In spherical coordinates Eqs. (6) and (7) become

$$-\frac{1}{u^2}\frac{\partial}{\partial u}u^3 g_e + \left(\cos\theta\frac{\partial g_e}{\partial u} - \frac{\sin\theta}{u}\frac{\partial g_e}{\partial \theta}\right) = \mathrm{St}\, g_e, \tag{8}$$

$$-\frac{1}{u^2}\frac{\partial}{\partial u}u^3 g_i = \frac{m^2}{M^2}\mathrm{St}\, g_i, \tag{9}$$

where

$$\mathrm{St}\, g_{e,i} = \frac{1}{u^2}\frac{\partial}{\partial u}u^2\left(\mathscr{D}_{uu}\frac{\partial g_{e,i}}{\partial u} + \frac{\mathscr{D}_{u\theta}}{u}\frac{\partial g_{e,i}}{\partial \theta}\right) + \frac{1}{u\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\left(\mathscr{D}_{u\theta}\frac{\partial g_{e,i}}{\partial u} + \frac{\mathscr{D}_{\theta\theta}}{u}\frac{\partial g_{e,i}}{\partial \theta}\right).$$

Exact and approximate expressions for the quantities $\mathscr{D}_{uu}$, $\mathscr{D}_{u\theta}$, and $\mathscr{D}_{\theta\theta}$ are given in the Appendix.

We can write Eqs. (6) and (7) in a form which enables us to give them a simple physical meaning:

$$\mathrm{div}_\mathbf{u}[(\mathbf{n} - \mathbf{u})g_e + \mathbf{Q}_e] = 0, \quad Q_{e\alpha} = -\mathscr{D}_{\alpha\beta}\frac{\partial g_e}{\partial u_\beta}, \tag{10}$$

$$\mathrm{div}_\mathbf{u}[-\mathbf{u}g_i + \mathbf{Q}_i] = 0, \quad Q_{i\alpha} = -\frac{m^2}{M^2}\mathscr{D}_{\alpha\beta}\frac{\partial g_i}{\partial u_\beta}. \tag{11}$$

The first of these equations describes a stationary distribution of a substance with concentration g_e in a gas with a stationary velocity field $\mathbf{n} - \mathbf{u}$ when there is a diffusion current $\mathbf{Q}_e$ present. The second one has a similar meaning for a substance with concentration g_i in a gas with a stationary velocity field $-\mathbf{u}$ (see Fig. 1).

The convective current (which corresponds to the term $-\mathbf{u}g_i$ in Eq. (11)) tends to concentrate the ions to the point $\mathbf{u} = 0$. The diffusion current counteracts this. As a result some stationary ion distribution will be established near the point $\mathbf{u} = 0$.

It is well known that vibrations with phase velocities less than the ion thermal velocity do not occur in a plasma.[11] This means that in some region near the point $\mathbf{u} = 0$ the quasi-linear diffusion coefficient $\mathscr{D}_{\alpha\beta}$ will be equal to zero. However, the convection current then leads to the result that the ions are concentrated in the point $\mathbf{u} = 0$ and their distribution takes the form of a delta-function $\delta(\mathbf{u})$. In reality, however, after part of the ions are concentrated in the point $\mathbf{u} = 0$ the dispersion changes and vibrations occur with arbitrarily small phase velocities so that the diffusion coefficient remains different from zero in any, however small, region around the point $\mathbf{u} = 0$. We are thus led to the conclusion that the ion distribution function necessarily has a delta-functionlike singularity (core) in the point $\mathbf{u} = 0$ (which corresponds to freely accelerated ions).

Similar considerations show that, generally speaking, there is also an electron core in the point $\mathbf{u} = \mathbf{n}$ (which corresponds to freely accelerated electrons). Those fractions of the particles which are in the electron and ion cores we shall denote, respectively, by X_e and X_i. The quantities X_e and X_i depend in the asymptotic regime clearly not on the time and on the electric field and are determined solely by the ratio of the electron to the ion mass.

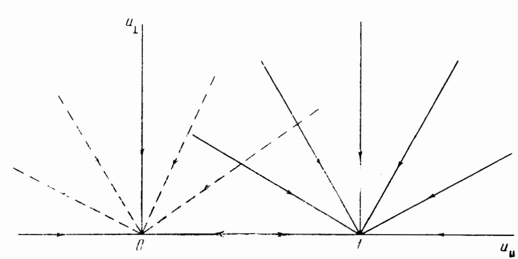

FIG. 1. Velocity field for Eqs. (10) and (11). The current lines for Eq. (11) are shown dashed.

To conclude this section we write down the self-similar equations for the one-dimensional solution which occurs when the ion cyclotron frequency ω_{Hi} is appreciably larger than the ion plasma frequency ω_{pi}. In that case the problem is described by one-dimensional distribution functions $f_{e,i}(v_z, t)$ and a one-dimensional spectral function $W(k_z, t)$ which satisfy the following normalization conditions:

$$\int f_{e,i}(v_z,t)\,dv_z = n, \quad \int W(k_z,t)\,dk_z = U(t),$$

where U(t) is the volume density of the electrostatic energy of the vibrations. We have then in the asymptotic regime

$$f_{e,i}(v_z,t) = \frac{mng_{e,i}(u)}{eEt}, \quad W(k_z,t) = \frac{eE^3t^2}{8\pi^2 m}\mathscr{W}(q), \tag{12}$$

$$u = \frac{mv}{ekt}, \quad q = \frac{eEtk_z}{m\omega_{pe}},$$

and

$$-\frac{d}{du}(u-1-\mu)g_e = \frac{d}{du}\mathscr{D}(u)\frac{dg_e}{du}, \tag{13}$$

$$-\frac{d}{du}ug_i = \mu^2\frac{d}{du}\mathscr{D}(u)\frac{dg_i}{du} \tag{14}$$

where

$$\mathscr{D}(u) = \frac{\mathscr{W}(q)}{|\omega/q - d\omega/dq|}\Bigg|_{\omega(q)/q=u} \tag{15}$$

is the quasi-linear diffusion coefficient and $\mu \equiv m/M$.

3. STUDY OF THE SELF-SIMILAR EQUATIONS FOR THE ONE-DIMENSIONAL MODEL

In the one-dimensional case the sum of the diffusion and the convection currents must in the stationary state vanish, i.e.,

$$-(u-1-\mu)g_e = \mathscr{D}(u)\frac{dg_e}{du}, \quad -ug_i = \mu^2\mathscr{D}(u)\frac{dg_i}{du}. \tag{16}$$

Hence it follows that in the regions $u < 0$ and $u > 1$ the condition for the vanishing of the increment of the vibrations, which in the one-dimensional case has the form

$$\frac{d}{du}(g_e + \mu g_i) = 0, \tag{17}$$

can be satisfied only when $g_e = g_i = 0$. As far as the region $0 < u < 1$ is concerned, in it the set of Eqs. (16), (17) enables us to find the functions g_e, g_i, and $\mathscr{D}$:[4]

$$g_e = \frac{Cu}{u+\mu^2}, \quad g_i = \frac{C\mu(1-u)}{u+\mu^2}, \quad \mathscr{D} = \mu^{-2}u^2(1-u);$$

here C is an arbitrary positive constant. From what was said in the preceding section it follows that we must add to the functions g_e and g_i a number of freely accelerated electrons and ions, while it follows from the normalization condition that $X_e + C = 1$, $X_i + 2C\mu \ln \mu^{-1} = 1$.

Knowing the functions g_e and g_i we can easily write down the dispersion relation

$$\varepsilon(\omega,q) = 1 - \frac{1-C}{(\omega-q)^2} - \frac{\mu}{\omega^2} + \frac{C}{\omega q} - \frac{C}{(\omega-q)q} = 0.$$

The function $\epsilon(\omega, q)$ must satisfy the following two requirements: 1) all vibrations must be stable, 2) vibrations must exist for all phase velocities in the interval (0,1).[5] From these conditions we can determine the constant C. To do this we must write the dispersion relation in the form

$$F(\tilde{u}) = \frac{1-C}{(\tilde{u}-1)^2} + \frac{\mu}{\tilde{u}^2} - \frac{C}{\tilde{u}} + \frac{C}{\tilde{u}-1} = q^2, \tag{18}$$

where $\tilde{u} = \omega/q$. Equation (18) is a fourth-degree equation in $\tilde{u}$ and for stability it is necessary that it have four real roots. In Fig. 2 we give the function $F(\tilde{u})$ for different values of the constant C. From this figure it is clear that when $C < 2\mu^{1/2}$ the system is unstable for small q while for $C > 2\mu^{1/2}$ there is a range of phase velocities inside the interval (0, 1) where there are no vibrations. We are thus led to the conclusion that $C = 2\mu^{1/2}$ and the distribution functions are thus uniquely determined.

We can find the spectral function from Eq. (15). As usual we divide all vibrations into two types: Langmuir waves (with phase velocities $\tilde{u} \gtrsim \mu^{1/2}$) and sound waves (with phase velocities $\tilde{u} \lesssim \mu^{1/2}$). For the first $\omega \approx q - 1$, and for the second $\omega \approx \mu^{1/2}$. We then get from (15) the following expressions for the energy densities of these vibrations:

$$U_l = 1/3\mu^2, \quad U_s = 1/2\sqrt{\mu}.$$

We list the main qualitative peculiarities of the solution obtained: 1) almost all electrons and ions are freely accelerated by the electric field; 2) notwithstanding this the system is on the threshold of stability which is secured by the presence of small groups of electrons and ions which are "smeared out" in velocity; 3) there are very "hot" ions with energies μ^{-1} times larger than the energy of the freely accelerated electrons (the relative concentration of such ions is approximately equal to $\mu^{3/2}$); 4) in the asymptotic regime the energy of the Langmuir vibrations appreciably exceeds the energy of the sound vibrations.

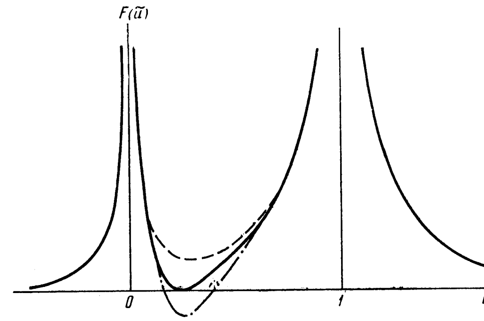

FIG. 2. The form of the function $F(\tilde{u})$ for three values of the constant C: dashed curve: $C < 2\mu^{1/2}$, full-drawn curve: $C = 2\mu^{1/2}$, dash-dot curve: $C > 2\mu^{1/2}$.

4. STUDY OF THE SELF-SIMILAR EQUATIONS FOR THE THREE-DIMENSIONAL MODEL

We shall publish the formal study of the self-similar equations for the three-dimensional case elsewhere.

[4] In the following calculations we shall assume for the sake of simplicity that $\mu \ll 1$ although one can also obtain an exact solution which is valid for any value of the electron to ion mass ratio.

[5] If the latter condition were not satisfied there would be sections of the interval (0, 1) where the diffusion coefficient $\mathscr{D}$ would vanish. The convection current would then remove ions to the left of this interval and electrons to the right and we could not satisfy Eq. (17).

Here we restrict ourselves to only a few qualitative considerations.

It follows from what we have said above that an important peculiarity of the one-dimensional solution is the presence of Langmuir vibrations. This is connected with the fact that ion-sound vibrations with a small phase velocity cannot guarantee a non-vanishing diffusion in the whole range of velocities (0, 1). In the three-dimensional case, however, where "skew" waves are also present even one ion sound wave is sufficient for the diffusion of particles in the whole velocity space. It is thus reasonable to consider the problem whether the anomalous resistance can be guaranteed solely by taking the excitation of ion-sound vibrations into account.

First of all we obtain exact relations which are analogs of the energy and momentum conservation laws. To do this we multiply Eqs. (10) and (11) by $\mathbf{u}^2$ and $\mathbf{n}\cdot\mathbf{u}$ and integrate them over $d^3\mathbf{u}$ bearing in mind that the function $\mathscr{W}$ is non-vanishing only for those values of $\mathbf{q}$ for which

$$\int \mathbf{q}\frac{\partial}{\partial \mathbf{u}}(g_e + \mu g_i)\delta(\omega - \mathbf{qu})\, d^3\mathbf{u} = 0.$$

As a result we get

$$\overline{u_e^2} - \bar{u}_e = -Av_{\text{ph}}, \tag{19}$$

$$\overline{u_i^2} = \mu A v_{\text{ph}}, \tag{20}$$

$$\bar{u}_e - 1 = -A, \tag{21}$$

$$\bar{u}_i = \mu A, \tag{22}$$

where we have written

$$A = \int \mathbf{n}Q_e d^3\mathbf{u}, \quad Av_{\text{ph}} = \int \mathbf{u}Q_e d^3\mathbf{u},$$

$$\bar{u}_{e,i} = \int g_{e,i}(\mathbf{nu})\, d^3\mathbf{u}, \quad \overline{u_{e,i}^2} = \int g_{e,i} u^2\, d^3\mathbf{u}.$$

The quantity v_{ph} has the meaning of a characteristic phase velocity of the vibrations. Adding (19) and (20) as well as (21) and (22) we easily obtain the required relations.

Since the phase velocity of the sound-wave-type vibrations is small compared with the electron thermal velocity we can use for the calculation of the tensor $\mathscr{D}_{\alpha\beta}$ occurring in the kinetic equation for the electrons the approximate Eq. (A.2) from which it follows that

$$\mathscr{D}_{\theta\theta} \sim \frac{B}{u}\chi_1(\theta), \quad \mathscr{D}_{u\theta} \sim \frac{v_{\text{ph}}}{u}\frac{B}{u}\chi_2(\theta), \quad \mathscr{D}_{uu} \sim \left(\frac{v_{\text{ph}}}{u}\right)^2 \frac{B}{u}\chi^3(\theta),$$

where B is a constant while χ_1, χ_2, and χ_3 are smooth functions of the angle θ of order of magnitude unity. For the main mass of electrons we have thus $\mathscr{D}_{\theta\theta} \gg \mathscr{D}_{u\theta}$, $\mathscr{D}_{uu}$ and the main processes of importance for them are elastic scattering processes. The scattering frequency ν increases with decreasing velocity ($\nu \propto u^{-3}$). There is thus always a region in velocity space (its boundaries will be indicated below) where the electron distribution is almost isotropic and where we can write the function g_e in the form $g_e(u, \theta) = \bar{g}_e(u) + \delta g_e(u, \theta)$, where

$$\delta g_e \ll \bar{g}_e(u) \equiv \frac{1}{2}\int_0^\pi g_e(u, \theta)\sin\theta\, d\theta.$$

We then get from Eq. (8)

$$\frac{\partial \delta g_e}{\partial \theta} = \frac{\sin\theta}{2D_{\theta\theta}} u^2 \frac{\partial \bar{g}_e}{\partial u} - u\frac{\mathscr{D}_{u\theta}}{\mathscr{D}_{\theta\theta}}\frac{\partial \bar{g}_e}{\partial u}. \tag{23}$$

Using the above given estimates for the diffusion coefficients we see that when $u \le u_1 = B^{1/3} v_{\text{ph}}^{1/3}$ the main term on the right-hand side of (23) is the second one and the anisotropy of the electrons in that region of velocities is determined by their interaction with the vibrations. When $u > u_1$ the electrons become anisotropic under the action of the electric field (first term on the right-hand side of (23)). Equation (23) is valid up to velocities $u \sim B^{1/2}$ when it is no longer possible to consider the electron distribution to be weakly anisotropic. Substituting Eq. (23) into Eq. (8) and integrating over the angles we get an equation for the isotropic part $\bar{g}_e$ of the distribution function:

when $u < u_1$ (region 1)

$$-\frac{1}{u^2}\frac{d}{du}u^3\bar{g}_e = \frac{1}{2u^2}\frac{d}{du}\left[u^2\frac{d\bar{g}_e}{du}\int_0\left(\mathscr{D}_{uu} - \frac{\mathscr{D}_{u\theta}^2}{\mathscr{D}_{\theta\theta}}\right)\sin\theta\, d\theta\right]$$

and

$$\bar{g}_e \propto \exp\left(-u^5/Bv_{\text{ph}}^2\right); \tag{24}$$

when $u_1 < u < B^{1/2}$ (region 2)

$$-\frac{1}{u^2}\frac{d}{du}u^3\bar{g}_e = \frac{1}{8u^2}\frac{d}{du}\left[u^4\frac{d\bar{g}_e}{du}\int_0^\pi \frac{\sin^3\theta}{\mathscr{D}_{\theta\theta}}d\theta\right],$$

and

$$\bar{g}_e \propto \exp(B/u). \tag{25}$$

The form of the distribution function and the relative number of electrons in the regions 1 and 2 are determined by the two parameters, B and v_{ph}, while in our units we have always $v_{\text{ph}} \ll 1$.

We first of all consider the case $B \gtrsim 1$ when clearly almost all electrons are in the regions 1 and 2 (from energetic considerations it is clear that only a small part of the electrons can have a velocity $u > 1$). One verifies easily that the majority of the electrons are then concentrated in the region 1. Indeed, as $u_1^5 \sim B^{5/3}v_{\text{ph}}^{5/3} > Bv_{\text{ph}}^2$ the solution (24) is for $u = u_1$ already exponentially small. The electron temperature ($\overline{u_e^2} \sim B^{2/5}v_{\text{ph}}^{4/5}$) and their directed velocity ($\bar{u}_e \sim v_{\text{ph}}$) are determined by their interaction with the vibrations and are not at all connected with the electric field, the influence of which one can neglect in region 1. This indicates already the physical lack of meaning of the solution obtained.

For a more rigorous proof we turn to Eqs. (19) to (21). As we are trying to construct a solution with $\bar{u}_e \ll 1$ it follows at once from (21) that the constant A is almost equal to unity. Taking this into account we get from Eq. (19)

$$\overline{u_e^2} = \alpha v_{\text{ph}}, \tag{26}$$

where $\alpha \lesssim 1$. On the other hand, we can determine the mean square electron velocity in the case considered from Eq. (24). We have

$$\overline{u_e^2} \sim B^{2/5}v_{\Phi}^{4/5},$$

which when $B \lesssim 1$ and $v_{\text{ph}} \ll 1$ is incompatible with (26). We have thus proved the unrealizability of ion-sound solutions with $\bar{u}_e \ll 1$ and $B \gtrsim 1$.

Let us now turn to a study of the case $B \ll 1$. In order that the electrons are weakly anisotropic they must nearly all be inside a sphere of radius $B^{1/2}$ in velocity space. If then $v_{\text{ph}} \ll B^2$, we have $u_1^5 \gg Bv_{\text{ph}}^2$ and again all electrons turn out to be in the region 1 which leads to the above described contradiction.

There remains thus the consideration of the solution when $v_{ph} \gtrsim B^2$. In that case $u_1^5 \lesssim Bv_{ph}^2$ and the distribution function $\bar{g}_e(u)$ does hardly change in the region 1. At the same time $u_1 \gtrsim B$ and the solution is also constant in the region 2. This constant solution, valid up to $u \sim B^{1/2}$ must be fitted to the solution in the remaining region of velocities where the anisotropy is no longer small. And as the size of this region is of order unity and much larger than $B^{1/2}$ most of the electrons found in it and therefore $\bar{u}_e \sim 1$. We have thus shown that when we take only one of the ion-sound vibrations into account we have necessarily $\overline{u_e^2} \sim 1$, $\bar{u}_e \sim 1$ (there is no anomalous resistance). Including the Langmuir vibrations in the problem cannot appreciably change this result as the Langmuir vibrations lead only to a redistribution of the energy and momentum inside the electron gas.

5. ANOMALOUS RESISTANCE TO A CURRENT AT RIGHT ANGLES TO THE MAGNETIC FIELD

The problem considered in this section is mainly of interest for the physics of shock waves propagating at right angles to the magnetic field. Trying to explain the anomalous resistance of a plasma in shock waves we take at once into account those concrete conditions which are usually satisfied in those experiments and which simplify the solution of the problem. Thus, the smallness of the electron cyclotron frequency compared to the electron plasma frequency enables us to neglect the influence of the magnetic field on the dispersion and as before we can use the usual dispersion relation (3). As ions in collisionless shock waves are non-magnetic the main part in the kinetic equation for the ions is played by the quasi-linear diffusion:

$$\frac{\partial f_i}{\partial t} = \mu^2 \frac{\partial}{\partial v_\alpha} D_{\alpha\beta} \frac{\partial f_i}{\partial v_\beta}. \tag{27}$$

On the other hand, the electron scattering frequency is much smaller than their cyclotron frequency so that the directed motion of the electrons is a drift motion and in the system of coordinates fixed to the drift we can assume their distribution to be axially symmetric around the magnetic field.

We verified in the preceding section that when $\mathbf{E} \parallel \mathbf{H}$ the majority of the electrons is nearly freely accelerated by the electric field. Now, however, the magnetic field prevents the "running away" of the electrons and we may expect that in a stationary state their directed velocity will be much smaller than their thermal velocity. To illustrate this last statement we consider an idealized model which allows an exact solution.

Let there be vibrations propagating only along the current the direction of which we shall assume to be along the z-axis. The component $D_{xx} = D(v_x)$ is the only non-vanishing one of the diffusion tensor. The ion distribution function is thus, as follows from (27), also one-dimensional while the electrons in the (v_x, v_y)-plane which is perpendicular to the direction of the magnetic field, are distributed axially symmetrically around the point $(\bar{v}, 0)$ ($\bar{v}$ is the drift velocity). Introducing polar coordinates (v, φ) in the (v_x, v_y) plane with the origin in that point we can write down the kinetic equation for the function $f_e(v, t)$:

$$\frac{\partial f_e}{\partial t} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \left[\frac{\partial}{\partial v_x} D(v_x) \frac{\partial f_e}{\partial v_x}\right] = \frac{\bar{D}}{v}\frac{\partial}{\partial v}\frac{1}{v^2}\frac{\partial f_e}{\partial v}, \tag{28}$$

where

$$\bar{D} = \frac{1}{\pi}\int_{v_{x_1}}^{v_{x_2}} D(v_x)(v_x - \bar{v})^2 dv_x,$$

while (v_{x_1}, v_{x_2}) is an interval of phase velocities where the vibrations and thus the diffusion coefficient $D(v_x)$ are non-vanishing. We took here into account the smallness of the phase velocities as compared to the electron thermal velocity; this will be confirmed in the solution obtained below.

We write the kinetic equation for the ions simply

$$\frac{\partial f_i}{\partial t} = \mu^2 \frac{\partial}{\partial v_x} D(v_x) \frac{\partial f_i}{\partial v_x}. \tag{29}$$

It is clear that in the asymptotic regime all velocities must be measured in units of the drift velocity $\bar{v}$. Corresponding to that we introduce self-similar variables $\xi = v/\bar{v}$ and $u_x = v/\bar{v}$ while we write the distribution function in the form

$$f_e(v, t) = \frac{n}{\bar{v}^2} g_e(\xi), \quad f_i(v_x, t) = \frac{n}{\bar{v}} g_i(u_x).$$

The self-similar diffusion coefficient $\mathscr{D}$ is connected with D through the relation $\mathscr{D} = D[\bar{v}d\bar{v}/dt]^{-1}$.

The functions g_e and g_i satisfy the equations

$$-\frac{1}{\xi}\frac{d}{d\xi}\xi^2 g_e = \frac{\bar{\mathscr{D}}}{\xi}\frac{d}{d\xi}\frac{1}{\xi^2}\frac{dg_e}{d\xi}, \quad \bar{\mathscr{D}} = \frac{1}{\pi}\int_{u_{x1}}^{u_{x2}} \mathscr{D}(u_x)(u_x - 1)^2 du_x,$$
$$-\frac{d}{du_x} u_x g_i = \mu^2 \frac{d}{du_x} \mathscr{D}(u_x) \frac{dg_i}{du_x}. \tag{30}$$

The first of these can easily be integrated:

$$g_e = C_1 \exp(-\xi^5 / 5\bar{\mathscr{D}}),$$

where the connection between the constants C_1 and $\bar{\mathscr{D}}$ can be found from the normalization condition

$$2\pi\int_0^\infty \xi g_e d\xi + X_e = 1,$$

where X_e is the number of particles in the electron core which is now in the point (1, 0).

The necessity that it exist in this model is clear from the following considerations. Let us suppose that $X_e = 0$. There are then in the system possible only ion-sound vibrations with a phase velocity which cannot be larger than some value u_{max}. Since the spectrum is one-dimensional, when the drift velocity is larger than u_{max} part of the electrons (those inside the circle in Fig. 3) will not interact with the vibrations. They will thus "cool off" and concentrate in the point $\xi = 0$. This will happen until the dispersion changes so much that vibrations occur with all phase velocities in the interval (0, 1). We can check that this occurs when $X_e \ll 1$. It

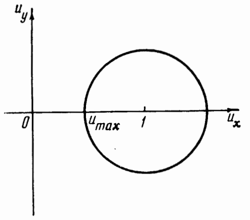

FIG. 3. Interaction of one-dimensional vibrations with electrons in a magnetic field.

then follows at once from the normalization condition that

$$\pi\Gamma(^7/_5)\,(5\overline{\mathscr{D}})^{2/5}C_1 = 1.$$

The contribution of the electrons to the dispersion relation is determined by their longitudinal velocity distribution function $h_e(u_x)$:

$$h_e(u_x) = 2\int\limits_{|u_x-1|}^{\infty}\frac{g_e(\xi)\xi d\xi}{[\xi^2-(u_x-1)^2]^{1/2}} = -2\int\limits_{|u_x-1|}^{\infty}\frac{dg_e}{d\xi}[\xi^2-(u_x-1)^2]^{1/2}d\xi. \quad (31)$$

From the condition $\gamma = 0$ in the phase velocity range (0, 1) follows that

$$\frac{dg_i}{du_x} = -\mu^{-1}\frac{dh_e}{du_x} \approx 2\mu^{-1}(1-u_x)\int\limits_0^{\infty}\frac{d\xi}{\xi}\frac{dg_e}{d\xi} = -\mu^{-1}(1-u_x)(5\overline{\mathscr{D}})^{-3/5}\frac{5\Gamma(^9/_5)}{2\pi\Gamma(^7/_5)}$$

(when evaluating dh_e/du_x from (31) we took into account that the electron thermal velocity is much larger than their drift velocity which in our units is simply equal to unity). Hence we find the ion distribution function:

$$g_i = \mu^{-1}(1-u_x)^2(5\overline{\mathscr{D}})^{-3/5}\frac{5\Gamma(^9/_5)}{4\pi\Gamma(^7/_5)}$$

and from Eq. (30) the diffusion coefficient $\overline{\mathscr{D}}$:

$$\overline{\mathscr{D}} = 1/40\pi\mu^2.$$

We can now find the root mean square electron velocity $(\overline{\xi^2})^{1/2}$ from the relation

$$\overline{\xi^2} = \int\limits_0^{\infty}\xi^3 g_e d\xi \Big/ \int\limits_0^{\infty}\xi g_e d\xi = (5\overline{\mathscr{D}})^{2/5}\frac{\Gamma(^9/_5)}{2\Gamma(^7/_5)}$$

It turns out to be equal to $0.38\,\mu^{-2/5}$. The ratio of the electron current velocity to their thermal velocity is in the model considered equal to $2.65\,\mu^{2/5}$.

The number of ions interacting with the vibrations is, as before, small:

$$1 - X_i = \int\limits_0^1 g_i(u_x)du_x = \mu^{1/5}\frac{10\Gamma(^9/_5)}{3(8\pi)^{2/5}\Gamma(^7/_5)} \approx 0{,}95\,\mu^{1/5},$$

and to determine X_e we must use the dispersion relation

$$\varepsilon(\omega,q) = 1 - \frac{1}{q^2}\int\limits_{-\infty}^{+\infty}\frac{du_x}{u_x}\frac{dh_e}{du_x} - \frac{X_e}{(\omega-q)^2} - \frac{\mu g_i(0)}{\omega q} - \frac{\mu X_i}{\omega^2} = 0,$$

or

$$1 + \frac{8.2\mu^{4/5}}{q^2} = \frac{X_e}{(\omega-q)^2} + \frac{\mu}{\omega^2} + \frac{2.86\mu^{6/5}}{\omega q}.$$

From the requirement of stability and that there are vibrations with phase velocities covering the whole interval (0, 1) we find, exactly as in Sec. 3, the number of particles in the electron core: $X_e \approx 8.2\,\mu^{4/5}$.

It is interesting to note that the ratio of the electron current velocity $\bar{v}$ to their thermal velocity v_{Te} and the number of resonant ions $1 - X_i$ can be estimated using simple considerations based on conservation laws. Due to the scattering of electrons by vibrations there appears an electron-ion friction force F_{fr} which transfers momentum from the electrons to the ions. If we denote the momentum of the latter by P_i we have $dP_i/dt = F_{fr}$. This equation is equivalent to the calculation of the first moment of the ion kinetic equation. Since $P_i \sim (1 - X_i)M\bar{v}$, we have $F_{fr} \sim (1 - X_i)Md\bar{v}/dt$. The work done by the friction force goes into heating the electrons: $dT\ /dt \sim \bar{v}F_{fr}$. Hence it follows that $v_{Te}^2 \sim (1 - X_i)M\bar{v}^2/m$. Equating the electron increment and the ion damping gives yet one more equation:

$$\frac{m}{M}\frac{1-X_i}{\bar{v}^2} \sim \frac{\bar{v}}{v_{Te}^3}.$$

From this we get at once that $1 - X_i \sim \mu^{1/5}$, $\bar{v}/v_{Te} \sim \mu^{2/5}$. This result is, of course, confirmed by the exact solution.

In our considerations we have so far not at all taken "skew" waves into account which propagate at an angle to the current direction. One verifies easily that the presence of a steep maximum in the electron distribution function in the point $\xi = 0$ leads to an instability of ion-sound waves with a wave vector directed almost at right angles to the current.

The electron core will under their influence be "smeared out" until these waves become stable. As a result of this the branch of vibrations with large phase velocities disappears and there will only be ion sound in the system. In the kinetic equations for the electrons and ions we can again change to self-similar variables but up to now we have not succeeded in solving them exactly. Nevertheless we can estimate the quantities which are of interest to us by considering the energy and momentum balance.[12] As a result we find that $1 - X_i \sim \mu^{1/4}$, $\bar{v}/v_{Te} \sim \mu^{1/4}$ (in satisfactory agreement with measurements of these quantities at the front of shock waves).

APPENDIX

When we use spherical coordinates u, θ, φ and q, θ', φ' in the velocity and wave vector spaces the diffusion tensor has the form

$$\mathscr{D}_{uu} = \frac{2}{u}\iint\Phi\frac{\omega^2}{qu^2}d\theta'\,dq,$$
$$\mathscr{D}_{u\theta} = \frac{2}{u}\iint\Phi\frac{\omega}{u}\frac{(\omega/qu)\cos\theta - \cos\theta'}{\sin\theta}d\theta'\,dq,$$
$$\mathscr{D}_{\theta\theta} = \frac{2}{u}\iint\Phi q\frac{[(\omega/qu)\cos\theta-\cos\theta']^2}{\sin^2\theta}d\theta'\,dq, \quad (A.1)$$

where

$$\Phi = \frac{W\sin\theta'}{\{\sin^2\theta\sin^2\theta' - [(\omega/qu)-\cos\theta\cos\theta']^2\}^{1/2}}.$$

The integration is over that region of the variables q and θ' where the expression under the radical sign is positive.

When $\omega/qu \ll 1$ we have

$$\mathscr{D}_{uu} = \frac{2}{u}\iint\Omega\frac{\omega^2}{qu^2}d\theta'\,dq,$$
$$\mathscr{D}_{u\theta} = \frac{2}{u}\iint\Omega\frac{-\omega\cos\theta'}{u\sin\theta}d\theta'\,dq, \quad (A.2)$$
$$\mathscr{D}_{\theta\theta} = \frac{2}{u}\iint\Omega q\frac{\cos^2\theta'}{\sin^2\theta}d\theta'\,dq,$$

where

$$\Omega = \frac{W \sin\theta'}{(\sin^2\theta \sin^2\theta' - \cos^2\theta \cos^2\theta')^{1/2}}$$

and the integration is over the region where $\sin\theta' > |\cos\theta|$.

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