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THE OSCILLATORY MODE OF APPROACH TO A SINGULARITY IN HOMOGENEOUS COSMOLOGICAL MODELS WITH ROTATING AXES

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We consider a homogeneous cosmological model of the Bianchi IX type which is more general than that previously considered[1] and in which the matrix of the coefficients $\gamma_{ab}(t)$ in the space metric (1.1) contains off-diagonal elements. The presence of these elements does not alter the general oscillatory character of evolution of the model on approach to a singularity but leads to a rotation of the axes of the alternating Kasner epochs. In homogeneous models, this phenomenon can exist only when matter is present; some observations are made regarding a possible connection with the properties of the general inhomogeneous solution of the Einstein equations for empty space as well as for space filled with matter.

1. INTRODUCTION

THE oscillatory mode of approach to a singularity in cosmological models (with Bianchi type-IX or type-VIII spaces) has been investigated by us in previous papers.[1)] These models are made especially important by the fact that they yield a prototype by which the most general cosmological solution of Einstein's equations near a temporal singularity should be constructed.

The earlier analysis we made was, however, incomplete in the following respect.

We recall that the metric of a homogeneous space may be represented in the form

$$dl^2 = \gamma_{ab}(t)\,(e_\alpha{}^a dx^\alpha)\,(e_\beta{}^b dx^\beta) \tag{1.1}$$

where $\mathbf{e}^1 \equiv \mathbf{l}$, $\mathbf{e}^2 \equiv \mathbf{m}$, $\mathbf{e}^3 \equiv \mathbf{n}$ are three basis vectors which are definite (for each of the Bianchi types) functions of the spatial coordinates while the coefficients γ_{ab} are functions of time (the Greek indices α, β number the spatial coordinates x^1, x^2, x^3 while the Latin indices a, b number the basis vectors $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$). In a synchronous system of coordinates the four-dimensional interval $ds^2 = dt^2 - dl^2$ and hence only the six functions $\gamma_{ab}(t)$ which should be determined by the Einstein equations remain unknown in such a system.

In the previously considered models the matrix of the coefficients γ_{ab} was assumed to be diagonal; the spatial metric was assumed to have the form:

$$dl^2 = (a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta)\,dx^\alpha dx^\beta, \tag{1.2}$$

which contains only three unknown functions of the time $\gamma_{11} = a^2$, $\gamma_{22} = b^2$, $\gamma_{33} = c^2$. Such an artificial reduction of the number of unknown functions did not lead to inconsistencies, owing to the fact that as a result of the specific symmetry of the types of spaces under consideration, the off-diagonal components of the Ricci tensor vanished identically while the rest of the Einstein equations (for a field in empty space) constituted a consistent system. Nevertheless, the limitation imposed on the metric by the requirement that the matrix γ_{ab} be diagonal could lead to the disappearance of some properties which are characteristic of the more general case. The present paper is devoted to the elimination of just this deficiency, and to the elucidation of the effect of nondiagonality of the matrix γ_{ab} on the behavior of the model near a singularity. We restrict ourselves here to the case of the more symmetric model of the IX type (although the extension of similar calculations to the case of the type-VIII model does not present any fundamental difficulties).

2. THE KASNER AXES

As always when operating with homogeneous spaces, all the three-dimensional vectors and tensors will sep-

1)The results of these investigations are given in [1] (which is referred to below as I). In this paper we follow, as far as possible, the notation and terminology used in I.

arate into sets of three basis vectors; the components of such decompositions are functions only of time (see I, Appendix C). We shall denote these components by the Latin indices a, b, c, ..., which take on the values 1, 2, 3.[2)] The raising and lowering of these indices are done with the aid of the matrices γ_{ab} and γ^{ab}.

The model being considered, understood as the exact solution of the Einstein equations, can exist only for a space filled with matter; otherwise the equations $R^0_\alpha = 0$ would automatically lead to the disappearance of the off-diagonal components of γ_{ab}. But from the more general point of view, if we consider the model only as the main terms of the limiting form of the metric near a singularity, then the presence of matter is not of decisive importance. Deferring the discussion of this question to Sec. 6, we shall consider first only the ab- and 00-components of the Einstein equations for a field in empty space, abstracting ourselves from the 0a-components; we recall that the last components, generally speaking, only play the role of conditions imposed on the initial values of the unknown functions.

The Einstein equations for the field in empty space in a synchronous reference frame have the form

$$R_0^0 = -\frac{1}{2}\dot{\varkappa}_a^a - \frac{1}{4}\varkappa_a^b\varkappa_b^a = 0, \tag{2.1}$$

$$R_a^b = -\frac{1}{2\sqrt{\Gamma}}(\sqrt{\Gamma}\varkappa_a^b)^{\cdot} - P_a^b = 0, \tag{2.2}$$

where the dot denotes differentiation with respect to t; P_a^b are the components—resolved in terms of the basis vectors—of the three-dimensional Ricci tensor; the quantities $\kappa_{ab} = \dot{\gamma}_{ab}$, $\kappa_a^b = \gamma_{ac}\gamma^{cb}$; Γ is the determinant of the matrix γ_{ab}.

The components P_a^b can be calculated (expressed in terms of the components of the matrix γ_{ab}) with the aid of the formulas of I (C.17) in terms of the given structural constants of the motion group of the space; a concrete application of these formulas to spaces of the types VIII and IX is made in the Appendix.

The values of the structural constants depend on the method by which the three basis vectors are chosen and are, in this sense, not unique. For the type-IX space of interest to us, the basis vectors may be chosen in such a way that the only nonvanishing structural constants are

$$C_{32}^1 = C_{13}^2 = C_{21}^3 = 1 \tag{2.3}$$

(and the constants which are obtained by interchanging the lower indices of these constants and differ from them by sign). Precisely such a choice will be implied everywhere below. We emphasize, however, that these conditions still do not determine the basis vectors uniquely: for a space of the type IX the structural constants preserve their values (2.3) in any orthogonal transformation of the vectors **l**, **m**, **n** which leaves the sum of the squares $\mathbf{l}^2 + \mathbf{m}^2 + \mathbf{n}^2$ unchanged.

With the structural constants (2.3) the Bianchi identity for the three-dimensional Ricci tensor is reduced to the form

$$P_a^b C_{cb}^a = 0. \tag{2.4}$$

By constructing similar combinations from the components R_a^b, we can write the Einstein equations as $R_a^b C_{cb}^a = 0$; by virtue of (2.2) and (2.4) their first integrals yield

$$\sqrt{\Gamma}\varkappa_a^b C_{cb}^a = 2C_c,$$

where C_c are arbitrary constants. In explicit form

$$\sqrt{\Gamma}(\varkappa_1^2 - \varkappa_2^1) = 2C_3, \quad \sqrt{\Gamma}(\varkappa_3^1 - \varkappa_1^3) = 2C_2, \quad \sqrt{\Gamma}(\varkappa_2^3 - \varkappa_3^2) = 2C_1. \tag{2.5}$$

A characteristic feature of the evolution of the models under consideration on approach to a singularity is the alternation of the "Kasner epochs" with a definite law of variation of the exponents p_1, p_2, p_3 which determine the variation of the distance scales in the three independent directions in space; in each transition from one epoch to the next a negative exponent is transferred from one direction to another. This property, which has been investigated by us in the diagonal case, is fully retained in the general nondiagonal case. At the same time, new features appear in it as well.

A separate Kasner epoch occurs during the period of time when the terms P_a^b in Eqs. (2.2) are small compared with the derivatives with respect to time and may be dropped. After this Eqs. (2.1–2) will, in the general case, have solutions of the form

$$\gamma_{ab} = a^2 L_a L_b + b^2 M_a M_b + c^2 N_a N_b, \tag{2.6}$$

with

$$a \sim t^{p_l},\ b \sim t^{p_m},\ c \sim t^{p_n}, \tag{2.7}$$

where p_l, p_m, p_n are any of the set of the three Kasner exponents p_1, p_2, p_3, while L_a, M_a, N_a are constant coefficients. The determinant of the matrix (2.6)

$$\Gamma = (abcV)^2,\ V = (\mathbf{L}[\mathbf{MN}]), \tag{2.8*}$$

the vector operations being performed as if the quantities L_a, M_a, and N_a formed Cartesian vectors **L**, **M**, and **N**; let us normalize these "vectors" in accordance with the relation $\mathbf{L}^2 = \mathbf{M}^2 = \mathbf{N}^2 = 1$. During a Kasner epoch the determinant Γ varies with time according to the law

$$\sqrt{\Gamma} = \Lambda t \tag{2.9}$$

where Λ is a constant.

The matrix γ^{ab}, which is the inverse matrix of (2.6), is given by

$$\gamma^{ab} = a^{-2}L^aL^b + b^{-2}M^aM^b + c^{-2}N^aN^b, \tag{2.10}$$

where

$$L^a = [\mathbf{MN}]_a/V,\ M^a = [\mathbf{NL}]_a/V,\ N^a = [\mathbf{LM}]_a/V. \tag{2.11}$$

Setting $\kappa_{ab} = \dot{\gamma}_{ab}$ and raising the index b with the aid of (2.10), we obtain

$$\varkappa_a^b = \frac{2}{t}(p_l L_a L^b + p_m M_a M^b + p_n N_a N^b) \tag{2.12}$$

Substituting (2.6) in (1.1), we obtain

$$dl^2 = (a^2 l_{\kappa\alpha} l_{\kappa\beta} + b^2 m_{\kappa\alpha} m_{\kappa\beta} + c^2 n_{\kappa\alpha} n_{\kappa\beta})\, dx^\alpha dx^\beta, \tag{2.13}$$

[2)]Everywhere below the numerical indices 1, 2, 3, will have just this meaning (and not the meaning of the numbers of the coordinates x^α).

*$(\mathbf{L}[\mathbf{MN}]) = \mathbf{L}\cdot(\mathbf{M}\times\mathbf{N})$.

where

$$\mathbf{l}_K = L_a \mathbf{e}^a, \ \mathbf{m}_K = M_a \mathbf{e}^a, \ \mathbf{n}_K = N_a \mathbf{e}^a. \tag{2.14}$$

We see from this that the laws (2.7) of temporal variation of the spatial scales pertain to the directions defined by the vectors (2.14). We shall call these directions the Kasner axes.

We emphasize that the transformation (2.14) from the vectors **l**, **m**, **n**, to $\mathbf{l}_K$, $\mathbf{m}_K$, $\mathbf{n}_K$ need by no means be orthogonal and therefore the vectors (2.14) cannot be chosen as basis vectors (under the condition of conservation of the values of the structural constants (2.3).[3]

The vectors (2.14) are also not the principal axes of the symmetric matrix γ_{ab}, and the quantities a^2, b^2, and c^2 are not its principal values. The latter are, generally speaking, expressible in terms of all the three functions a^2, b^2, and c^2. During a Kasner epoch the principal values vary and pass through a maximum or a minimum while the principal axes rotate with respect to the stationary reference frame **l**, **m**, **n**.[4]

Substituting the expressions (2.12) and (2.9) into Eqs. (2.5), we obtain three relations which may be written together in vector form:

$$\frac{1}{V}\{p_l[\mathbf{L}[\mathbf{MN}]] + p_m[\mathbf{M}[\mathbf{NL}]] + p_n[\mathbf{N}[\mathbf{LM}]]\} = \frac{\mathbf{C}}{\Lambda} \tag{2.15}$$

or

$$\frac{1}{V}\{\mathbf{N}(\mathbf{ML})(p_m - p_l) + \mathbf{M}(\mathbf{LN})(p_l - p_n) + \mathbf{L}(\mathbf{MN})(p_n - p_m)\} = \frac{\mathbf{C}}{\Lambda}, \tag{2.16}$$

where $\mathbf{C} = (C_1, C_2, C_3)$. Of the six quantities defining the directions of the Kasner axes (the six independent components of **L**, **M**, **N**), three are chosen arbitrarily after which the other three are found from (2.16).

Using this arbitrariness in the choice of the basis vectors, we choose **l** to coincide with $\mathbf{l}_K$, and the "plane" **l**, **m** to coincide with the plane $\mathbf{l}_K$, $\mathbf{m}_K$. This means that $L_2 = L_3 = 0$, $M_3 = 0$; let us designate the remaining components as

$$\begin{gathered}\mathbf{L} = (1, 0, 0), \quad \mathbf{M} = (\cos\theta_m, \sin\theta_m, 0), \\ \mathbf{N} = (\cos\theta_n, \ \sin\theta_n\cos\varphi_n, \ \sin\theta_n\sin\varphi_n).\end{gathered} \tag{2.17}$$

We then find from the three components of Eq. (2.16)

$$\begin{gathered}\operatorname{tg}\theta_m = \Lambda(p_m - p_l)/C_3, \quad \operatorname{tg}\varphi_n = \Lambda(p_n - p_m)/C_1, \\ (p_l - p_n)\operatorname{ctg}\theta_n = (C_2\sin\varphi_n - C_3\cos\varphi_n)/\Lambda.\end{gathered} \tag{2.18}$$

3. ROTATIONS OF THE KASNER AXES

In the diagonal case the Kasner axes are rigidly fixed to the reference vectors and do not change when the Kasner epochs are changed. In the nondiagonal case, however, the Kasner axes are not fixed beforehand and their directions change when one epoch is replaced by another. This change may be found with the aid of Eq. (2.16), bearing in mind that the constant **C** is an exact solution of the Einstein equations and remains the same (at a fixed reference frame) for alternating epochs.

Let us show first that the variation law for the Kasner exponents during the alternation of the epochs remains the same as in the diagonal case. This is easily verified by choosing the reference frame in the initial Kasner epoch by the method mentioned at the end of Sec. 2—in accordance with (2.17).

Let us consider an alternation of the epochs, in which γ_{11} goes through a maximum. This means that close to the transition a^2 is large compared with b^2 and c^2; we shall assume that $b^2 \gg c^2$ at the same time, i.e.,

$$a^2 \gg b^2 \gg c^2. \tag{3.1}$$

Then

$$\gamma_{11} \approx a^2, \quad \gamma_{22} \approx M_2^2 b^2, \quad \gamma_{33} \approx N_3^2 c^2 \tag{3.2}$$

while all the off-diagonal components are small in the sense that

$$\gamma_{12}^2 \ll \gamma_{11}\gamma_{22}, \quad \gamma_{13}^2 \ll \gamma_{11}\gamma_{33}, \quad \gamma_{23}^2 \ll \gamma_{22}\gamma_{33}. \tag{3.3}$$

These inequalities follow from (3.2) provided the ratios M_1/M_2, N_1/N_3, N_2/N_3 are not too large. Thus, $\gamma_{12} \approx \gamma_{22} M_1/M_2 = \gamma_{22}/\tan\theta_m$ and for (3.3) to be fulfilled we should have $\gamma_{11}/\gamma_{22} \gg (M_1/M_2)^2$. On account of (2.8) this requirement imposes on the constant C_3 the condition:

$$C_3/\Lambda \ll a/b, \tag{3.4}$$

where the right-hand side is the ratio of the values of a and b near the transition point. Analogous conditions are imposed on C_1 and C_2.

It is evident from continuity considerations in matching the solutions on both sides of the transition that the direction $\mathbf{l}_K$, which is connected with the largest (at the moment of replacement of one epoch by another) value of a^2, does not change: in the new Kasner epoch we have $\mathbf{l}'_K = \mathbf{l}_K$. For the same reason the direction $\mathbf{m}'_K$ in the new epoch, connected with the quantity $b^2 \gg c^2$, remains in the same plane ($\mathbf{l}_K$, $\mathbf{m}_K$). In other words, the chosen reference frame retains its properties in the new epoch, and with them are conserved the forms of (3.2) and the principal terms in γ_{ab}. Under the conditions (3.3) the off-diagonal components of γ_{ab} drop out completely from the Einstein equations and we return to the same situation (for the functions a, M_2b, N_3c), as obtained in the diagonal case for the functions a, b, c. The determinant is $\Gamma \approx \gamma_{11}\gamma_{22}\gamma_{33} \approx (abcM_2N_3)^2$, in the same way as in the diagonal case $\Gamma = (abc)^2$. It is clear therefore that the substitution rule for the exponents I (3.14), as well as the variation rule I (3.15) for the constant Λ, remains as before.

For two consecutive Kasner epochs pertaining to the same era (a succession of epochs with flipping of the negative exponent between a given pair of functions—in this case a and b), we thus have

$$\begin{gathered}p_l = p_1(u), \ p_m = p_2(u), \ p_n = p_3(u), \ p_l' = p_2(u-1), \\ p_m' = p_1(u-1), \ p_n' = p_3(u-1), \ \Lambda'/\Lambda = 1 + 2p_1(u),\end{gathered} \tag{3.5}$$

[3] With the vectors (2.14) as reference vectors the matrix γ_{ab} would be diagonal but the structural constants would be equal to $C^1_{12} = (\mathbf{N}\cdot\mathbf{L})/V$, $C^1_{13} = (\mathbf{M}\cdot\mathbf{L})/V$, $C^1_{32} = L^2/V$, etc.

[4] It is precisely this variation of the principal values and rotation of the principal axes that were observed by M. Rayan [2] and described by him as a reflection on a "centrifugal barrier" (introduced by Rayan in the spirit of the Hamiltonian method developed by Misner). It is clear from the foregoing that these phenomena are the algebraic consequence (the properties of the roots of the secular equation of a quadratic form) of an inadequate choice of the axes—rotating principal axes instead of stationary Kasner ones.

where the prime indicates quantities pertaining to the new epoch. Applying to these two epochs the equality (2.18), we obtain as a result the following relations for the changes in the relative orientation of the Kasner axes during the change of epochs:

$$\frac{\operatorname{tg}\theta_m'}{\operatorname{tg}\theta_m}=-\frac{2u-1}{2u+1},\quad \varphi_n'=\varphi_n,\quad \frac{\operatorname{tg}\theta_n'}{\operatorname{tg}\theta_n}=\frac{u-2}{u+2}. \tag{3.6}$$

We see that $|\theta_m'/\theta_m|<1$ and $|\theta_n'/\theta_n|<1$. This means that with each change of epochs the Kasner axes approach each other. It is easy to obtain similar formulas for a change of epochs in passing to the next era, i.e., to a series of oscillations of another pair of functions; these formulas reveal the same effect of approach to each other of the axes.[5)]

In the asymptotic limit of any arbitrary proximity to a singularity, the amplitude of the oscillations grows extremely rapidly; [3, 4] consequently the conditions (3.4) clearly can be fulfilled (although, according to the rules (3.5), the quantity Λ decreases somewhat from epoch to epoch). Also, the difference between the oscillating and the monotonically decreasing functions increases very rapidly during each era; hence the condition (3.1) will likewise be met. This condition could be violated on going from one era to the next if the change in the functions a, b, c, resulted in an accidental proximity of a and b at that moment; the probability of appearance of such "dangerous" cases tends, however, to zero asymptotically—by the same reasons as were expounded on a similar occasion in [3], Sec. 4. Thus, the asymptotic behavior of the general homogeneous model being considered will have the same properties as in the diagonal case, but to these properties is added now a new feature-a gradual approach of the Kasner axes to each other.

By themselves the formulas (3.5) do not still answer the question as to whether the common direction of the axes that are approaching each other tends to a definite limit. Such a limit, apparently, is the direction **C**, although the approach to it is not of a regular character. An indication to this may be discerned in the growth of the right hand side of Eq. (2.15) as a result of the systematic decrease of the quantity Λ; the same behavior, on the average, will be displayed by the left hand side of the equation in an irregular approach of **L, M, N** to the common direction **C**.

INTRODUCTION OF MATTER

It has already been shown in [5] that in the most general case of a generalized (inhomogeneous) Kasner metric, we may, near a singularity, neglect in the ab- and 00-components of the Einstein equations the terms that depend on matter. Then the laws of variation of the density of matter ϵ and its 4-velocity (u_0, u_a) with time are given (for the ultrarelativistic equation of state) by the formulas

$$\varepsilon=\varepsilon^{(0)}/a^2b^2,\quad u_a=u_a^{(0)}\sqrt{ab},\quad u_0=u_b^{(0)}N^b\sqrt{ab}/c, \tag{4.1}$$

where $\epsilon^{(0)}$ and u_a^0 are constants, while c is the least of the functions a, b, and c (cf. I, Sec. 2). As to the equations

$$R_a^0=T_a^0, \tag{4.2}$$

the presence of matter led only to a change in the relations imposed on the functions of the coordinates entering into the solution. A similar role is played by Eqs. (4.2) for the homogeneous model being considered here.

For a homogeneous space of the type IX the components of the four-dimensional Ricci tensor R_a^0 are given by the formula

$$R_a^0={}^1/_2\,\varkappa_b^{\ c}C_{ac}^b \tag{4.3}$$

(cf. I (C.18)). Comparing this expression with (2.5), we reduce Eqs. (4.2) to the form

$$C_a/\sqrt{\Gamma}=-4\varepsilon u_a u^0/3$$

and after substituting in this (2.8) and (4.1);

$$C_a=-{}^4/_3\,V\varepsilon^{(0)}u_a^{(0)}(u_b^{(0)}N^b). \tag{4.4}$$

We again see that Eqs. (4.2) establish only a relation between the constants in functions, whose form, however, is found without their aid.

In contrast to the diagonal case, a preferred direction exists in the general homogeneous model. This direction is expressed by the "vector" constant **C**, and from (4.4) it can be seen that it is connected with the motion of the matter.

5. THE CASE OF SMALL OSCILLATIONS

This section is devoted to the generalization to the nondiagonal case of the solution described in I, Sec. 4, and corresponding to a long era with small oscillations of the spatial scales in two directions as the scale in the third direction monotonically decreases.

In its turn the solution discussed below is a particular case of the general inhomogeneous solution previously found in [6], and the entire course of calculations follows closely the calculations in [6].

Let the monotonical decrease of the spatial scales take place in the direction e^3. Then γ_{33} is small compared with γ_{11}, γ_{22}, and γ_{12}, and γ_{11} and γ_{22} are close to each other. As is confirmed by the result, the nondiagonal components γ_{13}, $\gamma_{23}\sim\gamma_{33}$, so that besides the inequalities

$$\gamma_{33}\ll\gamma_{ab},\qquad |\gamma_{11}-\gamma_{22}|\ll\gamma_{11} \tag{5.1}$$

we have also the inequalities

$$\gamma_{a3}^2\ll\gamma_{aa}\gamma_{33} \tag{5.2}$$

(in this section the indices a, b assume only two values: 1, 2).

The inequalities (5.2) permit us in the first approximation to set $\gamma_{a3}=0$ everywhere. Then the determinant is

$$\Gamma=\gamma_{33}\Delta,\quad \Delta=\gamma_{11}\gamma_{22}-\gamma_{12}^2, \tag{5.3}$$

and the components of the inverse matrix are

$$\gamma^{11}=\gamma_{22}/\Delta,\quad \gamma^{22}=\gamma_{11}/\Delta,\quad \gamma^{12}=-\gamma_{12}/\Delta. \tag{5.4}$$

Discarding in the expressions (A.7) for P_a^b also γ_{33} in comparison with γ_{ab}, we obtain the Einstein equations

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5)To do that it is necessary to consider two cases: it may turn out that when the functions a, b and c change during the transition from the last epoch of one era to the first epoch of the next, $a^2\gg b^2\gg c^2$ or $a^2\gg c^2\gg b^2$; in the first case the axis m_k should be connected with b, while in the second—with c.

$R_1^1 + R_2^2 = 0$ in the form

$$[(\varkappa_1^1+\varkappa_2^2)\sqrt{\Gamma}]^{\cdot} = [\dot{\Delta}\sqrt{\gamma_{33}/\Delta}]^{\cdot} = 0.$$

If we introduce in place of t the variable ξ in accordance with

$$dt = {}^1/_2\sqrt{\gamma_{33}}d\xi, \tag{5.5}$$

then this equation simply yields

$$\Delta = f^2\xi^2 \tag{5.6}$$

where f is a constant.

It is convenient to write the totality of the components γ_{ab} in the form of a matrix $\hat{\gamma}$, which we represent (with allowance for (5.6)) in the form

$$\hat{\gamma} = f\xi e^{\hat{\chi}}, \tag{5.7}$$

where $\hat{\chi}$ is a symmetric matrix of zero trace. To the smallness of the oscillations corresponds the smallness of $\hat{\chi}$ for large values of ξ, so that

$$\hat{\gamma} \approx f\xi(1+\hat{\chi}+{}^1/_2\hat{\chi}^2). \tag{5.8}$$

Substitution of (5.7) into the equations $R_a^b = 0$ yields

$$\hat{\chi}''+\hat{\chi}'/\xi+\hat{\chi}=0 \tag{5.9}$$

(the prime denotes differentiation with respect to ξ). Here we have neglected only terms of the third or higher orders in $\hat{\chi}$. It follows from this that

$$\hat{\chi} = \frac{2}{\sqrt{\xi}}\begin{pmatrix} A\sin(\xi-\xi_0) & B\cos(\xi-\xi_0) \\ B\cos(\xi-\xi_0) & -A\sin(\xi-\xi_0)\end{pmatrix} + O(\xi^{-3/2}), \tag{5.10}$$

where A, B, and ξ_0 are arbitrary constants. The phases of the sine and cosine functions in (5.10) are assumed to be the same. This can always be achieved, owing to the arbitrariness which still remains in the choice of the basis vectors $\mathbf{e}^1$ and $\mathbf{e}^2$: these vectors may be subjected to an orthogonal transformation without violating the conditions (5.1–2) and without changing the selected values of the structural constants (2.3). The smallness of $\hat{\chi}$ is guaranteed by the factor $\xi^{-1/2}$, in which ξ is assumed large.

Thus, in the first approximation, we have for the matrix γ_{ab}:

$$\left.\begin{matrix}\gamma_{11}\\ \gamma_{22}\end{matrix}\right\} = f\xi\left[1\pm\frac{2A}{\sqrt{\xi}}\sin(\xi-\xi_0)\right], \quad \gamma_{12} = 2fB\sqrt{\xi}\cos(\xi-\xi_0). \tag{5.11}$$

When B = 0 we return to the previous result I (4.9) (with the constant $f = a_0^2/\xi_0$).

For the calculation of the function $\gamma_{33}(t)$ we use the equation

$$R_0^0 - R_3^3 = T_0^0 - T_3^3 \tag{5.12}$$

The case of small oscillations corresponds to the case when the Kasner exponents (p_1, p_2, p_3) are close to (0, 0, 1); under these conditions the components T_0^0, T_3^3 of the energy-momentum tensor of the matter turn out to be of the same order in 1/t as the left hand side of Eq. (5.12), as can in fact be seen from (4.1) (whereas the components T_a^b are, as before, small and may be discarded in the Einstein equations).

From the identity $u_0u^0 - u_au^a - u_3u^3 = 1$, we have in the first approximation

$$u_0^2 = u_3^2/\gamma_{33}.$$

The equation $R_3^0 - T_3^0$, on the other hand, yields

$$|\varepsilon u_0 u_3\sqrt{\gamma_{33}}\xi| = |6AB|$$

(the quadratic terms in (5.8) should be taken into account in the calculations here). Determining from this ϵu_0^2, we find

$$T_0^0 \approx T_3^3 = 8|AB|/\xi\gamma_{33}.$$

A simple calculation now reduces Eq. (5.12) to the form

$$(\ln\gamma_{33})' = 2(|A|+|B|)^2,$$

from which

$$\gamma_{33} = \mathrm{const}\cdot\exp[-2(|A|+|B|)^2(\xi_0-\xi)]. \tag{5.13}$$

Finally, using the definition (5.5) we find for the connection between ξ and the time t:

$$t = \mathrm{const}\cdot\exp[-(|A|+|B|)^2(\xi_0-\xi)]. \tag{5.14}$$

The verification of the assumption (5.2) about the smallness of γ_{a3} may now be carried out by considering the equations $R_a^0 = T_a^0$. These equations are themselves solved under the assumption (5.2), after which we find that the result confirms the assumption (cf. a similar analysis in [6]).

6. CONCLUDING REMARKS

Thus, the broadening of the class of homogeneous models leads to the appearance of a new characteristic phenomenon—the rotation of the Kasner axes in alternating epochs. At the same time, the general character of the oscillatory mode and the rule for the alternation of the Kasner epochs remain unchanged.

The homogeneous models with rotation of axes require the presence of matter; for empty space only homogeneous models with fixed axes are possible. It seems to us, however, that this circumstance is just connected with the homogeneity and is not of a fundamental nature from the point of view of the construction of a general inhomogeneous solution to the Einstein equations. It may be thought that the features which manifest themselves in homogeneous models in the presence of matter, are also characteristic of inhomogeneous models with matter as well as without. The role played in the Einstein equations by the terms of the energy-momentum tensor for matter, may be imitated by terms connected with the inhomogeneity of the space metric. The presence of matter is felt only in a change in the coupling between the arbitrary functions of the spatial coordinates appearing in the solution. We recall, in order to avoid any misunderstanding, that in speaking here about solutions to the Einstein equations, we have in mind their limiting form near a singularity.

In confirmation of the expressed point of view we may recall that such is precisely the state of affairs in the generalized (non-oscillatory) Kasner solution.[5] The same situation obtains for the general solution describing in the oscillatory mode a long era with small oscillations and thus generalizing the analogous solution for the homogeneous model considered here (Sec. 5).

APPENDIX

COMPUTATION OF THE RICCI TENSOR FOR HOMOGENEOUS SPACES OF THE BIANCHI VIII AND IX TYPES

For any homogeneous space the components of the three-dimensional Ricci tensor P_{ab} are expressed in terms of the structural constants of the motion group by the formulas I (C.17). For spaces of the VIII and IX types these formulas may, however, be reduced to a more convenient form.

Taking into consideration the antisymmetry of the structural constants with respect to their lower indices, we introduce the dual quantities C^{ab} in accordance with

$$C_{ab}^{c}=e_{abd}C^{dc}, \tag{A.1}$$

where e_{abd} is the unit antisymmetric symbol; according to I (C.15) these quantities are expressible in terms of the basis vectors by the formula

$$C^{ab}=-(\mathbf{e}^{a}\operatorname{rot}\mathbf{e}^{b})\;/\;(\mathbf{e}^{1}[\mathbf{e}^{2}\mathbf{e}^{3}]). \tag{A.2}$$

The metrics of type VIII and IX spaces are characteristic in that the determinant of the matrix C^{ab} is different from zero. The substitution of (A.1) into the Jacobi identity then yields

$$C^{ab}=C^{ba}. \tag{A.3}$$

Substituting now (A.1) into the formula I (C.17), we obtain

$$P_{a}^{b}=\frac{1}{2\Gamma}\{2C^{bd}C^{cf}(2\gamma_{dc}\gamma_{fa}-\gamma_{da}\gamma_{cf})+\delta_{a}^{b}[(C^{cd}\gamma_{cd})^{2}-2C^{df}C^{cg}\gamma_{fc}\gamma_{dg}]\}. \tag{A.4}$$

We draw attention to the fact that the components of only the matrix γ_{ab} itself, and not of its inverse γ^{ab}, enter into this expression.

Assuming that the matrix C^{ab} has been brought to its diagonal form, we denote

$$C^{11}=-\lambda,\quad C^{22}=-\mu,\quad C^{33}=-\nu. \tag{A.5}$$

The corresponding structural constants are:

$$C_{32}^{1}=\lambda,\quad C_{13}^{2}=\mu,\quad C_{21}^{3}=\nu. \tag{A.6}$$

For a type-IX space: $\lambda=\mu=\nu=1$ (in accord with the choice (2.3)), while for a space of the VIII type: $\lambda=-1$, $\mu=\nu=1$. With this choice of constants the final expressions for the components of the Ricci tensor are:

$$\begin{gathered}P_{1}^{1}=[\lambda^{2}\gamma_{11}^{2}-(\mu\gamma_{22}-\nu\gamma_{33})^{2}-4\mu\nu\gamma_{23}^{2}]/2\Gamma\\ P_{1}^{2}=\frac{\mu}{\lambda}P_{2}^{1}=[\mu\gamma_{12}(\lambda\gamma_{11}+\mu\gamma_{22}+\nu\gamma_{33})+2\mu\nu\gamma_{13}\gamma_{23}]/\Gamma\end{gathered} \tag{A.7}$$

the remaining components are obtained from here by a cyclic permutation of the indices 1, 2, 3 and the letters λ, μ, ν.

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