UNIQUENESS OF THE FIRSOV INVERSION METHOD AND FOCUSING POTENTIALS

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Submitted September 16, 1970

Zh. Eksp. Teor. Fiz. 60, 1604–1610 (May, 1971)

By using the Firsov inversion formula it is possible to construct spherically symmetric systems which have the properties of ideal focusing from the point of view of geometrical optics. The equivalent mechanical problem leads to fields of force focussing particles of a given energy E. In particular, the cut-off potential of the three-dimensional harmonic oscillator focusses at the edge of the well a parallel beam of particles of energy equal to the depth of the well. The cut-off potential for the Coulomb field possesses the property of reflecting particles of energy equal to one half the cut-off energy. In those cases when the trajectory varies discontinuously as the impact parameter is varied the problem of reconstructing the potential is nonunique, and by specifying in an almost arbitrary manner the index of refraction n(r) in one region we can satisfy the focusing condition by an appropriate choice of n(r) in another region. This arbitrariness can be utilized to eliminate chromatic aberration, to select the region of variation of n, etc. The advantage of the systems considered is their wide-angle characteristic.

1. THE INVERSION FORMULA

THE Firsov inversion formula[1,2] for the problem in classical mechanics of the scattering of a particle by a spherically symmetric field of force enables one to reconstruct the potential U(r) if we know the angle of deflection of the particle χ as a function of the impact parameter ρ for a given value of the total energy of the particle E.[1)] Instead of the potential energy U Firsov's formula contains the quantity

$$n(r)=\sqrt{1-\frac{U(r)}{E}}=\frac{v(r)}{v(\infty)} \tag{1}$$

(v(r) is the velocity of the particle at a given value of r) which has a particularly simple meaning if, using the opticomechanical analogy we go over to the equivalent problem in geometrical optics. Then n(r) will be the index of refraction at a given r, and from the condition $U(\infty) = 0$ it follows that $n(\infty) = 1$. From the formula for χ

$$\frac{\pi+\chi}{2}=\int\limits_{r_0}^{\infty}\frac{\rho dr}{r([rn(r)]^2-\rho^2)^{1/2}} \tag{2}$$

where r_0 is the greatest of the roots of the equation

$$rn(r)=\rho, \tag{3}$$

in [1] the following inversion formula was obtained

$$n(r)=\exp\left\{\frac{1}{\pi}\int\limits_{rn(r)}^{\infty}\frac{\chi(\rho)\,d\rho}{(\rho^2-[rn(r)]^2)^{1/2}}\right\}, \tag{4}$$

which defines the dependence of n(r) on r implicitly.[2)]

Formula (4) is derived on the assumption that the minimum value $r = r_0$ for the trajectory decreases continuously as ρ is decreased. This is satisfied only when rn(r) is a monotonically increasing function of r. In this case in going from a more distant trajectory to a closer one with impact parameters respectively given by ρ and $\rho - \Delta\rho$, we incorporate a new region $\Delta r = r_0(\rho) - r_0(\rho - \Delta\rho)$, which must tend to zero together with $\Delta\rho$. It is just in the case of such a "gradual probing" of the potential U(r) from the periphery towards the center that a unique reconstruction becomes possible and formula (4) is applicable.[3)]

Another natural condition consists of requiring that n(r) should be a single-valued function of r. For this it is necessary that the function

$$r(t)=t\exp\left\{-\frac{1}{\pi}\int\limits_{t}^{\infty}\frac{\chi(\rho)\,d\rho}{\sqrt{\rho^2-t^2}}\right\} \tag{5}$$

should not decrease with increasing t.[1] It can be easily verified that this condition (which limits the allowable functions $\chi(\rho)$) is equivalent to the condition of "gradual probing" formulated above.

We note, moreover, that for no n(r) can the function $\chi(\rho)$ decrease discontinuously and cannot be less than $-\pi$.

If in formula (4) we set r = 0, then we obtain a simple explicit formula for the value of the index of refraction at the origin

[1)]This formula was rederived in [3] (cf., [4] where there is also no reference to [1]). In Luneburg's book [5] containing lectures given in 1944 but not published at that time a similar inversion algorithm has also been described.

[2)]We note that in the book [2] fromula (4) is given with an incorrect sign in the exponent, since the definition of the angle χ there has the opposite sign. Our defintion ($-\pi < \chi < +\infty$; $\chi > 0$ in the case of attraction and $\chi < 0$ in the case of repulsion) is more convenient for attractive fields.

[3)]In [4] it is erroneously asserted that a monotonic dependence of $\chi(\rho)$ is necessary. We shall see later than n(r) can be easily reconstructed in accordance with formula (2) for nonmonotonic $\chi(\rho)$.

$$\sqrt{1-\frac{U(0)}{E}}=n(0)=\exp\left\{\frac{1}{\pi}\int_0^\infty \chi(\rho)\frac{d\rho}{\rho}\right\}, \tag{6}$$

which is applicable, of course, only when the trajectory with $\rho = 0$ passes through the origin, i.e., when $\chi(0) = 0$ and $U(r) < E$ for all values of r.

2. FOCUSING SYSTEMS WITH SPHERICAL SYMMETRY

We use the inversion formula in order to construct a spherical lens of radius R with an index of refraction n(r) (for $r > R$ we have $n(r) = 1$), which focuses the rays coming from the point A at a distance R_1 from the center of the lens to the point F at a distance R_2 from the center (Fig. 1). Such an optical system will image a sphere of radius R_1 on a sphere of radius R_2 and conversely and will be free from different aberrations characteristic of axially symmetric systems and associated with oblique rays (there remain only the chromatic aberration and the diffraction brought about by deviations from ray optics). From Fig. 1 we immediately obtain the formula for χ:

$$\chi(\rho)=\begin{cases}\alpha+\beta=\arcsin(\rho/R_1)+\arcsin(\rho/R_2), & \rho<R\\ 0 & \rho>R.\end{cases} \tag{7}$$

Substituting this relationship into (4) we can for given values of R, R_1, R_2 find an index of refraction n(r) which increases monotonically from a value equal to unity for r = R up to the value

$$n(0)=\exp\left\{\frac{1}{\pi}\int_0^{R/R_1}\arcsin x\frac{dx}{x}+\frac{1}{\pi}\int_0^{R/R_2}\arcsin x\frac{dx}{x}\right\} \tag{8}$$

for r = 0.

In the general case the integrals occurring above cannot be evaluated analytically but still they can all be expressed in terms of one universal function (cf., also [5,6]); some numerical results are given in [7].

We consider particular cases when the integral can be expressed in terms of elementary functions and the index of refraction n(r) can be obtained explicitly. Let $R_1 = \infty$, $R_2 = R$, i.e., the lens focuses a parallel beam of rays on its own surface (Fig. 2). Then utilizing the formula

$$\int_a^1 \frac{\arcsin x}{\sqrt{x^2-a^2}}dx=\frac{\pi}{2}\ln(1+\sqrt{1-a^2}), \tag{9}$$

we obtain the index of refraction considered by Luneburg:[5]

$$n(r)=\sqrt{2-(r/R)^2}, \tag{10}$$

falling off smoothly from the value $n = \sqrt{2}$ at the center of the lens to n = 1 for r = R. For the potential energy U(r) we obtain

$$U(r)=E(r^2/R^2-1),\ \ r<R;\ \ U(r)=0,\ \ r>R. \tag{11}$$

Thus, the spherically symmetric force field of a harmonic oscillator cut off at r = R focuses a plain parallel beam of particles if their energy is equal to the depth of the potential well. For charged particles such a field can be created by a uniformly charged sphere which is surrounded by a grounded conducting sphere of the same radius.4)

The trajectories of a particle in an oscillator field

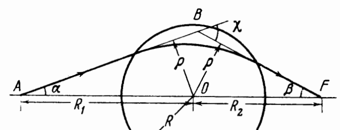

FIG. 1

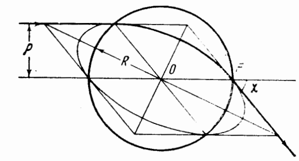

FIG. 2

FIG. 1. Focusing system with spherical symmetry.

FIG. 2. Focusing properties of a cut-off spherically symmetric harmonic oscillator potential.

will be, as is well known, ellipses. From Fig. 2 it can be seen that the semi-axes of the ellipse are equal to $R\sqrt{2}\cos(\chi/2)$ and $R\sqrt{2}\sin(\chi/2)$. Thus, the sum of the squares of the semi axes does not depend on χ and, consequently, all the ellipses correspond to motion with the same total energy E, i.e., the focusing property is easily proven also by a purely geometrical method.

Another simple case can be obtained if we set $R_1 = R_2 = R$, i.e., if we require that all the rays emerging from a certain point on the surface of the sphere should be brought together at the diametrically opposite point. Then $\chi = 2\arcsin(\rho/R)$ for $\rho < R$; $\chi = 0$, $\rho > R$ and we obtain

$$n=2/[1+(r/R)^2]. \tag{12}$$

This distribution of the index of refraction was first found by Maxwell in his problem of the "fish eye."[9]

The inversion problem can also be solved exactly if we require that scattering should occur at a constant angle $a\pi$, i.e., if we set

$$\chi=a\pi,\quad \rho<R;\quad \chi=0,\quad \rho>R. \tag{13}$$

Then we obtain the following dependence of r on n:

$$\frac{r}{R}=\frac{2}{n}(n^{1/a}+n^{-1/a})^{-1}=[n\,\mathrm{ch}(a^{-1}\ln n)]^{-1}. \tag{14}$$

In particular, for a = 1 we have a "cataphot"—a field which scatters all the particles (rays) backwards ($\chi = \pi$) for $\rho < R$:

$$n=\sqrt{2R/r-1}. \tag{15}$$

The corresponding potential energy for the particles is equal to

$$U(r)=2E(1-R/r),\ \ r<R;\qquad U(r)=0,\ \ r>R. \tag{16}$$

Thus, the cut-off Coulomb field possesses the property of reflecting particles incident on it with an energy equal to half the cut-off energy. For charged particles such a field can be created by a point charge surrounded by a grounded sphere of radius R. From Fig. 3 it can be seen that the major semiaxis of an ellipse with a focus at the center of the sphere corresponding to the trajectory with $\chi = \pi$, is equal to R and does not depend on ρ. From this it follows that all these trajectories correspond to one and the same energy which is equal to the energy of mo-

4)In the scattering of nucleons by nuclei the potential of a nucleus in the optical model can in a certain approximation be regarded as focusing, and this leads to the appearance of a maximum in the wave function at the edge of the nucleus (cf., for example, [8]).

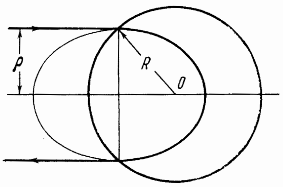

FIG. 3. Reflecting property of cut-off Coulomb potential.

tion along a circle of radius R. In virtue of this the focusing properties of this potential are also simply established geometrically.

Still more general will be the case when

$$\chi = a\pi + 2b \arcsin(\rho / R), \quad \rho < R; \quad \chi = 0, \quad \rho > R. \tag{17}$$

Substituting (17) into (4) and utilizing (9) after a simple transformation we obtain an implicit formula for n(r)

$$\frac{R}{rn} = \operatorname{ch}\left(\frac{b-1}{a+b}\ln n + \frac{b}{a+b}\ln\frac{r}{R}\right) \tag{18}$$

which contains all the special cases considered previously.

3. THE NONUNIQUE NATURE OF THE SOLUTION OF THE INVERSION PROBLEM

If the condition of increasing rn(r) is not satisfied, then the distance of closest approach $r_0(\rho)$ changes discontinuously at a certain $\rho = \tilde{\rho}$. We consider here the simplest case when there exists only one such discontinuity, so that

$$R' \equiv r_0(\tilde{\rho} - 0), \quad R'' \equiv r_0(\tilde{\rho} + 0), \quad R'' - R' > 0. \tag{19}$$

As a result of this the whole range of variation of r is divided into three parts: I ($r > R''$), II ($R'' > r > R'$), III ($R' > r > 0$) (cf., Fig. 4). This leads in turn to a discontinuous increase in $\chi(\rho)$ at the point $\tilde{\rho}$, provided the potential or its derivative have a discontinuity at the point R'', in which the function rn(r) attains a minimum.[5)]

In region I the potential can be uniquely reconstructed with the aid of formula (4) from a given $\chi(\rho)$ for $\rho > \tilde{\rho}$. However, then the range of values of r which is encompassed by a given ray undergoes a discontinuity at $\rho = \tilde{\rho}$ and Firsov's formula is inapplicable. We specify an index of refraction in region II in an arbitrary manner requiring only that the angle through which the trajectory is deflected in region II for $\rho = \tilde{\rho} - 0$ should be less than a given value of the discontinuous change in the angle of deflection $\Delta\chi$ for $\rho = \tilde{\rho}$, i.e.,

$$\Delta\chi = \chi(\tilde{\rho} - 0) - \chi(\tilde{\rho} + 0) \geqslant 2\tilde{\rho}\int_{R'}^{R''} \frac{dr}{r([rn(r)]^2 - \tilde{\rho}^2)^{1/2}}. \tag{20}$$

Evidently, the value of R' itself, i.e., the dimensions of region II, is determined simultaneously with the choice of the function n(r) in this region. If we now suppose that with a further decrease in ρ (i.e., for $\rho < \tilde{\rho}$) the rays encompass region III, then we can again use the Firsov algorithm and choose n(r) in region III in such a manner as to compensate for everything that has happened to the ray in region II (where n(r) was specified almost arbitrarily), and to guarantee a given angle of deflection χ for $\rho < \tilde{\rho}$. In order for this to occur the angle of deflection χ of the ray in region III must be equal to

$$\tilde{\chi} = \chi(\rho) - 2\int_{R'}^{\infty}\left\{\frac{1}{[(rn(r))^2 - \rho^2]^{1/2}} - \frac{1}{[(rn_1)^2 - \rho^2]^{1/2}}\right\}\frac{\rho}{r}\,dr, \tag{21}$$

$$n_1 \equiv n(R').$$

We introduce a new index of refraction $\tilde{n}$ which coincides with the desired n(r) for $r < R'$, while for $r > R'$ it is equal to n_1. If we now set $\chi(\rho) = 0$ for $\rho > \tilde{\rho}$, we can then utilize Firsov's formula,[6)] in order to obtain $\tilde{n}$ (and, consequently, also n for $r < R'$) in terms of χ. We obtain

$$n(r) = n_1 \exp\left\{\frac{1}{\pi}\int_{rn}^{\infty}\frac{\tilde{\chi}(\rho)\,d\rho}{[\rho^2 - (rn)^2]^{1/2}}\right\}, \quad r < R'. \tag{22}$$

Thus, if $\chi(\rho)$ has a discontinuity (or becomes infinite) the inverse problem does not have a unique solution. This arises because we did not in any way characterize the trajectories corresponding to total internal reflection of the ray and lying entirely within region II. The solution is determined uniquely by specifying the function n(r) satisfying only the condition (20) in region II, with at the same time also R being specified, i.e., the magnitude of the region II which can be arbitrary. If $R' = R''$, then region II disappears, (22) goes over into (4), n(r) is reconstructed uniquely by a simple application of the Firsov algorithm and the function rn(r) is continuous and everywhere increasing. In this case the discontinuity in $\chi(\rho)$ at $\rho = \tilde{\rho}$ occurs as a result of a temporary capture of the particle into a circular trajectory with $r = R''$, with U(r) and n(r) having a discontinuous derivative at $r = R''$.

In those examples of focusing fields which were considered in Sec. 2, we have just such a situation, since the angle $\chi(\rho)$ determined by formula (7) has a discontinuity at $\rho = R$. From the results of this section it follows that there exists a whole family of potentials with given focusing properties which have a discontinuity in the index of refraction at $r = R$. It is this particular behavior of n(r) which is the only possible one for real optical systems at the boundary between the lens and vacuum.

We restrict ourselves to the simplest example already discussed in Sec. 2, $R_1 = \infty$, $R_2 = R$ and we make the simplest choice of n(r) in region II: n(r) = const

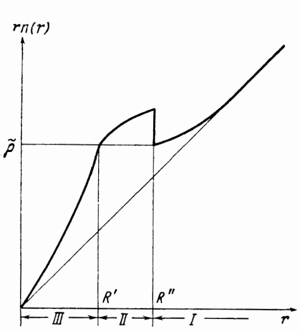

FIG. 4. Behavior of the function rn(r) in the simplest case of a non-unique solution of the inverse problem.

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5)If the function rn(r) is continuous at the point R″ together with its derivative, then $\chi(\rho)$ has at the point $\tilde{\rho}$ a logarithmic singularity associated with spiraling (cf., for example, [4]).

6)For the case when $n(\infty) \neq 1$, one should replace in formulas (2)–(4) n(r) by n(r)/n(∞).

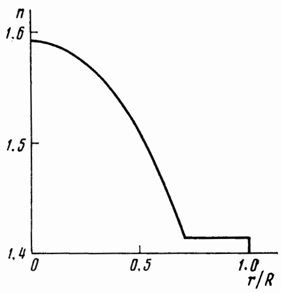

FIG. 5. The index of refraction of a lens with a homogeneous external layer which focuses a parallel beam of rays on its own surface.

$= n_1 = \sqrt{2}$. Then $R'' = R$, $R' = R/\sqrt{2}$ and the ray tangent to the surface of the lens is focused. In order to determine n(r) in region III, $R/\sqrt{2} > r > 0$, we obtain in accordance with formula (21)

$$\tilde{\chi} = 2\arcsin\frac{\rho}{R\sqrt{2}} - \arcsin\frac{\rho}{R}, \quad 0 < \rho < R. \tag{23}$$

Utilizing (22) and carrying out a numerical integration we obtain the index of refraction n(r) shown in Fig. 5. The maximum value of n(r) is attained at r = 0 and is equal to $n(0) = 2^{1/4}e^{G/\pi} \approx 1.5918$, where G is Catalan's constant (cf., [10], formula (3.513.6)). It is interesting that in this case the variation of the index of refraction $1.4142 < n < 1.5918$ is such that the lens can in principle be constructed from ordinary glass, although in practice it is apparently impossible at the present time to realize with a sufficient accuracy the variation of the index of refraction. However, it is possible to construct layered systems with spherical surfaces in which by a suitable choice of thicknesses and indices of refraction different aberrations can be eliminated.[11] Moreover, it is not even necessary to require ideal focusing for all, since the glancing rays are strongly reflected and their contribution to the f number of a lens is not great. We note that for paraxial rays a sphere with the index of refraction n = 2 has the same property as the lens described above.

Since the same focusing condition determining $\chi(\rho)$ corresponds to a large number of indices of refraction n(r) we can, in principle, utilize this nonuniqueness by striving to satisfy certain additional conditions, for example, by reducing to a minimum the chromatic aberration.

We have here considered the simplest case when $\chi(\rho)$ has only a single discontinuity, and the function rn(r) has only a single minimum. More complicated cases are, of course, also possible when there are several discontinuities, several minima in rn(r) and correspondingly several regions of arbitrary specification of n(r). In this case the problem of the reconstruction of n(r) will be solved by means of repeating the required number of times the procedure described above.

4. CONCLUSION

The principal advantage of spherically symmetric focusing systems is their wide-angle property which in the limit can be extended to 4π (if the lens is covered by a semitransparent light sensitive film and one observes distant luminous point objects—for example stars). The disadvantage is the nonflat spherical focal surface (but this is, in any case, inevitable for a sufficiently wide-angle system).

From the general theoretical point of view the problem considered here is a special case of a general (and quite complicated) inverse problem of geometrical optics—the determination of the index of refraction of a system from its focusing properties. The question of the uniqueness of this problem—of establishing what requirements it is necessary and sufficient to impose on the system in order that the condition of uniqueness be satisfied—is also a complicated one. From the foregoing it follows that this problem is sufficiently difficult even in the simplest spherically symmetric case, although here we can analyze the nature of the nonuniqueness, the method of removing it, and also the possibility of utilizing it.

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Translated by G. Volkoff
176