MAGNETIC QUANTIZATION AND ABSORPTION OF ULTRASOUND IN SUPERCONDUCTORS IN THE INTERMEDIATE STATE

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The wavefunctions and electron excitation spectrum in the normal layers in a superconductor in the intermediate state are found taking account of the coherent phase difference between the superconducting regions, which compensates the increment in the vector potential of the magnetic field on passing through the normal layers. The quantum absorption coefficient for a longitudinal sound wave propagating along the normal layers perpendicular to the magnetic field is calculated. The absorption displays characteristic oscillations due to magnetic quantization of the normal excitations. These oscillations are periodically modulated in frequency and amplitude.

THE phase relations for the complex superconducting order parameter for electrons leads to a whole series of distinctive phenomena in superconductors: magnetic flux quantization,[1] Abrikosov's singular vortex lines in the mixed state,[2] and the Josephson effect.[3] As was shown by one of the present authors,[4] the coherent phase difference, established in the presence of a magnetic field, between two superconductors separated by a layer of normal metal leads to a special magnetic quantization of the electron excitations in the normal layer, with energies less than the magnitude of the energy gap in the superconductors. These excitations, according to Andreev,[5] do not penetrate into the superconducting regions and are reflected from the boundary interface, the electron undergoing a transition to a hole and vice versa. In the absence of a magnetic field this leads to quantization of the excitations[5] trapped in the normal metal layer.

In a magnetic field the increment in the vector potential on passing through the normal layer ($\Delta A_y = HL$, L is the thickness of the layer) must be compensated in the phase of the order parameter in such a way as to ensure that the superconducting current and the magnetic field are zero in the interior of the superconducting regions. Taking account of this phase difference in the wavefunctions being reflected from the boundaries of the normal excitations changes the pattern of the quantization and leads to the result that the difference between the energy levels is equal to eH/m ($\hbar = c = 1$) even when the layer thickness is small compared with the radius of the electron orbit in the magnetic field ($eHL/p_F \ll 1$, e is the electron charge and p_F is the Fermi momentum), and Landau diamagnetic quantization cannot occur. In essence this effect is close to the well-known Aharonov-Bohm effect,[6] in which a shift is observed in the interference pattern obtained from two coherent electron beams as the magnitude of the magnetic flux penetrating the space between the beams is changed.[1)]

[1)] In [7] the Aharonov-Bohm effect is taken into account in a calculation of the scattering of electrons by Abrikosov's vortex lines.

The magnetic quantization being considered should lead to a whole series of singularities (of the de Haas–van Alphen oscillations[8] type) in the different kinetic coefficients, and this enables us to investigate experimentally both the phenomenon itself and the distinctive features of the energy spectrum of the electrons in the metal. From this point of view, the most convenient research tool is the quantum absorption of ultrasound, which penetrates easily into the interior of the metallic sample. The purpose of this paper is to calculate the absorption coefficient for ultrasound in a superconductor in the intermediate state, inasmuch as the intermediate state is the most natural realization of normal layers separated by superconducting regions.

In a previous paper,[4] the magnetic quantization was examined in a basically qualitative way in the limiting case $eHL/p_F \to 0$. To calculate the absorption of ultrasound in the intermediate state, we shall need electron wavefunctions and spectrum that are determined with greater exactness than in [4].

1. WAVEFUNCTIONS AND SPECTRUM

In the notation of [4], the system of equations for the two-component wavefunction of the "electron-hole" pair excitation has the following appearance (the x-axis is directed along the normal to the layers, the y-axis along a layer and perpendicular to the magnetic field, and the z-axis along the magnetic field **H**, which is parallel to the interfaces of the layers; we consider one normal layer ($0 < x < L$)):

$$x < 0:$$
$$\begin{pmatrix} \xi(\hat{p}_x, \hat{p}_y, \hat{p}_z), & \Delta \\ \Delta, & -\xi(\hat{p}_x, \hat{p}_y, \hat{p}_z) \end{pmatrix} \begin{pmatrix} \psi_1(x,y,z) \\ \psi_{-1}(x,y,z) \end{pmatrix} = \varepsilon \begin{pmatrix} \psi_1(x,y,z) \\ \psi_{-1}(x,y,z) \end{pmatrix},$$

$$0 < x < L:$$
$$\begin{pmatrix} \xi(\hat{p}_x, \hat{p}_y - \Phi x/L, \hat{p}_z), & 0 \\ 0, & -\xi(\hat{p}_x, \hat{p}_y + \Phi x/L, \hat{p}_z) \end{pmatrix} \begin{pmatrix} \psi_1(x,y,z) \\ \psi_{-1}(x,y,z) \end{pmatrix} = \varepsilon \begin{pmatrix} \psi_1(x,y,z) \\ \psi_{-1}(x,y,z) \end{pmatrix}, \quad (1)$$

$x > L$:

$$\begin{pmatrix} \xi(\hat{p}_x, \hat{p}_y - \Phi, \hat{p}_z), & \Delta e^{2i\Phi y} \\ \Delta e^{-2i\Phi y}, & -\xi(\hat{p}_x, \hat{p}_y + \Phi, \hat{p}_z) \end{pmatrix} \begin{pmatrix} \psi_1(x, y, z) \\ \psi_{-1}(x, y, z) \end{pmatrix}$$

$$= \varepsilon \begin{pmatrix} \psi_1(x, y, z) \\ \psi_{-1}(x, y, z) \end{pmatrix},$$

$$\Phi = eHL, \quad \xi(p) = \frac{p^2}{2m} - \mu, \quad \mu = \frac{p_F^2}{2m}, \quad \hat{p} = -i\nabla,$$

where Δ is the energy gap in the superconducting regions.

In view of the periodicity of the system (1) along the y-axis ($y \to y + \pi/\Phi$), separating out the free motion along the z-axis we can represent the wavefunction in the form of an expansion

$$\psi(x, y, z) = \sum_{\nu} \psi(x, \nu) \exp[i(p_z z + ky + 2\Phi\nu y)], \qquad (2)$$

after which, Eqs. (1) are rewritten as follows:

$x < 0$:

$$\begin{pmatrix} \xi(\hat{p}_x, k + 2\Phi\nu, p_z) - \varepsilon, & \Delta \\ \Delta, & -\xi(\hat{p}_x, k + 2\Phi\nu, p_z) - \varepsilon \end{pmatrix} \begin{pmatrix} \psi_1(x, \nu) \\ \psi_{-1}(x, \nu) \end{pmatrix} = 0,$$

$0 < x < L$:

$$\begin{pmatrix} \xi(\hat{p}_x, k + 2\Phi\nu - \Phi x/L, p_z) - \varepsilon, & 0 \\ 0, & -\xi(\hat{p}_x, k + 2\Phi\nu + \Phi x/L, p_z) - \varepsilon \end{pmatrix} \times \begin{pmatrix} \psi_1(x, \nu) \\ \psi_{-1}(x, \nu) \end{pmatrix} = 0, \qquad (3)$$

$x > L$:

$$\begin{pmatrix} \xi(\hat{p}_x, k + 2\Phi\nu - \Phi, p_z) - \varepsilon, & \Delta \\ \Delta, & -\xi(\hat{p}_x, k + 2\Phi\nu + \Phi, p_z) - \varepsilon \end{pmatrix} \times \begin{pmatrix} \psi_1(x, \nu) \\ \psi_{-1}(x, \nu - 1) \end{pmatrix} = 0.$$

One of the basic inequalities to be used in the following expresses the fact that in the intermediate state in a macroscopic sample $L \gg \xi_0$, where $\xi_0 \sim v_F/\Delta$ is the coherence length in the superconductor. Here it is assumed that the temperature $T \ll \Delta$ since it is in exactly these conditions that excitations with energy $\epsilon > \Delta$ are unimportant. Since the wavefunctions of excitations with energy $\epsilon < \Delta$ fall away in the superconducting regions in a length ξ_0, the wavefunction in the normal layer $0 < x < L$ plays the basic role. For this reason it is appropriate to exclude the regions $x < 0$ and $x > L$ from the treatment by writing down effective boundary conditions for the wavefunction of the normal layer; these follow from the requirement that the wavefunction and its derivatives be continuous at the points $x = 0$ and $x = L$. With this aim, we shall consider the solution of Eq. (3) in the region $x < 0$:

$k + 2\Phi\nu \gtrless \sqrt{p_F^2 - p_z^2}$:

$$\begin{pmatrix} \psi_1(x, \nu) \\ \psi_{-1}(x, \nu) \end{pmatrix} \approx 0;$$

$k + 2\Phi\nu < \sqrt{p_F^2 - p_z^2}$:

$$\begin{pmatrix} \psi_1(x, \nu) \\ \psi_{-1}(x, \nu) \end{pmatrix} = \sum_{s=\pm 1} A_s(\nu) \begin{pmatrix} 1 \\ e^{is\alpha} \end{pmatrix} \exp[(isp(\nu) + q(\nu))x],$$

$$p(\nu) \approx \sqrt{p_F^2 - p_z^2 - (k + 2\Phi\nu)^2}, \quad q(\nu) \approx m\sqrt{\Delta^2 - \varepsilon^2}/p(\nu),$$
$$e^{i\alpha} = (\varepsilon + i\sqrt{\Delta^2 - \varepsilon^2})/\Delta.$$

The characteristic phases associated with the motion of the "electron-hole" complex are of order $mL\epsilon/p_F \sim L/\xi_0 \gg 1$.[2] Therefore, in the expressions given, we can neglect the phase α, and in this case these expressions yield the following simple conditions at the point $x = 0$:

$$\psi_1(0, \nu) = \psi_{-1}(0, \nu), \quad \frac{d}{dx}\psi_1(0, \nu) = \frac{d}{dx}\psi_{-1}(0, \nu). \qquad (4)$$

Analogous arguments show that the boundary condition at the point $x = L$ has the form

$$\psi_1(L, \nu) = \psi_{-1}(L, \nu - 1), \quad \frac{d}{dx}\psi_1(L, \nu) = \frac{d}{dx}\psi_{-1}(L, \nu - 1). \qquad (5)$$

Because of the continuity of the function and its derivative, these boundary conditions (4) and (5) must be satisfied by the normal-layer wavefunction, which, according to (3), obeys the usual oscillator equations ($0 < x < L$):

$$\{p_x^2 - [p^2 + 2m\sigma\varepsilon - (k + 2\Phi\nu - \sigma\Phi x/L)^2]\}\psi_\sigma(x, \nu) = 0,$$
$$\sigma = \pm 1, \quad p = \sqrt{p_F^2 - p_z^2}. \qquad (6)$$

A simple analysis of Eqs. (6) and the boundary conditions (4) and (5) shows that the wavefunction in the normal layer for any x can be represented in the form ($0 < x < L$)

$$\psi_1(x, \nu) = u(k + 2\Phi\nu - \Phi x/L), \quad \psi_{-1}(x, \nu) = u(k + 2\Phi\nu + \Phi x/L) \qquad (7)$$

where the function $u(\xi)$ satisfies the equation

$$\{\hat{p}_\xi^2 - (L/\Phi)^2[p^2 - (-1)^{[(\xi - k)/\Phi]}2m\varepsilon - \xi^2]\}u(\xi) = 0 \qquad (8)$$

and the boundary conditions at the discontinuity points

$$u(k + 2\Phi\nu - 0) = u(k + 2\Phi\nu + 0),$$
$$\frac{d}{d\xi}u(k + 2\Phi\nu - 0) = -\frac{d}{d\xi}u(k + 2\Phi\nu + 0),$$

$$u(k + 2\Phi\nu + \Phi - 0) = u(k + 2\Phi\nu + \Phi + 0),$$
$$\frac{d}{d\xi}u(k + 2\Phi\nu + \Phi - 0) = -\frac{d}{d\xi}u(k + 2\Phi\nu + \Phi + 0). \qquad (9)$$

Here $[x]$ denotes the whole-number part of x, so that the fractional part $\{x\} = x - [x] > 0$.

Further simplifications are based on the fact that in the intermediate state in the normal layer the field is equal to the critical field: $H = H_c \sim \Delta\sqrt{mp_F}$. Therefore, in Eqs. (6) and (8) the condition for quasi-classical behavior $r_H/\lambda_F \sim p_F^2/eH \gg 1$ (r_H is the radius of an electron orbit in the magnetic field) must necessarily be fulfilled. Since in Eqs. (6) and (8) it is the motion of the excitations in the intervals $\Delta x \sim L$ and $\Delta\xi \sim \Phi$ that is important, the neighborhood of the turning points $\xi = \pm p$ requires a special treatment. As can be seen from Eqs. (6), the quasi-classical approximation is inapplicable in the interval $\Delta x \sim (L/p_F\Phi)^{1/3}$ in the neighborhood of the turning points. This interval is small compared with L:

$$\left(\frac{L}{p_F\Phi}\right)^{1/3} \Big/ L \sim [e\sqrt{v_F}(p_F\xi_0)^2]^{-1/3}\frac{\xi_0}{L} \sim 10^{-2}\frac{\xi_0}{L} \ll 1.$$

Thus, the quasi-classical approximation is everywhere

[2] We are considering energy values ϵ which are not to small, since otherwise a model with a step-like change in the parameter $\Delta(\Delta(x) = 0$ for $0 < x < L$; $\Delta(x) = \Delta$ for $x < 0$, $x > L$) is inapplicable and it is necessary to take the detailed behaviour of the function $\Delta(x)$ into account.

applicable in Eqs. (6) and (8). Further, for energies $\epsilon < \Delta$ we have $m\epsilon \ll p_F^2$ and, moreover, since we are interested in the case $L/r_H = \Phi/p_F \ll 1$, comparing the quantity $m\epsilon \sim m\Delta$ with Φp_F shows that the stronger inequality $m\Delta/\Phi p_F \sim (e\sqrt{v_F}\, p_F \xi_0)^{-1} \xi_0/L \ll 1$ is fulfilled. Therefore, all the subsequent expressions can be linearized in the energy ϵ.

With these remarks taken into account, the quasiclassical solution of Eqs. (8) with the boundary conditions (9) has, as is easily verified, the following form ($p = \sqrt{p_F^2 - p_Z^2}$):

$|\xi| > p$:
$$u(\xi) = 0;$$
$|\xi| < p$:
$$u(\xi) = A(p^2 - \xi^2)^{-1/4} \sin S(\xi);$$
$$S(\xi) = -\frac{L}{\Phi}\int_{-p}^{\xi} (-1)^{[(t-k)/\Phi]}\sqrt{p^2 - t^2}\, dt + \frac{mL\varepsilon}{\Phi}\int_{-p}^{\xi}\frac{dt}{\sqrt{p^2 - t^2}}. \quad (10)$$

The condition that the function $u(\xi)$ be continuous at the turning point $\xi = p$ gives $\sin S(p) = 0$, $S(p) = \pi n$, whence we obtain the spectrum of the allowed energy values:

$$\varepsilon = \Omega n + \varepsilon_0(k, p_z), \quad \Omega = eH/m, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$\varepsilon_0(k, p_z) = \frac{1}{\pi m}\int_{-p}^{p} (-1)^{[(t-k)/\Phi]}\sqrt{p^2 - t^2}\, dt. \quad (11)$$

The spectrum (11) has a band character and is periodic in the quasi-momentum k with period 2Φ, in agreement with the initial periodicity in Eqs. (1). It must be borne in mind that the allowed values of the quantities n, p_Z, and k in formula (11) are limited by the condition $-\Delta < \epsilon < \Delta$.

It is not difficult to find the normalization constant A in formula (10). Normalizing the wavefunction (2), (7), (10) to a finite volume, we have

$$\int dV(\psi^*(x, y, z), \psi'(x, y, z)) = L_y L_z \delta_{kk'}\delta_{p_z p_z'}$$
$$\times \sum_{\nu}\int_0 dx[u(k + 2\Phi\nu - \Phi x/L)u'(k + 2\Phi\nu - \Phi x/L) +$$
$$+ u(k+2\Phi\nu+\Phi x/L)u'(k+2\Phi\nu+\Phi x/L)] = \frac{V}{\Phi}\delta_{kk'}\delta_{p_z p_z'}\int d\xi u(\xi)u'(\xi).$$

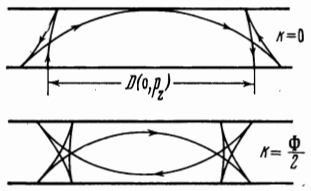
Omitting terms which oscillate over an electron wavelength, we obtain

$$\int dV(\psi^*(x, y, z), \psi'(x, y, z))$$
$$= \frac{A^2 V}{2\Phi}\delta_{kk'}\delta_{p_z p_z'}\int_{-1}\frac{d\eta}{\sqrt{1-\eta^2}}\cos\left[(n-n')\int_{-1}^{\eta}\frac{dt}{\sqrt{1-t^2}}\right] = \delta_{kk'}\delta_{p_z p_z'}\delta_{nn'},$$

whence follows

$$A = \sqrt{2\Phi/\pi V}. \quad (12)$$

The physical picture contained in this calculation is simple. If we put the free motion along the z-axis out of consideration, the electron and hole move in the normal layer along the arcs of circles. On reflection of the excitation from the boundary, the large normal momentum (of the order of the Fermi momentum p_F) of the excitation is conserved and an inversion occurs in the isotopic space of the "electron-hole": an excitation of the opposite sign moves in the reverse direction.[5] Since the canonical momentum along the y-axis is not conserved owing to the coherent phase difference of the order parameter (see Eqs. (1)), the center of the orbit undergoes a successive shift by $\pm 2L$ along the x-axis. At the same time, the ordinary momentum $k + 2\Phi\nu - \Phi x/L$ is conserved, but the velocity along the y-axis changes sign. The motion of the excitation as a whole can be turned around into the successive passage of an electron and a hole alternately along a complete arc of a circle (cf. the figure, where examples of the trajectory of an excitation are shown).



2. ABSORPTION OF ULTRASOUND

The absorption of ultrasound by electrons in superconductors in the intermediate state was calculated in [9] without taking account of quantum effects. At low temperatures when only the absorption in the normal layer is important, measurement of the ultrasound absorption enables us to judge the magnitude of the structure parameters of the intermediate state.

Turning to the calculation of the ultrasound absorption coefficient with allowance for the magnetic quantization, we note that for the results of the preceding section to be applicable to the intermediate state it is necessary that the length of the normal layers in the direction perpendicular to the magnetic field be great compared with the radius of an electron orbit in the magnetic field. Obviously, the electron mean free path must satisfy the same condition. On the other hand, as will be clear from the following, the characteristic sound wavelength must be small ($qL \gg 1$, q is the wavevector of the sound wave). Therefore, the absorption is essentially quantum absorption, and we can make use of the simplest "jellium" model[10, 11] to calculate the absorption coefficient of a longitudinal sound wave. In this model, the interaction between the electron and ion "liquids" is described by the introduction of a self-consistent electric potential φ and the interaction Hamiltonian for the electrons has the form

$$\mathcal{H}_{int} = \int dV e\varphi\hat{n}_e,$$

where $\hat{n}_e$ is the electron density operator. Correspondingly, the time-averaged energy losses are equal to

$$\bar{\mathcal{H}} = \int dV e\varphi\langle\hat{n}_e\rangle.$$

Here the dot denotes time-differentiation and the angular brackets denote statistical averaging. Hence, using known methods (cf. [12]) and the formula for the average energy of the ion "liquid," $\bar{\mathcal{H}} = \int dV\, \overline{n_i M u_i^2}$ (n_i is the density of the ions, $n_e = Zn_i$, M is the mass and u_i the velocity of the ions: $M\dot{u}_i = Ze\nabla\varphi$; the sound velocity in the "jellium" model is given by $s^2 = v_F^2 Zm/3M$), it is not difficult to obtain the electron-absorption coefficient for a plane sound wave (cf. the analogous formula in [13]):

$$\Gamma = \frac{\pi\omega}{4}\frac{mv_F^2}{3n_e V}\sum_{kk'}|M_{kk'}(q)|^2\left(\mathrm{th}\frac{\varepsilon}{2T} - \mathrm{th}\frac{\varepsilon'}{2T}\right)\delta(\varepsilon - \varepsilon' - \omega). \quad (13)$$

Here ω and $\mathbf{q}$ are the frequency and wave-vector of the sound wave, and λ is a complete set of electron quantum numbers. The matrix element $M_{\lambda\lambda'}(\mathbf{q})$ is given by the following formula:

$$M_{\lambda\lambda'}(\mathbf{q}) = \int dV(\psi^*(\mathbf{r},\lambda)\sigma_z e^{i\mathbf{q}\mathbf{r}}\psi(\mathbf{r},\lambda')), \qquad (14)$$

where σ_z is a Pauli matrix and $\psi(\mathbf{r}, \lambda)$ is the two-component wavefunction of the electron-hole excitation. In the normal state, the absorption given by formula (13) is equal to $\Gamma_n = \pi\omega^2/4v_F|q|$.[10,11]

In order to exclude absorption in the superconducting regions and threshold effects associated with the difference between the levels (11), we shall consider temperatures and frequencies satisfying the inequalities $\Omega \ll T \ll \Delta$, $\omega \ll \Delta$. Hence follows the inequality

$$\frac{q}{\Phi} \ll \frac{\Delta}{\Phi s} \sim \frac{v_F}{s}(\sqrt{v_F}ep_F\xi_0)^{-1}\frac{\xi_0}{L} \sim \frac{\xi_0}{L} \ll 1.$$

Then we consider the quantity

$$qL = \frac{\omega}{\Omega}\frac{\Phi}{ms} \sim \frac{\omega}{\Omega}\frac{L}{r_H}\frac{v_F}{s} \sim 10^3\frac{L}{r_H}\frac{\omega}{\Omega},$$

where r_H is the electron orbit radius in the magnetic field. It is found that, in most cases in the intermediate state, $L/r_H \sim 10^{-1}$ and for frequencies ω that are not too small (compared with Ω), the relation $qL \gg 1$ is valid. The above inequalities simplify the calculations of the matrix element (14) and the absorption coefficient (13).

We shall consider the case of propagation of a wave over the normal layer along the y-axis perpendicular to the magnetic field ($\mathbf{q}\cdot\mathbf{r} = qy$). Putting the expressions (2) and (7) for the wavefunction into formula (14) for the matrix element and proceeding in the same way as in the calculation of the normalization integral in Sec. 1, we obtain

$$M_{\lambda\lambda'}(q) = -\delta_{p_z p_z'}\delta_{k,k'+2\Phi\{q/2\Phi\}}J(q),$$

$$J(q) = \frac{V}{\Phi}\int d\xi(-1)^{[(\xi-k)/\Phi]}u(\xi)u'(\xi-q). \qquad (15)$$

Hence, taking account of the explicit expressions (10) and (12) for the function $u(\xi)$ (after discarding the terms oscillating over an electron wavelength), it follows, in view of the inequality $q \ll \Phi$, that

$$J(q) = \frac{1}{\pi}\int_{-p}^{p} d\xi\frac{(-1)^{[(\xi-k)/\Phi]}}{\sqrt{p^2-\xi^2}}\cos\chi(\xi),$$

$$\chi(\xi) = \frac{qL}{\Phi}\int_{-p}^{\xi}(-1)^{[(t-k)/\Phi]}\frac{t\,dt}{\sqrt{p^2-t^2}} + \frac{mL\omega}{\Phi}\int_{-p}^{\xi}\frac{dt}{\sqrt{p^2-t^2}}. \qquad (16)$$

In the latter expression the difference $\epsilon - \epsilon'$ is replaced by ω in accordance with formula (13).

The phase $\chi(\xi)$ in formula (16) is large and to calculate the integral J(q) in (16) it is sufficient to find the stationary points $\chi(\xi)$. There is only one such point, $\xi_0 \approx (-1)\exp[k/\Phi]\, m\omega/q \ll \Phi$. (The extremities of the interval $\xi = \pm p$ give a small contribution to the integral J(q). In the neighborhood of the point ξ_0

$$\chi(\xi) \approx \chi(0) - (-1)^{[k/\Phi]}\frac{qL}{2\Phi p}(\xi-\xi_0)^2.$$

The subsequent calculation of the integral (16) is elementary and gives the following result:

$$J(q) \approx -(-1)^{[k/\Phi]}\sqrt{\frac{2\Phi}{\pi|q|Lp}}\cos\chi(0). \qquad (17)$$

Finally, we calculate the sum of the hyperbolic tangents occurring in expression (13):

$$\sum_n\left(\mathrm{th}\frac{\Omega n + \varepsilon_0(k,p_z)}{2T} - \mathrm{th}\frac{\Omega n + \varepsilon_0(k,p_z) - \omega}{2T}\right),$$

$$-\Delta < \Omega n + \varepsilon_0(k,p_z) < \Delta.$$

In view of the inequalities $\Omega \ll T \ll \Delta$, this sum can be replaced by the integral:

$$\frac{1}{\Omega}\int_{-\infty}^{+\infty}dx\left(\mathrm{th}\frac{x}{2T} - \mathrm{th}\frac{x-\omega}{2T}\right) = \frac{2\omega}{\Omega}.$$

Putting this quantity and formulas (11), (15), and (17) into expression (13) for the absorption coefficient and discarding the oscillating term, we obtain

$$\Gamma = \Gamma_n\frac{1}{\pi}\int_{-p_F}^{p_F}\frac{dp_z}{\sqrt{p_F^2-p_z^2}}\int\frac{dk}{2\Phi}\Omega\sum_n\delta\left(\Omega n + \frac{\partial\varepsilon_0(k,p_z)}{\partial k}q - \omega\right),$$

$$\frac{\partial\varepsilon_0(k,p_z)}{\partial k} = -\frac{1}{\pi m}\int_{-p}^{p}(-1)^{[(t-k)/\Phi]}\frac{t\,dt}{\sqrt{p^2-t^2}}. \qquad (18)$$

In the latter expression for Γ, we can neglect the frequency ω in comparison with the second term in the argument of the δ-function. Using Poisson's summation formula, we find the oscillating part of the absorption (18):

$$\left(\frac{\Gamma}{\Gamma_n}\right)_{\mathrm{osc}} = \sum_{n\neq 0}\frac{1}{\pi}\int_{-p_F}^{p_F}\frac{dp_z}{\sqrt{p_F^2-p_z^2}}\int\frac{dk}{2\Phi}\exp[inf(k,p_z)],$$

$$f(k,p_z) = \frac{2qL}{\Phi}\int_{-p}^{p}(-1)^{[(t-k)/\Phi]}\frac{t\,dt}{\sqrt{p^2-t^2}}, \qquad p = \sqrt{p_F^2-p_z^2}. \qquad (19)$$

The phase $f(k, p_z)$ is large and in the successive integration over k and p_z in expression (19) it is again necessary to make use of the stationary-phase method. As is shown in the Appendix, within the limits of a period in k the only stationary points are $k = 0$ and $k = \Phi$, while the function $f(0, p_z)$ is stationary at the point $p_z = 0$. Taking account of this, it is not difficult to write the formal result of the integration and summation in formula (19):

$$\left(\frac{\Gamma}{\Gamma_n}\right)_{\mathrm{osc}} = -\frac{2(-1)^{[p_F/\Phi]}U(f(0,0))}{p_F\Phi}\left(\left|\frac{\partial^2 f(0,0)}{\partial p_z^2}\frac{\partial^2 f(0,0)}{\partial k^2}\right|\right)^{-1/2} \qquad (20)$$

where U(x) is a periodic function:

$$U(x) = -\sum_{n\neq 0}\frac{\sin nx}{n}, \quad 0 < x < 2\pi, \quad U(x) = \frac{1}{2}(x-\pi). \qquad (21)$$

The function f(0, 0) occurring in expression (20) and its derivatives are given, according to (19), by the following relations:

$$f(0,0) = \frac{4qL}{\Phi}\sum_{n=-[p_F/\Phi]}^{[p_F/\Phi]}(-1)^n\sqrt{p_F^2-(\Phi n)^2},$$

$$\frac{\partial^2 f(0,0)}{\partial p_z^2} = -\frac{4qL}{\Phi}\sum_{n=-[p_F/\Phi]}^{[p_F/\Phi]}\frac{(-1)^n}{\sqrt{p_F^2-(\Phi n)^2}},$$

$$\frac{\partial^2 f(0,0)}{\partial k^2} = -\frac{4qL}{\Phi}\sum_{n=-[p_F/\Phi]}^{[p_F/\Phi]}(-1)^n\frac{p_F^2}{[p_F^2-(\Phi n)^2]^{3/2}}.$$

In view of the inequality $\Phi \ll p_F$, further simplifications are possible in these formulas, giving as a result

$$f(0,0) \approx -2qL\sqrt{\frac{2p_F}{\Phi}}\Pi\left(\frac{p_F}{\Phi}\right),$$

$$p_F\Phi\frac{\partial^2 f(0,0)}{\partial p_z^2} \approx 2qL\sqrt{\frac{2p_F}{\Phi}}\Pi'\left(\frac{p_F}{\Phi}\right), \qquad (22)$$

$$\Phi^2\frac{\partial^2 f(0,0)}{\partial k^2} \approx -2qL\sqrt{\frac{2p_F}{\Phi}}\Pi''\left(\frac{p_F}{\Phi}\right).$$

The function $\Pi(x)$ and its derivatives are periodic in x and are determined by the following relations:

$$\Pi(x) = \int_0^\infty (-1)^{[t-x]}\frac{dt}{\sqrt{t}}, \quad \Pi'(x) = -2(-1)^{[x]}\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+\{x\}}}, \qquad (23)$$

$$\Pi''(x) = (-1)^{[x]}\sum_{n=0}\frac{(-1)^n}{(n+\{x\})^{3/2}}.$$

Putting the relations (22) into formula (20), we obtain the final result:

$$\left(\frac{\Gamma}{\Gamma_n}\right)_{osc} = -\frac{(-1)^{[r_H/L]}U(2q\sqrt{2r_HL}\Pi(r_H/L))}{|q|r_H\sqrt{2}\,|\Pi'(r_H/L)\Pi''(r_H/L)|},$$
$$U(x) = -\sum_{n\neq 0}\frac{\sin nx}{n}, \quad 0<x<2\pi, \quad U(x) = \frac{1}{2}(x-\pi). \qquad (24)$$

The ultrasound absorption oscillations described by formulas (23) and (24) are of a comparatively complicated character. Since in the intermediate state the field in the normal layer is constant and the electron orbit radius $r_H = p_F/eH_c$ is fixed, the most convenient experimentally varying parameter is the thickness of the normal layer as a function of the external magnetic field. According to formulas (24) and (23), on change of the thickness L, periodic "bursts" in the ultrasound absorption should be observed, with characteristic period determined from the condition

$$2q\sqrt{2r_HL}\left|\Pi'\left(\frac{r_H}{L}\right)\right|\frac{r_H\Delta L}{L^2} = 2\pi,$$

$$\frac{\Delta L}{L} \sim \frac{1}{qr_H}\sqrt{\frac{L}{r_H}}.$$

These oscillations are modulated in amplitude and frequency by the periodic function $\Pi(r_H/L)$ (23) with significantly larger period: $\Delta L/L = L/r_H$; as can be seen from the formulas (23), the function $\Pi(x)$ contains root singularities, and this should lead, according to (24), to a sharp decrease in the amplitude of the oscillations on increase of the frequency. Rough estimates show that the fundamental period, in the magnetic field, of the oscillations is about 1 Oe, whereas the modulation period is of the order of 10–100 Oe.

The physical meaning of the modulation period is clear. The periodicity of the function $\Pi(x)$ (23) is associated with the commensurability or non-commensurability of the layer thickness L and the radius of the electron orbit in a magnetic field. As regards the fundamental period of the oscillations, this is more simply interpreted if in formula (24) we consider the oscillations in the absorption as the wave-vector q is changed. From this point of view, we shall consider the trajectories of the excitations in coordinate space (cf. the figure). From the calculation of the matrix element $M_{\lambda\lambda'}$ (15), (16), it follows that an electron makes the principal contribution to the ultrasound absorption when it moves perpendicularly to the direction of propagation of the wave (i.e., at the ends of the trajectories depicted in the figure). This corresponds to a well-known fact; it is mainly the electrons in a narrow "band" ($\mathbf{q}\cdot\mathbf{v} \sim 0$) at the Fermi surface which absorb. Resonance in the absorption sets in when (for longitudinal sound) a whole number of waves fits in between two points of efficient absorption of sound by the electron. For the trajectories drawn in the figure, this distance $D(k, p_Z)$, as an elementary calculation shows, is equal to

$$D(k,p_z) = 2\sum_{n=-[N_+]}^{[N_-]}(-1)^n[r_H^2(p_z) - L^2(n+k/\Phi)^2]^{1/2},$$
$$N_\pm = r_H(p_z)/L \pm k/\Phi.$$

Or, alternatively,

$$D(k,p_z) = \int_{-r_H(p_z)}^{r_H(p_z)}(-1)^{[y/L-k/\Phi]}\frac{y\,dy}{\sqrt{r_H^2(p_z)-y^2}}, \quad r_H(p_z) = \frac{\sqrt{p_F^2-p_z^2}}{eH} = \frac{Lp}{\Phi}.$$

Comparing the latter expression with formula (19), it is not difficult to see that the oscillating part of the ultrasound absorption (20) can be written in the form:

$$\left(\frac{\Gamma}{\Gamma_n}\right)_{osc} = -\frac{(-1)^{[r_F/\Phi]}U(2qD(0,0))}{p_F\Phi|q|}\left(\left|\frac{\partial^2 D(0,0)}{\partial p_z^2}\frac{\partial^2 D(0,0)}{\partial k^2}\right|\right)^{-1/2} \qquad (25)$$

Thus, the period of the oscillations is determined by the extremal "orbit diameter" and the effect is in some sense analogous to the oscillations in the ultrasound absorption in normal metals in a magnetic field predicted by Pippard.[14] However, in the given case, in contrast to the Pippard oscillations, according to formula (25) a doubling of the oscillation period occurs, due to the correlation between the electron and hole.

APPENDIX

To prove the statements, made in the text of the article, concerning the function $f(k, p_Z)$ (19), we represent it in the following form:

$$f(k,p_z) = 4qLh(p/\Phi,\ k/\Phi), \quad p = \sqrt{p_F^2-p_z^2},$$

where the function h(x, y) is defined by the formula

$$h(x,y) = \sum_{n=-[x-y]}^{[x+y]}(-1)^n\sqrt{x^2-(n-y)^2}, \quad x>0. \qquad (A.1)$$

Since $h(x, y+1) = -h(x, y)$ and $h(x, -y) = h(x, y)$, it is sufficient to study the properties of h(x, y) as a function of y in the interval $(0 < y < \frac{1}{2})$. Differentiating (A.1) with respect to y, we obtain

$$\frac{\partial h(x,y)}{\partial y} = \sum_{n=-[x-y]}^{[x+y]}(-1)^n\frac{n-y}{\sqrt{x^2-(n-y)^2}}. \qquad (A.2)$$

We note that at the point $y = 0$ we have $\partial h(x, y)/\partial y = 0$.

We consider first the case when the sum $[x+y] + [x-y]$ is odd. A simple regrouping of the terms in

formula (A.2) leads to the following result ($L = \frac{1}{2}([x+y]+[x-y]-1)$):

$$(-1)^{[x-y]}\frac{\partial h(x,y)}{\partial y}=\sum_{l=0}^{L}\left[\frac{2l-[x-y]-y}{\sqrt{x^2-(2l-[x-y]-y)^2}}-\frac{2l+1-[x-y]-y}{\sqrt{x^2-(2l+1-[x-y]-y)^2}}\right]<0.$$

In the case when the sum $[x+y]+[x-y]$ is even, the inequalities $y<\{x\}$ and $y<1-\{x\}$ must be fulfilled. In this case

$$\frac{\partial h(x,y)}{\partial y}=\frac{1}{2}\sum_{n=-[x]}^{[x]}(-1)^n\left[\frac{n-y}{\sqrt{x^2-(n-y)^2}}-\frac{n+y}{\sqrt{x^2-(n+y)^2}}\right] \tag{A.3}$$

$$=-\frac{y}{\sqrt{x^2-y^2}}-\sum_{n=1}^{[x]}(-1)^n\left[\frac{n+y}{\sqrt{x^2-(n+y)^2}}-\frac{n-y}{\sqrt{x^2-(n-y)^2}}\right]$$

If $[x]$ is even, then

$$\frac{\partial h(x,y)}{\partial y}=-\frac{y}{\sqrt{x^2-y^2}}-\sum_{l=1}^{\frac{1}{2}[x]}\left\{\left[\frac{2l+y}{\sqrt{x^2-(2l+y)^2}}-\frac{2l-y}{\sqrt{x^2-(2l-y)^2}}\right]-\left[\frac{2l-1+y}{\sqrt{x^2-(2l-1+y)^2}}-\frac{2l-1-y}{\sqrt{x^2-(2l-1-y)^2}}\right]\right\}<0.$$

For odd $[x]$, the expression (A.3) can be represented in another form:

$$\frac{\partial h(x,y)}{\partial y}=\frac{y}{\sqrt{x^2-y^2}}+\sum_{l=0}^{\frac{1}{2}([x]-1)}\left\{\left[\frac{2l+1+y}{\sqrt{x^2-(2l+1+y)^2}}-\frac{2l+1-y}{\sqrt{x^2-(2l+1-y)^2}}\right]-\left[\frac{2l+y}{\sqrt{x^2-(2l+1+y)^2}}-\frac{2l-y}{\sqrt{x^2-(2l+1-y)^2}}\right]\right\}>0.$$

Thus, the derivative $\partial h(x, y)/\partial y \neq 0$ everywhere, apart from the point $y = 0$, in the region of regularity.

The uniqueness of the stationary point $p_z = 0$ of the function $f(0, p_z)$ is proved analogously.

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